CHAPTER 1

Additive Schwarz Methods for Elliptic Finite Element Problems in Three Dimensions

Maksymilian Dryja*
Olof B. Widlund†

Abstract. Many domain decomposition algorithms and certain multigrid methods can be described and analyzed as additive Schwarz methods. When designing and analyzing domain decomposition methods, we encounter special difficulties in the case of three dimensions and if the coefficients are discontinuous and vary over a large range. In this paper, we first introduce a general framework for Schwarz methods. Three classes of applications are then considered: certain wire basket based iterative substructuring methods, Neumann-Neumann algorithms with low dimensional, global subspaces and a modified form of a multilevel algorithm introduced by Bramble, Pasciak and Xu.

Introduction. In this paper, we discuss additive Schwarz methods for solving systems of linear algebraic equations, which result from finite element approximations of second order, elliptic problems in three dimensional, bounded regions. A general framework is presented which is quite useful in the design and analysis of a variety of domain decomposition and some multigrid methods. Three methods are then described and analyzed. They are extensions of methods previously considered in the literature for solving the systems of algebraic equations which correspond to the

* Department of Mathematics, Warsaw University, 2 Banach Street, 00-913 Warsaw, Poland. This work was supported in part by the National Science Foundation under Grant NSF-CCR-8903003, and in part by Polish Scientific Grant #R.R.I. 10.
† Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, N.Y. 10012. This work was supported in part by the National Science Foundation under Grant NSF-CCR-8903003 and, in part, by the U. S. Department of Energy under contract DE-FG02-88ER250553 at the Courant Mathematics and Computing Laboratory.
interfaces between the substructures. We also consider problems with discontinuous coefficients with a great variation in the values.

The first method considered is an iterative substructuring algorithm recently developed by B. Smith [9], [10]. The second is a domain decomposition method developed by R. Glowinski, P. Le Tallec, Y.-H. De Roeck et al., see [1], [3]. Finally, we consider a variant of a multigrid-like method discovered by J. H. Bramble, J. E. Pasciak and J. Xu [2].

The paper is organized as follows. In Section 2 a finite element approximation of second order, elliptic problems is considered. A system of algebraic equations corresponding to the discrete problem is reduced to a system defined on the interfaces of the substructures after eliminating the interior variables associated with the interior nodal points of each substructure. This reduced system with the Schur complement matrix can be solved by a variety of iterative methods.

In Section 3, a general abstract framework for additive Schwarz methods is introduced, see also [5], [6]. In Sections 4, 5 and 6 the three algorithms mentioned above are described and analyzed inside this framework. In particular, we discuss problems with discontinuous coefficients and show that the rate of convergence of certain variants of the methods can be made independent of the variation of the coefficients.

The results of Section 4 of this paper have been obtained jointly with Barry Smith; see further [4].

2. Differential and finite element model problems. To simplify the presentation, we discuss only two model problems, a standard Poisson equation and a special second order problem with discontinuous, piecewise constant coefficients. We call them model problem I and II, respectively. The continuous model problem I is of the form:

Find \( u \in H^1_0(\Omega) \) such that

\[
(2.1) \quad a(u, v) = f(v), \quad v \in H^1_0(\Omega) 
\]

where

\[
(2.2) \quad a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, ds, \quad f(v) = \int_{\Omega} fv \, dx 
\]

For simplicity, we let \( \Omega \) be a bounded polyhedral region in three dimensions. A coarse triangulation of \( \Omega \) is introduced by dividing the region into nonoverlapping simplices \( \Omega_i, i = 1, \ldots, N \), which are also called substructures. The substructures \( \Omega_i \) are further divided into elements \( e_j \). We associate parameters \( H \) and \( h \) with the coarse and fine triangulations and assume that these triangulations are shape regular in the sense common to finite element theory.

Let \( V^h(\Omega) \) be the finite element space of continuous, piecewise linear functions defined on the fine triangulation and which vanish on \( \partial \Omega \), the boundary of \( \Omega \). The discrete model problem I is of the form:

Find \( u_h \in V^h \) such that

\[
(2.3) \quad a(u_h, v_h) = f(v_h), \quad v_h \in V^h 
\]
or alternatively, find the vector $x$ of nodal values such that

$$Kx = b$$

(2.4)

Here $K$ is the stiffness matrix, $x$ is the vector of nodal values and $b$ the load vector.

Our goal is to describe and analyze iterative methods for solving (2.2) which can be regarded as additive Schwarz methods.

All our results can be extended to general conforming finite element approximations of any self-adjoint, second order, elliptic problem. This includes the case when there is a great variation in the values of the coefficients. Here we consider only the case when

$$a(u, v) = \int_{\Omega} \rho(x) \nabla u \cdot \nabla v \, dx$$

(2.5)

where $\rho(x) > 0$ is a piecewise constant function. Equation (2.1) with the bilinear form (2.5) is called model problem II. We assume that the jumps of $\rho(x)$ occur only at substructure boundaries. Thus, $\rho(x) = \rho_i = \text{constant} > 0$ on the substructure $\Omega_i$. The methods discussed can be generalized to the case when $\rho(x)$ varies continuously in each subregion.

Let $K$ be the stiffness matrix given by the bilinear form (2.2) or (2.5). In the first step of many iterative substructuring methods, the unknowns in the interior of the substructures are eliminated. This reduces the system (2.4) to a system of linear algebraic equations associated with the interfaces only. We now describe this procedure.

