CHAPTER 16

Application of Domain Decomposition Techniques to Modal Synthesis for Eigenvalue Problems

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Abstract: component mode (or modal) synthesis is a Rayleigh-Ritz method that enables computation of the normal modes of a linearly elastic structure that can be subdivided into several substructures whose normal modes are partly known. Information transfer between substructures is achieved by the introduction in the Ritz procedure of mode shapes defined on the whole structure and called here coupling modes. Three nonconventional "fixed interface" methods are presented in a continuous framework. Once discretized, they extend more classical ones. The definition of the coupling modes relies on the spectral properties of suitable interface operators that naturally arise when designing iterative substructuring algorithms for source problems. Several approximation errors will be given and different algorithms compared.

1) Introduction to modal synthesis.

Let us consider the usual model eigenvalue problem, posed over a domain $\Omega \subset \mathbb{R}^d$, partitioned in two nonoverlapping subregions Ω_1 and Ω_2 separated by an interface Γ:

$$(1) \qquad \begin{cases} \textit{find } (\lambda, u) \in \mathbb{R} \times H_0^1(\Omega) \textit{ such that} \\ \displaystyle\int_\Omega \nabla u \nabla v = \lambda \int_\Omega uv \quad \forall\, v \in H_0^1(\Omega). \end{cases}$$

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This so-called global problem admits a countable family of solutions $(\lambda_k, u_k)_{k=1}^{j_0} \in \mathbb{R}^+ \times H_0^1(\Omega)$ arranged in such a way that the eigenvalues form an increasing sequence. The first j_0 eigenpairs are of interest here.

The subsequently developed theory works as well for any strongly elliptic operator of any order with symmetric boundary conditions. In particular, this is the case of the two- or three-dimensional Lamé system, the plate operator, and many other systems of interest. The right-hand side could even be nonsymmetric, as for instance in the neutron equation. The domain may be partitioned into $p > 2$ subdomains.

In a first place, for each subdomain Ω_i, the local problem

$$(2) \qquad \begin{cases} \textit{find } (\lambda, u) \in \mathbb{R} \times H_0^1(\Omega_i) \textit{ such that} \\ \displaystyle\int_{\Omega_i} \nabla u \nabla v = \lambda \int_{\Omega_i} uv \qquad \forall\, v \in H_0^1(\Omega_i), \end{cases}$$

admits a family $(\lambda_{ij}, u_{ij})_{j=1}^{+\infty} \in \mathbb{R}^+ \times H_0^1(\Omega_i)$ of solutions such that $(\lambda_{ij})_{j=1}^{+\infty}$ forms an increasing sequence. The functions u_{ij} are extended onto the other subdomain by zero. They are called "fixed interface modes".

In a second place, we introduce an a priori arbitrary basis $(u_{\Gamma\ell})_{\ell=1}^{+\infty}$ of the space $H_{00}^{1/2}(\Gamma) = tr_\Gamma H_0^1(\Omega)$, where tr_Γ denotes the trace operator on Γ. Letting $R : H_{00}^{1/2}(\Gamma) \longrightarrow H_0^1(\Omega)$, $v \longrightarrow \tilde{v}$, denote the harmonic lifting operator, such that

$$(3) \qquad \begin{cases} \Delta\tilde{v} = 0 \quad \textit{in } \Omega_i, \quad 1 \le i \le 2\,, \\ \tilde{v} = v \quad \textit{on } \Gamma\,, \\ \tilde{v} = 0 \quad \textit{on } \partial\Omega\,, \end{cases}$$

we define the "coupling modes" $\tilde{u}_{\Gamma\ell} = Ru_{\Gamma\ell}$, and the family

$$(4) \qquad \mathcal{F} = \bigcup_{i=1,2} (u_{ij})_{j=1}^{+\infty} \cup (\tilde{u}_{\Gamma\ell})_{\ell=1}^{+\infty}\,,$$

that forms a basis of the space $H_0^1(\Omega)$(cf. Bourquin [1990a,b]). This is a Riesz basis whenever $(u_{\Gamma\ell})_{\ell=1}^{+\infty}$ forms a Riesz basis of the space $H_{00}^{1/2}(\Gamma)$.

