

## Decomposition and Fictitious Domains Methods for Elliptic Boundary Value Problems

S. V. Nepomnyaschikh\*

**Abstract.** Boundary value problems for elliptic second order equations in three-dimensional domains with piecewise smooth boundaries are considered. Discretization of the problem is performed using a conventional version of the finite element method with piecewise linear basis functions. The main purpose of the paper is the construction of a preconditioning operator for the resulting system of grid equations. The method is based on two approaches: decomposition of the domain into subdomains and using a new version of the method of fictitious domains. The rate of convergence of the corresponding conjugate gradient method is independent of both the grid size and the number of subdomains.

**1. Introduction.** In this paper, we consider preconditioning operators for the system of grid equations approximating the following boundary value problem:

$$\left\{ \begin{array}{l} -\sum_{i,j=1}^3 \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} + a_0(x)u = f(x), \quad x \in \Omega, \\ u(x) = 0, \quad x \in \Gamma_0, \\ \frac{\partial u}{\partial n} + \sigma(x)u = 0, \quad x \in \Gamma_1. \end{array} \right. \quad (1.1)$$

Here,  $\frac{\partial u}{\partial n}$  is the derivative in the conormal direction. We assume that  $\Omega$  is a polyhedron. Let  $\Omega$  be a union of  $n$  nonoverlapping subdomains  $\Omega_i$ ,

$$\bar{\Omega} = \bigcup_{i=1}^n \bar{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset, \quad i \neq j$$

where  $\Omega_i$  are polyhedrons with diameters on the order of  $H$ . Let  $\Omega^h$

$$\Omega^h = \bigcup_{i=1}^n \Omega_i^h$$

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\*USSR Academy of Sciences, Computing Center, Lavrentiev av. 6, Novosibirsk, 630090, USSR.

be a regular triangulation of  $\Omega$  which is characterized by a parameter  $h$ .

We use  $H^1(\Omega, \Gamma_0)$  to denote a subspace of the Sobolev spaces  $H^1(\Omega)$  of real functions

$$H^1(\Omega, \Gamma_0) = \{v \in H^1(\Omega) \mid v(x) = 0, x \in \Gamma_0\}$$

and we introduce the bilinear form

$$a(u, v) = \int_{\Omega} \left( \sum_{i,j=1}^3 a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + a_0(x)uv \right) dx + \int_{\Gamma_1} \sigma(x)uv dx$$

and the linear functional

$$\ell(v) = \int_{\Omega} f(x)v dx .$$

We assume that the coefficients of the problem (1.1) are such that  $a(u, v)$  is a symmetric, coercive and a continuous form in  $H^1(\Omega, \Gamma_0) \times H^1(\Omega, \Gamma_0)$ , i.e.

$$a(u, v) = a(v, u) , \quad \forall u, v \in H^1(\Omega, \Gamma_0) ,$$

$$\mu_0 \|v\|_{H^1(\Omega)}^2 \leq a(v, v) \leq \mu_1 \|v\|_{H^1(\Omega)}^2 , \quad \forall v \in H^1(\Omega, \Gamma_0) ,$$

and that the linear functional  $\ell(v)$  is continuous in  $H^1(\Omega, \Gamma_0)$ , i.e.,

$$|\ell(v)| \leq \mu \|v\|_{H^1(\Omega)} .$$

Here  $\mu_0, \mu_1, \mu$  are positive constants. A weak formulation of (1.1) is:

Find  $u \in H^1(\Omega, \Gamma_0)$  such that

$$a(u, v) = \ell(v) , \quad \forall v \in H^1(\Omega, \Gamma_0) . \tag{1.2}$$

Denote by  $W$  the space of real-valued continuous piecewise linear functions. Using the finite element method we can pass from (1.2) to the linear algebraic system

$$Au = f \tag{1.3}$$

Our purpose is the construction of a preconditioner  $B$  for the problem (1.3) such that the following inequalities are valid:

$$c_1(Bu, u) \leq (Au, u) \leq c_2(Bu, u) , \quad \forall u \in R^N . \tag{1.4}$$

Here  $N$  is the dimension of  $W$ , the positive constants  $c_1, c_2$  are independent of  $h$  and  $H$ , and the multiplication of  $B^{-1}$  by the vector can be at a low cost.