Let $K^{(i)}$ be the stiffness matrix of the bilinear $a_{\Omega_i}(u_h, v_h)$ which represents the contribution of the substructure $\Omega_i$ to the integral $a_{\Omega_i}(u_h, v_h) = a(u_h, v_h)$. Let $x$ and $y$ be the vectors of nodal values that correspond to the finite element functions $u_h$ and $v_h$, respectively. Then the stiffness matrix $K$ of the entire problem can be obtained by using the method of subassembly,

$$x^T Ky = \sum_i x^{(i)T} K^{(i)} y^{(i)}$$

(2.6)

Here $x^{(i)}$ is the subvector of nodal parameters associated with $\overline{\Omega_i}$, the closure of $\Omega_i$. We represent $K^{(i)}$ as

$$
\begin{pmatrix}
K^{(i)}_{ii} & K^{(i)}_{iB} \\
K^{(i)T}_{iB} & K^{(i)}_{BB}
\end{pmatrix}
$$

(2.7)

dividing the subvector $x^{(i)}$ into two, $x^{(i)}_I$ and $x^{(i)}_B$, corresponding to the variables which are interior to the substructure and those which are shared with other substructure, i.e. they are associated with the nodal points of $\partial \Omega_i$. Since the interior variables are associated with only one of the substructures, they can be eliminated locally and in parallel. The resulting reduced matrix is a Schur complement and is of the form

$$S^{(i)} = K^{(i)}_{BB} - K^{(i)T}_{iB} K^{(i)T}_{iI} K^{(i)}_{iB}$$

(2.8)
From this it follows that the Schur complement corresponding to the global stiffness matrix $K$ is given by $S$ where

$$x_B^T S y_B = \sum_i x_B^{(i)T} S^{(i)} y_B^{(i)}$$

If the local problems are solved exactly what remains is to find a sufficiently accurate approximation of solution of the linear system

$$S x_B = g_B$$

Note that the elimination of the interior variables of a substructure can be viewed in terms of an orthogonal projection, with respect to the bilinear form, of the solution $u_h$ of (2.3) onto the subspace $H^1_0(\Omega_i) \cap V^h$. It is easy to show that these subspaces are orthogonal, in the sense of $a(u, v)$, to the piecewise discrete harmonic functions given by

$$a(u_h, v_h) = 0, \quad v_h \in H^1_0(\Omega_i) \cap V^h, \quad i = 1, \ldots, N$$

or alternatively by

$$K^{(i)}_{II} x^{(i)} + K^{(i)}_{IB} x_B^{(i)} = 0$$

It is convenient to rewrite (2.10) in variational form. Let $s_i(u, v)$ and $s(u, v)$ denote the forms defined by (2.9), i.e.

$$s_i(u, v) = x_B^{(i)} S^{(i)} y_B^{(i)} \quad \text{and} \quad s(u, v) = x_B^T S y_B$$

Equation (2.10) can then be rewritten as

$$s(u, v) = (g, v)_{L^2(\Gamma)}, \quad v \in V^h(\Gamma)$$

Here $u$ is the discrete harmonic part of the solution and $V^h(\Gamma) \subset H^{1/2}(\Gamma)$ the restriction of $V^h(\Omega)$ to $\Gamma$.

Problem (2.14) can be solved by different iterative methods of additive Schwarz type.

3. **An abstract additive Schwarz method.** We now describe and analyze the convergence of abstract additive Schwarz methods. We note that the theory presented here is a modified version of the theory developed in our previous papers, see [5], [6], [7]. The difference is that we now include the effects of inexact solvers from the very beginning.

Let $V$ be a finite dimensional space with the scalar product $a(u, v)$. We consider the abstract problem

$$a(u, v) = f(v), \quad v \in V$$

Let

$$V = V_0 + V_1 + \cdots + V_N$$
and let the $b_i(u,v)$ be symmetric, positive definite bilinear forms on $V_i \times V_i$. We introduce operators $T_i: V \to V_i$, by

$$b_i(T_iu,v) = a(u,v), \quad v \in V_i$$

and put

$$T = T_0 + T_1 + \cdots + T_N$$

We replace (3.1) by

$$Tu = g, \quad g = \sum_{i=0}^N g_i, \quad g_i = T_iu$$

**Theorem 3.1.** Let

(i) there exist a constant $C_0$ such that for all $u \in V$ there exists a decomposition $u = \sum_{i=0}^N u_i, \ u_i \in V_i$, such that

$$\sum_{i=0}^N b_i(u_i,u_i) \leq C_0^2 a(u,u)$$

(ii) there exist a constant $\omega$ such that for $i = 0, \ldots, N$,

$$a(u,u) \leq \omega b_i(u,u), \quad u \in V_i$$

(iii) there exist constants $\epsilon_{ij}$, for $i, j = 1, \ldots, N$, such that

$$a(u_i,u_j) \leq \epsilon_{ij} a(u_i,u_i)^{1/2} a(u_j,u_j)^{1/2}, \quad u_i \in V_i, \quad u_j \in V_j$$

Then

$$C_0^{-2} a(u,u) \leq a(Tu,u) \leq (\rho(\epsilon) + 1)\omega a(u,u), \quad u \in V$$

where $\rho(\epsilon)$ is the spectral radius of the matrix $\epsilon = \{\epsilon_{ij}\}_{i,j=1}^N$.

A proof of this theorem can be found in [8].