In a third place, once three numbers $N_1, N_2,$ *and* N_Γ are given, we define the finite-dimensional space

$$(5) \qquad V_N = Span \left\{ \bigcup_{i=1,2} (u_{ij})_{j=1}^{N_i} \cup (\tilde{u}_{\Gamma\ell})_{\ell=1}^{N_\Gamma} \right\}.$$

Modal synthesis consists in computing the j_0 first solutions $(\lambda_k^N, u_k^N)_{k=1}^{j_0} \in \mathbb{R}^+ \times V_N$ of the well-posed discrete variational eigenvalue problem:

$$(6) \qquad \begin{cases} \textit{find } (\lambda^N, u^N) \in \mathbb{R} \times V_N \textit{ such that} \\ \displaystyle\int_\Omega \nabla u^N \nabla v^N = \lambda^N \int_\Omega u^N v^N \qquad \forall\, v^N \in V_N. \end{cases}$$

The convergence result $\lambda_k^N \longrightarrow \lambda_k$, $u_k^N \longrightarrow u_k$(with an appropriate scaling) follows from the above mentioned completeness property in addition to standard results in eigenvalue approximation theory.

How to choose the coupling modes? In his pioneering work on modal synthesis, Hurty [1965] picked out any basis and took $N_\Gamma = +\infty$ in (4). In view of (5), this choice is forbidden in a continuous framework since V_N becomes infinite-dimensional. However, it makes sense in a finite element or any discrete setting. If h denotes a discretization parameter, the function $u_{\Gamma\ell}^h$ thus coïncides with the ℓth shape function spanning the interfacial interpolation space, and the number of coupling modes is equal to the number of interfacial degrees of freedom. The coupling modes are defined as the discrete harmonic liftings $\tilde{u}_{\Gamma\ell}^{hh} = R^h u_{\Gamma\ell}^h$ through discretization of problem (3).

It should be kept in mind that modal synthesis is an approximation method rather than a domain decomposition algorithm, since the problem really solved is not equivalent to the original one.

This method has been widely used in aerospace engineering practice during the last three decades, because of its accuracy and suitability to the requirements of industrial project management. More generally speaking, modal synthesis looks quite attractive for different reasons, among which the reduction of computer memory storage requirements and amenability to coarse grain parallel computing are to be found. Furthermore, it allows to decouple virtually every coupled mechanical system made of heterogeneous subsystems. Last but not least, it allows for cheap parametric studies, since recomputations due to local geometrical or mechanical modifications can be performed locally, and also for integration of experimental results in the Ritz procedure. The advantages of modal synthesis are discussed in Bourquin[1991a].

There exists many variants of Hurty's method. We refer to Craig [1985], Gibert[1988], Imbert [1979], Jézéquel [1985], Meirovitch [1980], Morand [1977], and Valid [1977] for detailed reviews or analyses on this topic.

Although Hurty's method is around thirty years old, to our knowledge the first error bounds are derived in Bourquin [1991c] and read as follows, where the discretized version of the method is considered, that leads to the approximate eigenvalues $\lambda_k^{N,h}$:

$$(7) \qquad \lambda_k^{N,h} - \lambda_k \leq C(k) \left[\sum_{i=1,2} \frac{1}{C_i N_i^{\frac{2}{n}}} + \left\| (I - P^h) u_k \right\|_a^2 \right] ,$$

if P^h denotes the orthogonal projection mapping on the finite element space used, and $\| \ \|_a$ the energy norm. As usual, similar estimates hold regarding the eigenfunction approximation.