As a rule, the most efficient preconditioners for solving boundary value problems in domains with complex geometry can be constructed by simplifying geometry of the original domain. We use two approaches: decomposition of the domain into subdomains [5–7] and a version of the method of fictitious domains; cf. [8–10].

The remainder of the paper is organized as follows. In section 2, we describe abstract results which are useful in constructing the preconditioning operator. Using Lemma 2.1 we split the original space of grid functions into a sum of subspaces and using Lemma 2.2 and Lemma 2.3, on the fictitious domains method, we give equivalent norms in subspaces. In section 3, we construct a decomposition of the original

mesh space into a sum of two subspaces. The functions of the first subspace are equal to zero at the boundaries of subdomains. The second subspace corresponds to the values of mesh functions at the boundaries of subdomains extended inside with conservation of the norm. The second subspace has a complex structure, and to simplify it, we decompose it into a sum of a coarse space and local subspaces [1–3,9,11]. In section 4, using the abstract results, we design preconditioning operators for these subspaces. This design uses the fictitious domain (space) method to overcome two obstacles: complex geometry of the substructures and chaotic distribution of the nodes of the triangulation.

**2. General Framework.** In this section, we outline an abstract method for constructing a preconditioning operator by splitting the original Hilbert space into a sum of subspaces. Here, we completely follow [5–10].

**Lemma 2.1.** *Let the Hilbert space  $H$  with the scalar product  $(u, v)$  be split into a sum of subspaces*

$$H = H_1 + H_2 + \dots + H_m ,$$

let  $A: H \rightarrow H$  be a linear, self-adjoint, continuous and positive definite operator, and  $P_i: H \rightarrow H_i$ ,  $i = 1, 2, \dots, m$ , be the orthogonal projections with respect to the scalar product  $(u, v)_A$  generated by the operator  $A$

$$(u, v)_A = (Au, v) .$$

Assume that positive constants  $\alpha$  and  $\beta$  exist such that for any element  $u \in H$  there exists  $u_i \in H_i$  such that

$$u_1 + u_2 + \dots + u_m = u$$

$$\alpha((u_1, u_1)_A + (u_2, u_2)_A + \dots + (u_m, u_m)_A) \leq (u, u)_A$$

and

$$((P_1 + P_2 + \dots + P_m)u, u)_A \leq \beta(u, u)_A$$

for any  $u \in H$ . Assume further that there are selfadjoint operators  $B_i$ ,  $i = 1, 2, \dots, m$ , such that

$$B_i: H \rightarrow H_i ,$$

$$c_1(B_i u, u) \leq (Au, u) \leq c_2(B_i u, u) , \quad \forall u \in H_i .$$

Then,

$$\alpha c_1(A^{-1}u, u) \leq (B^{-1}u, u) \leq \beta c_2(A^{-1}u, u) , \quad \forall u \in H ,$$

where  $B^{-1} = B_1^\dagger + B_2^\dagger + \dots + B_m^\dagger$ , and  $B_i^\dagger$  is a pseudoinverse of  $B_i$ .

Using Lemma 2.1, we will define the preconditioner in the following way. Split the original space of grid functions  $W$  into a sum of subspaces, each having a definite structure which makes it possible to determine easily invertible equivalent norms (i.e., operators  $B_i$ ) in these subspaces. The following lemmas will be used in constructing operators  $B_i$ , which give equivalent norms in subspaces, and for which the multiplication of  $B_i^\dagger$  by the vector can be performed efficiently.

**Lemma 2.2.** *Let  $R^m$  and  $R^n$  be real Euclidean spaces, and  $\Sigma$  and  $S$  be symmetric positive definite matrices of order  $m$  and  $n$ , respectively. Denote by*

$$\begin{aligned}(\phi, \psi)_{\Sigma} &= (\Sigma\phi, \psi) , \\(u, v)_{\mathcal{S}} &= (Su, v)\end{aligned}$$

the scalar products generated by these matrices in  $R^m$  and  $R^n$ , and let  $T: R^m \rightarrow R^n$ , be a linear operator such that

$$\alpha(\phi, \phi)_{\Sigma} \leq (T\phi, T\phi)_{\mathcal{S}} \leq \beta(\phi, \phi)_{\Sigma}$$

for any  $\phi \in R^m$ . Here,  $\alpha$  and  $\beta$  are positive constants. Let

$$C = T\Sigma^{-1}T^* ,$$

where  $T^*$  is an operator adjoint to  $T$  with respect to the Euclidean scalar products in  $R^m$  and  $R^n$ . Then,

$$\alpha(C^+u, u) \leq (u, u)_{\mathcal{S}} \leq \beta(C^+u, u)$$

for any  $u \in E = \text{Im } T$ .  $C^+$  is a pseudo-inverse of  $C$ .