4. A wirebasket based method. In this section, we describe and analyze an iterative substructuring method, recently developed by B. Smith, see [9], [10]. For the description and analysis, we use the general framework of Section 3. Here $V = V^h(\Gamma)$ and

$$a(u,v) = s(u,v)$$

4.1. A method without a vertex space. To describe the method, we need some notations. Let $F_{ij}$, be the open faces and let $W_i$ be the wirebasket of the substructure $\Omega_i$. $F_{ij}$ is the closure of the face common to the substructures $\Omega_i$ and $\Omega_j$. The wirebasket is the union of the closures of the edges of $\Omega_i$. Let $W_{ih}$ and $\partial F_{ijh}$ denote the set of nodal points belonging to $W_i$ and $\partial F_{ij}$, the boundary of $F_{ij}$, respectively. Let

$$\bar{u}_i = \frac{1}{n_i} \sum_{x \in W_{ih}} u(x), \quad \bar{u}_{ij} = \frac{1}{n_{ij}} \sum_{x \in \partial F_{ijh}} u(x)$$

(4.2)
where \( n_i \) and \( n_{ij} \) are the number of nodal points on \( \mathcal{W}_i \) and \( \partial \mathcal{F}_{ij} \), respectively. We can now introduce the representation

\[
V^h(\Gamma) = V_0(\Gamma) + \sum_{ij} V_{ij}(\Gamma)
\]

Here \( V_{ij}(\Gamma) \) is the space of functions \( v \in V^h(\Gamma) \) which vanish at all nodal points not on \( \mathcal{F}_{ij} \). The space \( V_0 \) is defined in a special way. It is a space of continuous, piecewise linear functions defined, on \( \partial \Omega_i \), by its values given on the wirebasket \( \mathcal{W}_i \) and by constant values at the nodal points of each face \( \mathcal{F}_{ij} \). The constant associated with \( \mathcal{F}_{ij} \) is the average of the nodal values on \( \partial \mathcal{F}_{ij} \). The basis functions of \( V_0 \) are of the form

\[
\phi_i^{(b)}(x) = \phi_i(x) + \frac{1}{n_{ij}} \theta_{ij}(x) \quad \text{on} \quad \mathcal{F}_{ij}
\]

Here \( \phi_i(x) \) is the standard nodal basis function for \( x_i \in \mathcal{W}_i \) and \( \theta_{ij}(x) \in V^h \) is given by

\[
\theta_{ij}(x) = \begin{cases} 
1 & \text{at the nodal points } x \in \mathcal{F}_{ij} \\
0 & \text{at the nodal points } x \in \partial \mathcal{F}_{ij}
\end{cases}
\]

It is also possible to consider the coarse space \( V_0 \) as the range of an interpolation operator defined on \( \mathcal{F}_{ij} \) by

\[
\tilde{I}_h u = \sum_{x_k \in \mathcal{F}_{ij}} u(x_k) \phi_k(x) + \bar{u}_{ij} \theta_{ij}(x)
\]

We now define the quadratic forms corresponding to the different subspaces. Let \( b_0(u,v) : V_0 \times V_0 \to R \), be of the form

\[
b_0(u,v) = (1 + \log \frac{H}{h}) \sum_i \sum_{x \in \mathcal{W}_i} h(u(x) - \bar{u}_i)(v(x) - \bar{v}_i)
\]

and

\[
b_0(u,v) = (1 + \log \frac{H}{h}) \sum_i \rho_i \sum_{x \in \mathcal{W}_i} h(u(x) - \bar{u}_i)(v(x) - \bar{v}_i)
\]

for model problems I and II, respectively. To define \( b_{ij}(u,v) : V_{ij} \times V_{ij} \to R \), let \( \Omega_{ij} = \Omega_i \cup \Omega_j \cup \mathcal{F}_{ij} \) and introduce the discrete harmonic extension operator \( \mathcal{H}_{ij} \) from \( \mathcal{F}_{ij} \) to \( \Omega_{ij} \) by

\[
a_{ij}(\mathcal{H}_{ij}u,v) = 0, \quad v \in V^h(\Omega_i) \cup V^h(\Omega_j)
\]

\[
\mathcal{H}_{ij}u = u \quad \text{on} \quad \mathcal{F}_{ij}, \quad \mathcal{H}_{ij}u = 0 \quad \text{on} \quad \partial \Omega_{ij}
\]

Here \( V^h(\Omega_i) = V^h(\Omega) \cap H^1_0(\Omega_i) \) and \( \mathcal{H}_{ij}u \in V^h(\Omega_{ij}) \). We define the bilinear form as

\[
b_{ij}(u,v) = a(\mathcal{H}_{ij}u, \mathcal{H}_{ij}v)
\]
ELLiptic Finite Element Problems

We have now defined all our subspaces and the associated bilinear forms. As in the abstract theory, we define the operators by

\[(4.9a)\quad b_0(T_0u, v) = s(u, v), \quad v \in V_0\]

and

\[(4.9b)\quad b_{ij}(T_{ij}u, v) = s(u, v), \quad v \in V_{ij}\]

and let

\[T = T_0 + \sum_{ij} T_{ij}\]

**Theorem 4.1.** For all \(u \in V^h(\Gamma)\), with \(T_0\) and \(T_{ij}\) defined by (4.9), we have

\[(4.10)\quad \gamma_0(1 + \log \frac{H}{h})^{-2} s(u, u) \leq s(Tu, u) \leq \gamma_1 s(u, u)\]

Here \(\gamma_0\) and \(\gamma_1\) are constants independent of \(H\) and \(h\) and the jumps of the coefficients.

To prove this theorem, we use Theorem 3.1; cf. Dryja, Smith and Widlund [4]. For an alternative proof, see Smith [9], [10].