The stiffness matrix K^N of the resulting Ritz method (5) writes

$$(8) \qquad K^N = \begin{bmatrix} \begin{bmatrix} \lambda^h_{11} & & & & 0 \\ & \ddots & & & \\ & & \lambda^h_{1N_1} & & \\ & & & \lambda^h_{21} & \\ & & & & \ddots \\ 0 & & & & \lambda^h_{2N_2} \end{bmatrix} & 0 \\ 0 & \left[\int_\Omega \nabla \tilde{u}^{hh}_{\Gamma \ell} \nabla \tilde{u}^{hh}_{\Gamma k} \right] \end{bmatrix}$$

Because of harmonicity of the coupling modes, the last block of K^N coïncides with the Schur complement matrix associated with the interface. The mass matrix

$$(9) \qquad M^N = \begin{bmatrix} \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ 0 & & & & 1 \end{bmatrix} & \left[\int_\Omega u^h_{ij} \tilde{u}^{hh}_{\Gamma k} \right] \\ \left[\int_\Omega \tilde{u}^{hh}_{\Gamma \ell} u^h_{ij} \right] & \left[\int_\Omega \tilde{u}^{hh}_{\Gamma \ell} \tilde{u}^{hh}_{\Gamma k} \right] \end{bmatrix}$$

exhibits a similar structure. Of course, solving problem (5) becomes expensive when the number of degrees of freedom describing the interface increases. In particular, this feature forbids mesh refinements in the vicinity of the interface, intricate domain partitions, and certain large three-dimensional computations.

In addition, Hurty's method results from a mixture of two different discretization processes: the first one is of physical nature since a basis of special normal modes of vibration is truncated, and the second one is purely numerical because it is tied to the finite element (or whatever) method used in the practical computation.

We claim that there exists a mesh-independent way to define coupling modes, yielding an overall mesh-independent accuracy when the mesh size is sufficiently small. The proposed strategies take advantage of recent developments in the field of domain decomposition methods for static problems. They aim first of all to shed light on the theory of modal synthesis, as far as the formulation and convergence properties are concerned, and second of all to lead to new practical methods of general use that prove accurate and robust.

2) An intrinsic choice of coupling modes based on the Poincaré-Steklov operator.

A quick look at the stiffness and mass matrices K^N and M^N indicates that the subblocks of the global stiffness matrix that correspond to internal degrees of freedom of each subdomain have been diagonalized. This suggests to diagonalize also the remaining part of the overall stiffness, that is the Schur complement matrix, and to choose the resulting eigenvectors as discrete coupling modes.

The corresponding modal synthesis algorithm bears a strong resemblance to an iterative substructuring procedure for a source problem. As a matter of fact, both algorithms start with the resolution of a *local* eigenvalue (resp., source) problem on each subdomain with an arbitrarily prescribed Dirichlet boundary condition on the interface. Iterative substructuring then uses the residual gap of stresses X_0 along the interface that has been computed at the first step as the source term of the problem $S^h X = X_0$, where S^h denotes the Schur complement matrix. This problem is usually solved thanks to a preconditioned conjugate gradient procedure. In the same way, the proposed modal synthesis method will need the computation of the low-frequency spectrum of S^h.

At the continuous level, let $S : H_{00}^{1/2}(\Gamma) \longrightarrow (H_{00}^{1/2}(\Gamma))'$ denote the Poincaré-Steklov operator associated with the operator describing the mechanics of the problem (here the Laplacian) and the domain decomposition, in such a way that $Sv = \sum_{i=1}^{2} \frac{\partial}{\partial n_i}(\tilde{v}_{|\Omega_i})_{|\Gamma}$, where n_i denotes the unit outer normal vector along $\partial\Omega_i$. It follows from standard compactness arguments that the problem

$$(10) \qquad \begin{cases} \textit{find } (\lambda, u) \in \mathbb{R} \times H_{00}^{1/2}(\Gamma) \textit{ such that} \\ \\ Su = \lambda u \end{cases}$$

admits a family $(\lambda_{\Gamma\ell}, u_{\Gamma\ell})_{\ell=1}^{+\infty}$ of solutions, arranged in such a way that the sequence $(u_{\Gamma\ell})$ is increasing. The family $(u_{\Gamma\ell})$ forms an orthogonal basis of both spaces $H_{00}^{1/2}(\Gamma)$ and $L^2(\Gamma)$. Then the family

$$(11) \qquad \mathcal{F}_S = \bigcup_{i=1,2} (u_{ij})_{j=1}^{+\infty} \cup (\tilde{u}_{\Gamma\ell})_{\ell=1}^{+\infty}$$

forms an orthogonal basis of the space $H_0^1(\Omega)$.