**Lemma 2.3.** Let  $H_0$  and  $H$  be Hilbert spaces with the scalar products  $(u_0, v_0)_{H_0}$  and  $(u, v)_H$ , respectively, and let  $A$  and  $B$  be selfadjoint positive definite and continuous operators in the spaces  $H_0$  and  $H$ :

$$A: H_0 \rightarrow H_0 , \quad B: H \rightarrow H .$$

Let  $R$  be a linear operator such that

$$R: H \rightarrow H_0 ,$$

$$(ARv, Rv)_{H_0} \leq c_R(Bv, v)_H$$

for any element  $v \in H$ . Furthermore, let there exist an operator  $T$  such that

$$T: H_0 \rightarrow H , \quad RTu_0 = u_0 ,$$

$$c_T(BTu_0, Tu_0)_H \leq (Au_0, u_0)_{H_0}$$

for any element  $u_0 \in H_0$ . Here,  $c_R$  and  $c_T$  are positive constants. Then

$$c_T(A^{-1}u_0, u_0) \leq (RB^{-1}R^*u_0, u_0)_{H_0} \leq c_R(A^{-1}u_0, u_0)$$

for any element  $u_0 \in H_0$ . Here,  $R^*$  is the operator adjoint to  $R$  with respect to the scalar products  $(u_0, v_0)_{H_0}$  and  $(u, v)_H$  such that

$$R^*: H_0 \rightarrow H ,$$

$$(R^*u_0, v)_H = (u_0, R_0v)_{H_0} .$$

**3. Decomposition into subspaces.** To design the preconditioning operator  $B$ , we will completely follow [9] and decompose the space  $W$  into a sum of subspaces

$$W = W_0 + W_1 .$$

To this end, divide the nodes of the triangulation  $\Omega^h$  into two groups: those which lie inside of  $\Omega_i^h$  and those which lie on boundaries of  $\Omega_i^h$ . The subspace  $W_0$  corresponds to the first set. Let

$$S^h = \bigcup_{i=1}^n \partial\Omega_i^h ,$$

$$\begin{aligned} W_0 &= \{u^h \in W \mid u^h(x) = 0, x \in S^h\} , \\ W_{0,i} &= \{u^h \in W_0 \mid u^h(x) = 0, x \in \Omega_i^h\} , \quad i = 1, 2, \dots, n . \end{aligned}$$

It is clear that  $W_0$  is the direct sum of the orthogonal subspaces  $W_{0,i}$  with respect to the scalar product in  $H^1(\Omega)$ :

$$W_0 = W_{0,1} \oplus \dots \oplus W_{0,n} .$$

The subspace  $W_1$  corresponds to the second group of nodes  $\Omega^h$  and can be defined in the following way. First, define  $W_{1/2}$  which is the space of traces of functions from  $W$  on  $S^h$ :

$$W_{1/2} = \{\phi^h \mid \phi^h(x) = u^h|_{S^h}, u^h \in W\}$$

To define the subspace  $W_1$ , we need a norm-preserving extension operator of functions given at  $S^h$  into  $\Omega^h$ . Let us consider the subdomain  $\Omega_i^h$ . Let  $(s_1, s_2, n)$  be a near-boundary coordinate system [7,8] which is defined in a  $\delta$ -neighborhood of  $\partial\Omega_i^h$ . Here  $s_1, s_2$  define a point  $P$  at  $\partial\Omega_i^h$  and  $n$  is the distance between the given point and  $\partial\Omega_i^h$ . Set

$$\begin{aligned} t: H^{1/2}(\partial\Omega_i^h) &\rightarrow H^1(\Omega_i^h) , \\ t\phi &= u , \\ u(s_1, s_2, n) &= \left(1 - \frac{n}{\delta}\right) \frac{1}{\text{meas } K(P, n)} \int_{K(P, n)} \phi(x) dx , \\ P \in \partial\Omega_i^h , \quad 0 \leq n &\leq \delta , \end{aligned} \quad (3.1)$$

where

$$K(P, n) = \{(s'_1, s'_2, 0) \in \partial\Omega_i^h \mid |s'_i - s_i| \leq n, i = 1, 2\} .$$