We now briefly discuss how the method can be implemented. Problem (2.14) has been replaced by

\[(4.11)\quad Tu = g\]

where \(g = g_0 + \sum g_{ij}, g_0 = T_0 u_h,\) and \(g_{ij} = T_{ij} u_h.\)

To solve (4.11), we can use the conjugate gradient method since \(T\) is symmetric, positive definite and well conditioned. For simplicity, we only consider the first Richardson method. Thus

\[u^{n+1} = u^n - \tau T(u^n - u), \quad \tau = 2/(\gamma_0(1 + \log \frac{H}{h})^{-2} + \gamma_1)\]

Let \(r^n = T(u^n - u) = r^n_0 + \sum_{ij} r^n_{ij},\) where \(r^n_0 = T_0(u^n - u)\) and \(r^n_{ij} = T_{ij}(u^n - u).\) To find \(r^n_0\) and \(r^n_{ij},\) we solve

\[(4.12a)\quad b_0(r^n_0, v) = s(u^n, v) - (g, v) \equiv F(v), \quad v \in V_0\]

and

\[(4.12b)\quad b_{ij}(r^n_{ij}, v) = F(v), \quad v \in V_{ij}\]

To compute \(s(u^n, v),\) we solve the Dirichlet problems

\[(4.13)\quad a(H_k u^n, v) = 0, \quad v \in H^1(\Omega_k) \cap V^h(\Omega)\]

with \(u^n\) given on \(\partial \Omega_k.\) \(H_k u^n\) is the discrete harmonic extension of \(u^n\) from \(\partial \Omega_k\) to \(\Omega_k.\)

The problem (4.12a) reduces to a system with a sparse matrix and block Gaussian
elimination is used to find \( r_0^2(x) \) at the nodal points of \( x \in \mathcal{W}_i \); cf. Dryja, Smith and Widlund [4]. To find \( r_0^n \), we solve a problem similar to (4.7), after replacing \( \Omega_k \) with \( \Omega_{ij} \).

4.2. The method with vertex spaces. In this subsection, we discuss a variant of a method discussed in Chapter 4 of Smith [9]. The estimate in Theorem 4.1 contains two log factors. We can remove one of them for model problem I by adding vertex spaces to the representation (4.3) of \( V^h(\Gamma) \); we call this a wirebasket based method with vertex spaces. For each substructure vertex \( x_k \), we define a space \( V_k(\Gamma) \) as follows. Let \( \bar{\mathcal{F}}_k \) be the intersection of the faces, which have the vertex \( x_k \) in common, with a neighborhood of \( x_k \). We assume that \( \text{dist}(x_k, \partial \bar{\mathcal{F}}_k) \) is on the order of \( H \). The space \( V_k(\Gamma) \) is the subspace of functions belonging of \( V^h(\Gamma) \) which vanish at the nodal points not in \( \bar{\mathcal{F}}_k \). Clearly

\[
V^h(\Gamma) = V_0 + \sum_k V_k + \sum_{ij} V_{ij}
\]

Let \( \bar{\mathcal{H}}_k u \) be the extension of \( u \) from \( \bar{\mathcal{F}}_k \) to \( \bar{\Omega}_k \) where \( \bar{\Omega}_k \) is a neighborhood of \( x_k \) which contains \( \bar{\mathcal{F}}_k \). Let

\[
b_k(u, v) = a(\bar{\mathcal{H}}_k u, \bar{\mathcal{H}}_k v), \quad u, v \in V_k(\Gamma)
\]

where \( a(u, v) \) is given by (2.2). Let \( T_k \) be given by

\[
b_k(T_k u, v) = s(u, v), \quad v \in V_k(\Gamma)
\]

and let

\[
T = T_0 + \sum_k T_k + \sum_{ij} T_{ij}
\]

**Theorem 4.2.** For all \( u \in V^h(\Gamma) \), with \( T_0, T_{ij} \) and \( T_k \) defined by (4.9) and (4.15) and for model problem I,

\[
\gamma_0(1 + \log \frac{H}{h})^{-1} s(u, u) \leq s(Tu, u) \leq \gamma_1 s(u, u)
\]

where \( \gamma_0 \) and \( \gamma_1 \) are constants independent of \( H, h \).

A proof of this theorem is based on Theorem 3.1 of Section 3; cf. Dryja, Smith and Widlund [4].

5. The Neumann-Neumann method. In this section, we give a description of a method, introduced by R. Glowinski, P. L. Tallec, Y.-H. De Roeck et al., see [1], [3], as an additive Schwarz method. We then extend the method to the case of a large number of substructures introducing a coarse space \( V_0 \), which is similar but not identical to that of Section 4. We also modify this method to handle problems with discontinuous coefficients.