The coupling modes enjoy a mechanical interpretation: they coïncide with the free vibrations of the structure Ω whose mass would be lumped on the interface (cf. Bourquin and d'Hennezel [1991a]).

If V_N is defined as in (4), solving problem (5) leads to a sequence of solutions $(\lambda_k^N, u_k^N)_{k=1}^{j_0}$ satisfying the following property (cf. Bourquin[1989,1991a,b]):

$$(12) \qquad \lambda_k^N - \lambda_k = o\left(\sum_{i=1}^{2} \frac{1}{\lambda_{iN_i}^s} + \frac{1}{\lambda_{\Gamma N_\Gamma}^{2\alpha}}\right)$$

for any $s < 3/2$ and any α such that the solution of the Poisson-Dirichlet problem in each subdomain with a $L^2(\Omega_i)$ source term belongs to $H^{1+\alpha}(\Omega_i)$, $i = 1, 2$. Lower bounds for the eigenvalues of the Poincaré-Steklov operator, in the spirit of Weyl's formula, can be derived and yield

$$\lambda_k^N - \lambda_k = o\left(\sum_{i=1}^{2} \frac{1}{N_i^{\frac{2s}{d}}} + \frac{1}{N_\Gamma^{\frac{2\alpha}{d-1}}}\right) . \tag{13}$$

The discretized version of the method can be formulated in the same way and leads to similar error bounds (cf. Bourquin [1991b]). The convergence speed appearing in (12), (13) is optimal. It should be noticed that the right-hand side of those inequalities do not depend, to some extent, on the regularity of the global solutions u_k. A thorough study of how these estimates behave when the number of substructures increases is carried out in Bourquin[1991a].

One of the most striking features of the proposed method lies in the small and mesh-independent number of coupling modes to be taken into account in practice in order to achieve a prescribed accuracy. It is usually comparable to the number of global modes to be computed, as demonstrated in Bourquin and d'Hennezel [1991a,b]. For example, let us consider a three-dimensional elastic beam made of 4 unit cubes sequentially glued to one another. The beam is clamped at one end and free everywhere else. Then only 3 fixed interface modes per unit cube and 3 coupling modes are sufficient to yield a **.5%** accuracy regarding the 3 first global normal modes. Furthermore, the first coupling modes and corresponding global normal modes exhibit a very similar shape. This means that an intrinsic representation of the interface displacements corresponding to low-frequency vibrations of the whole structure is provided thanks to the superposition of very few eigenfunctions of the Poincaré-Steklov operator.

One possible algorithm to compute the coupling modes consists of a Lanczos procedure applied to the operator S^{-1} , or to $\left(S^h\right)^{-1}$ for the discretized version (see Bourquin [1991c] for other ideas). At each step of the latter, a source problem of the type $Su = f$ must be solved. But most iterative substructuring methods aim to realize this program. Any algorithm seems a priori suitable, the faster it is, the faster will be the resulting modal synthesis. The preconditioned conjugate gradient algorithm designed by Bourgat et al. [1988a,b] has been chosen.

3) Other choices based on iterative substructuring preconditioners. Of course, using the eigenfunctions of the Poincaré-Steklov operator as coupling modes is not mandatory, except if the orthogonality of the family $\mathcal{F}$ is requested.