The function  $u$  is extended by zero in the rest of  $\Omega_i$ . Using Hardy's inequality [4], it is not difficult to see that  $t$  is a norm-preserving operator, i.e.

$$\|u\|_{H^1(\Omega_i)} = \|t\phi\|_{H^1(\Omega_i)} \leq c_1 \|\phi\|_{H^{1/2}(\partial\Omega_i)} , \quad \forall \phi \in ,$$

where  $c_1$  is a positive constant. Using the auxiliary mesh, which is topologically equivalent to a uniform mesh, we can define the finite-element analogy  $t_h$  of the operator  $t$  from (3.1), such that

$$\|t_h \phi^h\|_{H^1(\Omega_i)} \leq c_2 \|\phi^h\|_{H^{1/2}(\partial\Omega_i)} , \quad \forall \phi^h \quad (3.2)$$

The cost of the multiplication of  $t$  and  $t^*$  by a vector is proportional to the number of mesh domain nodes. Here  $c_2$  independent of  $h$ ,  $H$  and  $t^*$  is an adjoint operator to  $t$  with respect to the Euclidean scalar product (see [8,9] for more details). Set

$$W_1 = t_h W_{1/2} .$$

It is obvious that  $W = W_0 + W_1$  and according to (3.2) this decomposition of  $W$  satisfies Lemma 2.1 with  $\alpha$  independent of  $h$  and  $H$ .

According to Lemma 2.2, to define the preconditioning operator  $B_1$  for the subspace  $W_1$  we need to define the preconditioning operator  $B_{1/2}$  which generates the norm in the space  $W_{1/2}$ . The set  $S^h$  has a complex structure. Therefore we will split  $W_{1/2}$  into subspaces associated with the following substructures. Let  $S^h$  be a union of  $k$  nonoverlapping triangles  $T_i$ :

$$S^h = \bigcup_{i=1}^k \bar{T}_i, \quad T_i \cap T_j = \emptyset, \quad i \neq j. \quad (3.3)$$

We assume that  $k$  and  $n$  are of the same order of magnitude. Split the space  $W_{1/2}$  into a sum of subspaces

$$W_{1/2} = V_0 + V_1 + \cdots + V_k, \quad (3.4)$$

where  $V_0$  is the coarse space which consists of continuous functions linear on the triangles  $T_i$ ,  $i = 1, 2, \dots, k$ , and  $V_i$ ,  $i = 1, \dots, k$  correspond to  $T_i$  and are defined below. Denote by

$$\begin{aligned} W_{1/2,i} &= \{\phi_i^h \in H^{1/2}(T_i) \mid \phi_i^h(x) = \phi^h(x), \quad x \in T_i, \quad \phi^h \in W_{1/2}\}, \\ \dot{W}_{1/2,i} &= \{\phi_i^h \in W_{1/2,i} \mid \phi_i^h(x) = 0, \quad x \in \partial T_i\}. \end{aligned}$$

For any triangle  $T_i$ , we define the explicit extension operator  $\tau_i$ :

$$\tau_i: W_{1/2,i} \rightarrow W_{1/2}.$$

Denote by  $T'_i$  the union of adjacent triangles:

$$T'_i = \bigcup_{\bar{T}_j \cap \bar{T}_i \neq \emptyset} \bar{T}_j.$$

Let  $\phi^h$  be a function given on  $T_i$ . We will define the extension of the function  $\phi^h$  using two tricks: *reflection* and *semi-reflection*. First let us define reflection  $\tilde{\phi}^h$  of the function  $\phi^h$ . Let  $T_j$  be a triangle from  $T'_i$ , which has a common side with  $T_i$  (see Fig. 3.1a).