5.1. The method without a coarse space. We first consider model problem I, i.e. when \( s(u, v) \) corresponds to the bilinear form \( a(u, v) \) given by (2.2). Let \( V_i(\Gamma) \)
be the subspace of functions $v \in V^h(\Gamma)$ which vanish at all nodal points on $\Gamma \setminus \partial \Omega_i$. It is easy to verify that

\begin{equation}
V^h(\Gamma) = V_1(\Gamma) + \cdots + V_N(\Gamma)
\end{equation}

Let $s_i(u, v)$ be the bilinear form on $V_i \times V_i$ defined in Section 2, i.e.

\begin{equation}
s_i(u, v) = x^{(i)}_B \cdot y^{(i)}_B
\end{equation}

For the method considered, using the notations from Section 3, $V = V^h(\Gamma)$, $V_i = V_i(\Gamma)$, $i = 1, \ldots, N$, and $a(u, v) = s(u, v)$. There is no coarse space $V_0$. To define $b_i(u, v): V_i(\Gamma) \times V_i(\Gamma) \to R$, we introduce a counting function $\alpha_i \in V^h(\Gamma)$, associated with $\partial \Omega_i$.

\[ \alpha_i(x) = \begin{cases} 2 & x \in F_{ij}h \\ n_i(x) & x \in W_{ih} \\ 0 & \text{all other nodal points on } \Gamma \end{cases} \]

Here $n_i(x)$ is the number of substructures which have the nodal point $x \in W_i$ in common. Let

\begin{equation}
b_i(u, v) = s_i(I_h(\alpha_i u), I_h(\alpha_i v)) \quad u, v \in V^h(\Gamma)
\end{equation}

Here $I_h$ is the linear interpolation operator on the fine triangulation. We note that $b_i(u, v)$ is symmetric and positive definite on $V_i \times V_i$.

We introduce an operator $T_i: V^h \to V_i$, by

\begin{equation}
b_i(T_i u, v) = s(u, v) \quad v \in V_i
\end{equation}

and put

\[ T = T_1 + \cdots + T_N \]

**Theorem 5.1.** For all $u \in V^h(\Gamma)$, with $T_i$ defined by (5.4) and for model problem I

\begin{equation}
\gamma_0 s(u, u) \leq s(T_i u, u) \leq \gamma_1 \frac{(1 + \log \frac{H}{h})^2}{H^2} s(u, u)
\end{equation}

where $\gamma_0$ and $\gamma_1$ are constants independent of $H$ and $h$.

This result is proved in [3] using other tools. We believe that our proof, which is based on Theorem 3.1, is simpler; cf. [8]. We note that an estimate such as (5.5) but with four log factors was given already in [4]. For the case when there is a red-black ordering of the substructures, we also showed that three log factors are enough and we also derived similar estimates for algorithms with coarse spaces.

**5.2. A method with a coarse space.** We now describe the Neumann-Neumann method with a coarse space $V_0$. The factor $1/H^2$ can now be removed from the estimate (5.5). A function $v \in V_0$ is a continuous, piecewise linear function defined by the nodal values on the wirebasket $W_i$ and it is constant on the faces $F_{ij}$ with different constants on different faces. This space is different from the coarse space
used in Section 4. The basis functions of $V_0$ are of the following form. For a nodal point $x_k \in W_{ih}$, we use a standard nodal basis function $\phi_k(x)$ and for a face $F_{ij}$ the function $\theta_{ij}(x)$ defined in Section 4. $V_0$ can be regarded as the range of an interpolation operator defined on $\partial \Omega_i$ by

$$\tilde{I}_k u = \sum_{x_k \in W_{ih}} u(x_k) \phi_k(x) + \sum_{F_{ij} \subset \partial \Omega_i} u(\bar{x}_{ij}) \theta_{ij}(x)$$

on $\partial \Omega_i$. Here $\bar{x}_{ij}$ is a fixed nodal point on $F_{ij}$, and the summation in the second term is taken over the faces $F_{ij}$. The spaces $V_i(\Gamma)$, $i = 1, \ldots, N$, are defined by

$$V_i = \{v \in V^h(\Gamma); v(x) = 0, x \in \Gamma \setminus \partial \Omega_i, \bar{v}_i = 0\}$$

Here $\bar{v}_i$ is a weighted discrete average of $u$ on $W_{ih}$, i.e.

$$\bar{v}_i = \frac{1}{n_i} \sum_{x \in W_{ih}} \alpha_i(x) u(x)$$

It is easy to verify that

$$V^h(\Gamma) = V_0 + V_1 + \cdots + V_N$$

We now introduce bilinear forms $b_i(u, v): V_i \times V_i \to \mathbb{R}$. For $i = 1, \ldots, N$, they are defined as in (5.4). For $i = 0$, we use

$$b_0(u, v) = (1 + \log \frac{H}{h})^{-1} \sum_t \left\{ h \sum_{x \in W_{ih}} (u(x) - \bar{u}) (v(x) - \bar{u}) + \sum_{F_{ij} \subset \partial \Omega_i} H (u(\bar{x}_{ij}) - \bar{u}) (v(\bar{x}_{ij}) - \bar{u}) \right\}$$

Here $\bar{u}_i$ is the discrete average of $u$ on $W_{ih}$ introduced in (4.2). Let

$$T = T_0 + T_1 + \cdots + T_N$$

where

$$b_i(T_i u, v) = s(u, v), \quad v \in V_i, \quad i = 0, \ldots, N$$

**Theorem 5.2.** For all $u \in V^h(\Gamma)$, with $T_i$ defined by (5.10) and for model problem I

$$\gamma_0 s(u, u) \leq s(T_i u, u) \leq \gamma_1 (1 + \log \frac{H}{h})^2 s(u, u)$$

where $\gamma_0$ and $\gamma_1$ are constants independent of $H$ and $h$.

A proof of this theorem is given in [8].