First of all, assume that the first coupling modes corresponding to problem (1) have been computed, then it is possible to reuse them for treating the eigenvalue problem

$$\left\{ \begin{array}{c} \textit{find } (\lambda, u) \in \mathbb{R} \times H_0^1(\Omega) \textit{ such that} \\ \displaystyle\int_{\Omega_1} \nabla u \nabla v + \sigma \int_{\Omega_2} \nabla u \nabla v = \lambda \left\{ \int_{\Omega_1} uv + \eta \int_{\Omega_2} uv \right\} \quad \forall\, v \in H_0^1(\Omega) , \end{array} \right. \tag{14}$$

where σ and η denote two positive constants. Notice that the latter do not modify the fixed interface modes. In general, the original family $\mathcal{F}_S$ is not orthogonal any more, but it remains a Riesz basis of $H_0^1(\Omega)$ and similar (nevertheless slightly worse) error bounds as (13) can be derived (cf. Bourquin[1991a,b]): more precisely, if the function u_k satisfies $u_k \in H^{1+\beta}(\Omega_i)$, $i = 1, 2$, $\beta > 0$ (such a β exists), inequalities (12) and (13) still hold if α is replaced by δ=min(α,β). From the purely numerical standpoint, the accuracy of the modal synthesis does not deteriorate significantly.

The case of stiff eigenvalue problems should be treated in a different way, relying on the asymptotic analysis quoted in Sanchez Hubert and Sanchez Palencia [1989]. Their results seem particularly interesting in view of elastoacoustic analysis.

Let us now return to the case where σ=η=1. Based on the very promissing accuracy of the method relying on the spectrum of the Poincaré-Steklov operator, an idea that shows up naturally for generating coupling modes is to look for the spectrum of an operator T that is spectrally close to S and cheap to invert. We recognize the properties of a preconditioner.

The classical Neumann-Dirichlet preconditioner can be considered. In the case of two subdomains, coupling modes are chosen as the eigenfunctions $u_{\Gamma,\ell}$ of one of the two operators $S_i : H_{00}^{1/2}(\Gamma) \longrightarrow (H_{00}^{1/2}(\Gamma))'$ such that $S_i v = \frac{\partial}{\partial n_i}(\tilde{v}_{|\Omega_i})_{|\Gamma}$. The resulting family

$$\mathcal{F}_{S_i} = \bigcup_{m=1,2} (u_{mj})_{j=1}^{+\infty} \cup (\tilde{u}_{\Gamma_i \ell})_{\ell=1}^{+\infty} \tag{15}$$

forms a Riesz basis of $H_0^1(\Omega)$, and error bounds analogous to (12) and (13) still hold. They rely on the property that the coupling modes are orthogonal in $H^1(\Omega_i)$.

Of course, this preconditioner has the same spectrum as the Poincaré-Steklov operator whenever the domain decomposition is symmetric with respect to the interface. Nevertheless, in the case of stiff problems with subdomains of arbitrary shape, the operator S_i corresponding to the stiffest subregion replaces the Poincaré-Steklov operator in a first order approximation. Corrector functions of any order may be computed very easily. This procedure appears to be justified by the arguments developed in Sanchez Hubert and Sanchez Palencia [1989].

The same preconditioner can be used for general domain partitions whenever a suitable ordering of the subdomains exists. In the case of *tree-like partitions*, the a priori error estimates do not blow up when the number of substructures increases. This configuration is of interest in Aerospace or Civil Engineering. Since the coupling modes and of course the fixed interface modes are defined in a totally local way, the effect of adding an appendage to an existing structure can be assessed by just adding two blocks in the mass and stiffness matrices of problem (5). This should enable for example to take care of all intermediate stages of a building process without resorting every time to a global recomputation.