**5.3. A method with a coarse space for model problem II.** In this subsection, we describe a Neumann-Neumann method for model problem II. The bilinear form is now given by

$$a(u, v) = \int_\Omega \rho(x) \nabla u \cdot \nabla v \, dx$$
where $\rho(x) = \rho_i = \text{constant} > 0$ on substructure $\Omega_i$. Let $V_i, i = 1, \ldots, N$, associated with $\partial \Omega_i$, be a space of functions $v \in V^h(\Gamma)$ which vanish on the wirebasket $\mathcal{W}_i$ of $\Omega_i$ and at all nodal points on $\Gamma$ which are not on $\partial \Omega_i$. The space $V_0$ is the same as in Section 4, i.e. it is the range of the interpolation operator $I_h$; cf. (4.5). We have

\begin{equation}
V^h(\Gamma) = V_0 + V_1 + \cdots + V_N
\end{equation}

To define $b_i(u, v): V_i \times V_i \rightarrow R$, $i = 1, \ldots, N$, we introduce a piecewise constant function $\bar{\rho}_i$, defined on $\partial \Omega_i$, which is equal to $(\rho_i + \rho_j)^{1/2}$ on the face $F_{ij}$ common to $\Omega_i$ and $\Omega_j$. Let

\begin{equation}
b_i(u, v) = s_i(I_h(\bar{\rho}_i u), I_h(\bar{\rho}_i v)), \quad u, v \in V_i
\end{equation}

where $s_i(u, v)$ corresponds to the form $a(u, v) = (\nabla u, \nabla v)_{L^2(\Omega)}$; cf. (2.2) and (2.13). For $i = 0$

\begin{equation}
b_0(u, v) = \sum_i \rho_i \sum_{x \in \mathcal{W}_h} h(u(x) - \bar{u}_i)(v(x) - \bar{v}_i)
\end{equation}

where $\bar{u}_i$ is defined in (4.2). Note that for $u \in V_i$, $I_h(\bar{\rho}_i u)$ is defined uniquely on the wirebasket $\mathcal{W}_i$ since $u = 0$ there. Let $T_i: V^h(\Gamma) \rightarrow V_i$, be an operator defined by

\begin{equation}
b_i(T_i u, v) = s(u, v), \quad v \in V_i, \quad i = 0, 1, \ldots, N
\end{equation}

and let

$$T = T_0 + T_1 + \cdots + T_N$$

**Theorem 5.3.** For all $u \in V^h(\Gamma)$, with the $T_i$ defined by (5.16), and for model problem II

\begin{equation}
\gamma_0(1 + \log \frac{H}{h})^{-2}s(u, u) \leq s(Tu, u) \leq \gamma_1(1 + \log \frac{H}{h})^2s(u, u)
\end{equation}

Here $\gamma_0$ and $\gamma_1$ are constants independent of $H$ and $h$ and the jumps of $\rho(x)$.

A proof of this theorem is given in [8].

6. A multilevel method. In this section, we discuss a modification of the multilevel method suggested in J. Bramble et al. [2] for the problem (2.14) in terms of the additive Schwarz method described in Section 3. We first consider model problem I and then model problem II. We note that we have previously described this multilevel method as an additive Schwarz method for the original problem (2.3); cf. Dryja and Widlund [7].

6.1. The method for model problem I. We discuss problem (2.3) with the bilinear form given by (2.2) defined on a triangulation of $\Omega$ obtained by successive refinements. We consider $(\ell + 1)$ levels of triangulations of $\Omega$, associated with parameters $h_k$, which satisfy $V^{h_{k-1}}(\Omega) \subset V^h(\Omega)$. Here the $V^h(\Omega)$ are standard finite element spaces with $V^{h_0}(\Omega)$ and $V^{h_k}(\Omega)$ the coarsest and finest spaces, respectively.
\( V^{h_k}(\Gamma) \) is the restriction of \( V^h(\Omega) \) to \( \Gamma \). The problem considered, (2.3), is defined in \( V^{h_k}(\Gamma) \). As a representation of \( V^h = V^{h_\ell} \), we take

\begin{equation}
V^h = V_0 + V_1 + \cdots + V_\ell, \quad V_k = V^{h_k}
\end{equation}

We introduce, for \( k \geq 1 \), \( b_k(u,v): V_k \times V_k \rightarrow \mathbb{R} \), by

\begin{equation}
b_k(u,v) = \sum_{x \in N_k} h_k u(x) v(x)
\end{equation}

Here \( N_k \) is the set of nodal points of the level \( k \) triangulation on \( \Gamma \). We introduce \( T_k: V^h \rightarrow V_k \), by

\begin{equation}
T_0 = T_0 + T_1 + \cdots + T_\ell
\end{equation}

**Theorem 6.1.** Let \( h_k/\ell h_{k+1} \) be uniformly bounded. Then for all \( u \in V^h(\Gamma) \), with \( T_k \) defined by (6.3), and for model problem I

\begin{equation}
\gamma_0 (\ell + 1)^{-1} s(u,u) \leq s(Tu,u) \leq \gamma_1 (\ell + 1) s(u,u)
\end{equation}

If \( \Omega \) is convex then

\begin{equation}
\gamma_2 s(u,u) \leq s(Tu,u) \leq \gamma_3 (\ell + 1) s(u,u)
\end{equation}

Here \( \gamma_i, i = 0, \ldots, 3 \), are constants independent of \( h_0, \ldots, h_\ell \).

**Proof:** The proof uses Theorem 3.1 of Section 3.

**Assumption (i):** Let \( \mathcal{H}u \) be the discrete harmonic extension in the sense of \( a(u,v) \) of \( u \) from \( \partial \Omega_i \) to \( \Omega_i \), for all \( i \). We use the partitioning

\[ u = \sum_{k=0}^\ell u_k, \quad u_k|_{\Gamma} = Q_k \mathcal{H}u|_{\Gamma} \in V_0(\Gamma), \quad u_k|_{\Gamma} = (Q_k - Q_{k-1}) \mathcal{H}u|_{\Gamma} \in V_k(\Gamma), \quad k = 1, \ldots, \ell \]

Here \( Q_k \) is the \( L_2 \)-projection from \( V^h(\Omega) \rightarrow V_k(\Omega) \), i.e.,

\[ (Q_k u, v)_{L^2(\Omega)} = (u, v)_{L^2(\Omega)}, \quad v \in V_k, \quad k = 0, \ldots, \ell \]

It is well known that

\begin{equation}
|Q_i u|_{H^1(\Omega)} \leq C |u|_{H^1(\Omega)}
\end{equation}

and

\begin{equation}
|u - Q_i u|_{L^2(\Omega)} \leq h_i |u|_{H^1(\Omega)}
\end{equation}
Using (6.6), we have

$$s(u_0, u_0) \leq C|Q_0 \mathcal{H}u|^2_{H^1(\Omega)} \leq C|\mathcal{H}u|^2_{H^1(\Omega)} \leq Cs(u, u)$$

For $k = 1, \ldots, \ell$, we obtain, by using an elementary inequality and (6.7),

$$b_k(u_k, u_k) \leq C h_k^{-1}||u_k||^2_{L^2(\Gamma)} \leq C h_k^{-2}||Q_k \mathcal{H}u - Q_k-1 \mathcal{H}u||^2_{L^2(\Omega)}$$
$$\leq C h_k^{-2} \{||\mathcal{H}u - Q_k \mathcal{H}u||^2_{L^2(\Omega)} + ||\mathcal{H}u - Q_k-1 \mathcal{H}u||^2_{L^2(\Omega)}\}$$
$$\leq C||\mathcal{H}u||^2_{H^1(\Omega)} \leq Cs(u, u)$$

Thus

$$s(u_0, u_0) + \sum_{i=1}^N b_i(u_i, u_i) \leq C(\ell + 1)s(u, u)$$

which proves Assumption (i).

For a convex region $\Omega$, we can instead use $u_0 = P_0 \mathcal{H}u$ and $u_k = (P_k - P_k-1)\mathcal{H}u$, where $P_k$ is the $H^1$-projection and use the fact that the resulting $u_k$ are $H^1$-orthogonal; cf. Dryja and Widlund [5]. For more details see below.

**Assumption (ii)** follows from the inverse inequality.

**Assumption (iii)** is proved by using standard arguments.

### 6.2. A new method for model problem II

We now discuss a new method for model problem II, i.e. when the bilinear form is given by

$$a(u, v) = \int_\Omega \rho(x) \nabla u \cdot \nabla v \, dx,$$

where $\rho(x) = \rho_i = \text{constant} > 0$ on $\Omega_i$. The decomposition of $V^h(\Gamma)$ is now of the form

$$V^h(\Gamma) = V_0(\Gamma) + V_1(\Gamma) + \cdots + V_\ell(\Gamma)$$

Here $V_0$ is the space from Section 4, i.e. the range of the interpolation operator $I_h$; cf. (4.5).

We let $V_k$, $k = 1, \ldots, \ell$, be the space of functions in $V^h(\Gamma)$ which vanish at the nodal points of the wirebasket of $\partial \Omega_i$. It is easy to verify that the representation (6.8) of $V^h$ holds.

We introduce $b_k(u, v) : V_k(\Gamma) \times V_k(\Gamma) \to R$, by

$$b_0(u, v) = (1 + \log \frac{H}{h}) \sum_i \sum_{x \in W_{ih}} h \rho_i(u(x) - \bar{u}_i)(v(x) - \bar{v}_i),$$

cf. (4.6), and for $k = 1, \ldots, \ell$,

$$b_k(u, v) = \sum_{x \in S_k} h_k \rho(x) u(x) v(x), \quad u, v \in V_k(\Gamma)$$

Here $\rho(x)$ is piecewise constant and equal to $(\rho_i + \rho_j)/2$ on the face $F_{ij}$ which is common to the substructures $\Omega_i$ and $\Omega_j$. Note that for $u \in V_k$, $\rho(x)u(x) = 0$ on $W_i$ since $u(x) = 0$ there.
We now introduce $T_k: V^h(\Gamma) \to V_k$, by
\begin{equation}
(6.12) \quad b_k(T_k u, v) = s(u, v), \quad v \in V_k, \quad k = 0, \ldots, \ell
\end{equation}
and put
\begin{equation}
(6.13) \quad T = T_0 + T_1 + \cdots + T_{\ell}
\end{equation}

**Theorem 6.2.** Let $h_{k}/h_{k+1}$ be uniformly bounded. Then for all $u \in V^h(\Gamma)$, with $T_k$ defined by (6.12), and for model problem II
\begin{equation}
(6.14) \quad \gamma_0(\ell + 1)^{-2} s(u, u) \leq s(T u, u) \leq \gamma_1(\ell + 1) s(u, u)
\end{equation}
where $\gamma_0$ and $\gamma_1$ are constants independent of $h_0, h_1, \ldots, h_{\ell}$ and the jumps of $\rho(x)$.