One can also think of using a generalized Neumann-Dirichlet preconditioner as in Bourgat et al.[1988a,b]. When a complex interface Γ with cross-points has to be

coped with, we set $V_\Gamma = tr_\Gamma H_0^1(\Omega)$, and for each subdomain Ω_i, we define $W_i = tr_{\partial\Omega_i}(H^1(\Omega_i) \cap H_0^1(\Omega))$. We then let $P_i : W_i \longrightarrow V_\Gamma$ denote a continuous extension operator and $T_i : W_i' \longrightarrow W_i$ the Neumann-Dirichlet mapping via harmonic lifting (also called Calderon operator). Then the operator

$$T = \sum_{i=1}^{p} P_i T_i P_i^t : V_\Gamma' \longrightarrow V_\Gamma \tag{16}$$

is a compact self-adjoint operator over $L^2(\Gamma)$. The computation of its eigenfunctions $(u^{\Gamma\ell})_{\ell=1}^{+\infty}$ gives rise to a new Riesz basis

$$\mathcal{F}_T = \bigcup_{i=1,p} (u_{ij})_{j=1}^{+\infty} \cup (\tilde{u}^{\Gamma\ell})_{\ell=1}^{+\infty}, \tag{17}$$

and to a new modal synthesis method for which we didn't succeed so far in proving error estimates, but that looks quite promising from the purely numerical point of view, since the number of coupling modes $\tilde{u}^{\Gamma\ell}$ to take into account to achieve a prescribed accuracy is slightly greater than, but close to, the number of coupling modes $\tilde{u}_{\Gamma\ell}$ (tied to the Poincaré-Steklov operator) that is necessary in view of the same level of accuracy. This feature has been checked for the two-dimensional Laplacian on a quare divided in four subdomains with internal vertices, and on an L-shaped membrane subdivided in two subdomains. In the first case, the number of coupling modes increases by less than 20%.

A Lanczos algorithm for computing the coupling modes is still chosen. However, each step is now very cheap because any one of the mentioned preconditioners is explicitely invertable. More precisely, if N_{it} denotes the number of iterations needed for solving the source problem $Tu = f$, each step costs approximately $2N_{it}$ less than the corresponding step of the algorithm proposed in §2. This property may tremendously reduce the computer time, especially when large three-dimensional elasticity problems are tackled, because we have noticed that $N_{it} > 50$ may occur. For the L-shaped membrane, a time reduction factor of 7 has been observed.

A systematic theoretical and numerical study of these variants seems worth to be undertaken.

4) Concluding remarks.

i) The choice of the Poincaré-Steklov operator as a candidate for generating coupling modes leads to a very accurate modal synthesis method and seems locally optimal, at least from the numerical point of view. As a matter of fact, computing a rough approximation of the functions $u_{\Gamma\ell}$ reduces the marginal CPU time but degrades the final accuracy, in such a way that additional coupling modes are needed! In the same way, the few numerical tests already performed indicate that the coupling modes $u^{\Gamma\ell}$ based on preconditioners (§3) behave slightly worse than the ones based on the Poincaré-Steklov operator from the viewpoint of modal synthesis. Up to now, we do not clearly understand those facts.

ii) The various definitions of coupling modes presented here require to consider more or less complicated eigenvalue problems posed over the interface. One could rather think of using tabulated coupling modes or of generating them by hand. This strategy looks attractive when a reasonably good approximation of the mode shapes of the Poincaré-Steklov operator or of its preconditioners can be easily found. For instance, if a software is designed to compute always the same kind of structures made of an arbitrary number of substructures connected together in a predetermined way (cf. Blanc Sommereux et al. [1989]), a modal toolbox can be developed that comprises suitable interfacial modes computed once and for all with a general method. Based on our experience, the low-frequency interfacial modes $u_{\Gamma\ell}$ undergo little change when the geometry or the mechanics of one substructure changes, so that the procedure may encompass various practical problems.

On the other hand, in very simple situations, the computation of the functions $u_{\Gamma\ell}$ can be carried out analytically (cf. Bourquin[1991b]).

However, this procedure does not extend to more realistic elliptic problems, even to scalar ones, and it seems quite hard to figure out what analytical expressions could be used to represent the coupling modes related to the operator of Kirchhoff plates or of two- or three-dimensional elasticity.

In this respect, the full generality of our method should be highlighted.

iii) The modal synthesis method does not depend on the algorithm used to compute the coupling modes. Therefore, current progresses in iterative substructuring theory and practice can be taken advantage of. In particular, any enhancement of the preconditioning techniques should provide the same enhancement of the proposed modal synthesis methods as far as CPU time or approximation properties are concerned.

References.

BLANC SOMMEREUX, I., CAPLAIN, G., EBERT, R., FLAMENT, B., LEFEBVRE, G., NEVEU, A., PEUPORTIER, B., Projet SYMBOL (Synthèse Modale et Boîte à Outils Logiciels), *Rapport AFME/ARMINES*, CENERG-GISE (1989). (In French.)

BOURGAT, J.F. , GLOWINSKI, R. , LE TALLEC, P., Formulation variationnelle et algorithme de décomposition de domaines pour les problèmes elliptiques, *C.R. Acad. Sci. Paris,* t. 306, Série I, (1988a), pp. 569-572. (In French.)

BOURGAT, J.F. , GLOWINSKI, R. , LE TALLEC, P., VIDRASCU , M., Variational formulation and algorithm for trace operator in domain decomposition calculations, *rapport INRIA* 804, (1988b).

BOURQUIN, F., Synthèse modale d'opérateurs elliptiques du second ordre, *C. R. Acad. Sci.*, Paris, t. 309, Série I, (1989) pp. 919-922. (In French.)

BOURQUIN, F., Synthèse modale et analyse numérique des multistructures élastiques, doctoral dissertation, Université Pierre et Marie Curie, Paris (1991a).

BOURQUIN, F., Domain decomposition and eigenvalues of second order operators: convergence analysis, *to appear in Jour. de Math. Pures et Appl.*(1991b).

BOURQUIN, F., Component mode synthesis and eigenvalues of second order operators: discretization and algorithms, *to appear in RAIRO Modél. Math. Anal. Num.* (1991c).

BOURQUIN, F., d'HENNEZEL, F.: Numerical study of an intrinsic component mode synthesis method, *to appear in Comp. Meth. Appl. Mech. Eng.* (1991a).

BOURQUIN, F., d'HENNEZEL, F., Intrinsic component mode synthesis and plate vibrations, *to appear in the proceedings of the Second World Congress on Computational Mechanics,* Stuttgart, (1991b).

CRAIG, R.R. Jr., A review of time domain and frequency domain component mode synthesis methods, *joint ASCE/ASME mechanics conference*, Albuquerque, New Mexico, June 24-26, A.M.D. vol. 67, (1985).

GIBERT, R. J., Vibrations des Structures, Interactions avec les fluides, Sources d'excitations aléatoires, cours de l'école d'été d'analyse numérique CEA-INRIA-EDF, 1986, collection de la Direction des Etudes et Recherches d'Electricité de France, Eyrolles, 1988.

HURTY, W.C., Dynamic analysis of structural systems using component modes, *AIAA Journal,* vol 4 (4), (1965), pp. 678-685.

IMBERT, J.F. , Calcul des structures par éléments finis, Cépadues, 1979.

JEZEQUEL, L., Synthèse modale : théorie et extensions, Doctoral dissertation, université Claude Bernard, Lyon (1985), (In French.)

MEIROVITCH, L., Computational Methods in Structural Dynamics, SIJTHOFF & NOORDHOFF, 1980.

MORAND, H., Méthodes de détermination approchée des modes propres de vibration en calcul des structures , sous-structuration dynamique. *ONERA R.T.* n° 2 | 3238 RY OOOR, (1977) (In French.)

SANCHEZ HUBERT, J., SANCHEZ PALENCIA, E., Vibration and coupling of continuous systems, asymptotic methods, Springer-Verlag, USA, 1989.

VALID, R., La mécanique des Milieux continus et le calcul des structures, Collection de la Direction des Etudes et Recherches d'Electricité de France, Eyrolles, 1977.