**Proof:** We again use Theorem 3.1.

**Assumption (i):** Let $u_0 = \tilde{I}_k u$ where $\tilde{I}_k u$ is defined in (4.5). It can be established that
\begin{equation}
(6.15) \quad b_0(u_0, u_0) \leq C(\ell + 1)^2 s(u, u)
\end{equation}
see Dryja, Smith and Widlund [4] and Smith [9]. We now define $u_k$. Let $w = u - u_0$ and note that it vanishes on the wirebasket $W_i$. The function $w$, given on $F_{ij}$, the face common to $\Omega_i$ and $\Omega_j$, is extended to $\Omega_i$ as a discrete harmonic function $w_{ij}$, i.e.
\begin{equation}
(6.16) \quad (\nabla w_{ij}, \nabla v)_{L^2(\Omega_i)} = 0, \quad v \in V^h(\Omega_i)
\end{equation}
\quad $w_{ij} = w$ on $F_{ij}$ and $w_{ij} = 0$ on $\partial \Omega_i \setminus F_{ij}$
Let $\tilde{V}^h(\Omega_i), k = 0, \ldots, \ell$, be the subspace of $V^h(\Omega_i)$, (the restriction of $V^h$ to $\Omega_i$) of functions which vanish on $\partial \Omega_i \setminus F_{ij}$. We note that $w_{ij} \in \tilde{V}^h(\Omega_i)$. This function is represented as
\begin{equation}
(6.17) \quad w_{ij} = P_{ij,1} w_{ij} + (P_{ij,2} - P_{ij,1}) w_{ij} + \cdots + (P_{ij,\ell} - P_{ij,\ell-1}) w_{ij}
\end{equation}
where $P_{ij,k}$ is the $H^1$-projection: $\tilde{V}^h(\Omega_i) \to \tilde{V}^h(\Omega_i)$, i.e.
\begin{equation}
(\nabla P_{ij,k} v, \nabla \phi)_{L^2(\Omega_i)} = (\nabla v, \nabla \phi)_{L^2(\Omega_i)}, \quad \phi \in \tilde{V}^h(\Omega_i)
\end{equation}
It is known that, since $\Omega_i$ is convex,
\begin{equation}
(6.18) \quad \|u - P_{ij,k} u\|_{L^2(\Omega_i)} \leq M_{h_k} |u|_{H^1(\Omega_i)}
\end{equation}
On $F_{ij}$ we define $u_k$ as the traces of the functions defined in (6.17)
\begin{equation}
(6.19) \quad u_1 = P_{ij,1} w_{ij}, \quad u_k = (P_{ij,k} - P_{ij,k-1}) w_{ij}, \quad k = 2, \ldots, \ell
\end{equation}
It is easy to see that $P_{ij,k} - P_{ij,k-1}$ is an orthogonal projection from $V^h(\Omega_i)$ into $\tilde{V}^h(\Omega_i) = \text{Range}(P_{ij,k} - P_{ij,k-1}) \subset V^h(\Omega_i)$. We now show that
\begin{equation}
\sum_{k=1}^{\ell} b_k(u_k, u_k) \leq C(\ell + 1)^2 s(u, u)
\end{equation}
for the $u_k$ defined above. We obtain, by using the definition of $\tilde{\rho}(x)$,

$$
(6.20) \quad b_k(u_k, u_k) = \sum_{x \in N_k} h_k \tilde{\rho}(x) u_k^2(x) = \frac{1}{2} \sum_i h_k \sum_{j \in N_{ij,k}} \rho_i u_k^2(x)
$$

where $N_{ij,k}$ is the set of nodal points of the $k$-level triangulation of $\mathcal{F}_{ij}$ and the summation with respect to $j$ is over the different faces of $\partial \Omega_i$. For one face, we obtain

$$
(6.21) \quad h_k \sum_{x \in N_{ij,k}} \rho_i u_k^2(x) \leq Ch_k^{-2} \rho_i \|u_k\|_{L^2(\Omega_i)}^2 = Ch_k^{-2} \rho_i \|(P_{ij,k} - P_{ij,k-1})u_k\|_{L^2(\Omega_i)}^2 \leq Ch_k^{-2} \rho_i \{\|P_{ij,k} u_k - u_k\|_{L^2(\Omega_i)} + \|P_{ij,k-1} u_k - u_k\|_{L^2(\Omega_i)}\} \leq C \rho_i \|u_k\|_{L^2(\Omega_i)}^2
$$

Adding these inequalities with respect to $k$, we obtain

$$
\rho_i \sum_{k=1}^{\ell} h_k \sum_{x \in N_{ij,k}} u_k^2(x) \leq C \rho_i \sum_{k=1}^{\ell} \|\nabla u_k\|_{L^2(\Omega_i)} \leq C \rho_i \|\nabla P_{ij,k} \omega_{ij}\|_{L^2(\Omega_i)} + \sum_{k=2}^{\ell} \|\nabla (P_{ij,k} - P_{ij,k-1}) \omega_{ij}\|_{L^2(\Omega_i)}\} \leq C \rho_i \|\omega_{ij}\|_{H_0^1(\mathcal{F}_{ij})} \leq C (\ell + 1)^2 s_i(u, u)
$$

For the last inequality, see for example Dryja and Widlund [6]. Combining this inequality with (6.20), we obtain

$$
\sum_{k=1}^{\ell} b_k(u_k, u_k) \leq C (\ell + 1)^2 s(u, u) .
$$

Together with (6.15), this provides the bound for Assumption (i).

**Assumption (ii):** It is known that for $u \in V_0$

$$
a(u, u) \leq C b_0(u, u) ,
$$

cf. [4]. For $k = 1, \ldots, \ell,$

$$
a(u, u) \leq C b_k(u, u) , \quad u \in V_k
$$

follows from an inverse inequality.

**Assumption (iii)** is proved by using standard arguments.

References