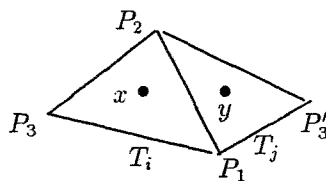


Figure 3.1

Let  $u, v, w$ ,  $u + v + w = 1$  be the barycentric coordinates on the triangle  $T_i$ . Then any point  $x$  from  $T_i$  can be represented in the following form

$$x = u \cdot P_1 + v \cdot P_2 + w \cdot P_3$$

where  $P_1, P_2, P_3$  are the vertices of the triangle  $T_i$ . Set

$$\tilde{\phi}(y) \equiv \tilde{\phi}(u \cdot P_1 + v \cdot P_2 + w \cdot P_3) = \phi(c) \equiv \phi(u \cdot P_1 + v \cdot P_2 + w \cdot P_3)$$

Now let us define semi-reflection  $\tilde{\phi}$  of the function  $\phi$  given on  $T_i$  for the case when a triangle  $T_j$  has a vertex common with the triangle  $T_i$  (see Fig. 3.2).

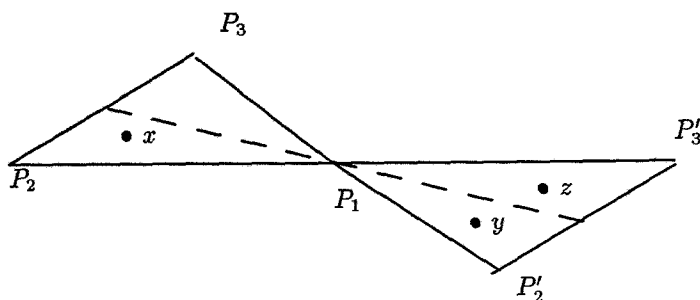


Fig. 3.2

Set

$$\begin{aligned} \tilde{\phi}(z) &\equiv \tilde{\phi}(u \cdot P_1 + v \cdot P'_3 + w \cdot 0.5 \cdot (P'_2 + P'_3)) \\ &= \tilde{\phi}(y) \equiv \tilde{\phi}(u \cdot P_1 + v \cdot P'_2 + w \cdot 0.5 \cdot (P'_2 + P'_3)) \\ &= \phi(x) \equiv \phi(u \cdot P_1 + v \cdot P_2 + w \cdot 0.5 \cdot (P_2 + P_3)) . \end{aligned}$$

It is clear that we can define the “semi-reflection” for the case when a triangle  $T_j$  has a side common with the triangle  $T_i$  too. Combining both approaches, we can define an extension  $\tilde{\phi}^h$  of the function  $\phi^h$  onto  $T_i$ . Using the piecewise linear cutoff function  $\xi_i$  defined by

$$\xi_i(x) = \begin{cases} 1, & x \in T_i \\ 0, & x \notin T_i, \end{cases}$$

we can define the extension operator  $\tau_i$  by

$$\tau_i \phi^h = \xi_i \tilde{\phi}^h .$$

Set

$$V_i = \tau_i W_{1/2,i} + \hat{W}_{1/2,i}, \quad i = 1, 2, \dots, k . \tag{3.5}$$

Using Lemma 4.1 from [9] it is easy to see that the decomposition (3.4), (3.5) satisfies Lemma 2.1 with  $\alpha$  independent of  $h$  and  $H$ .

**4. The Fictitious Space Method.** In this section we will design a preconditioning operators for the subspaces  $W_{0,i}$ ,  $i = 1, 2, \dots, n$ , and  $V_i$ ,  $i = 0, 1, \dots, k$  constructed in Section 3 using Lemmas 2.2 and 2.3.

There are two obstacles to using the finite-element approximation of the Laplace operator as a preconditioning operator for  $W_{0,i}$ . The first is related to the geometry

of  $\Omega_i$  while the second relates to the chaotic distribution of nodes of the triangulation  $\Omega_i^h$ .

Let the domain  $\Omega_i$  be embedded in the cube  $\Pi_i$ . We extend the triangulation  $\Omega_i^h$  to the entire cube  $\Pi_i^h$  in such a way that  $\Pi_i^h$  is a quasi-uniform. We will denote by  $W(\Pi_i^h)$  the space of continuous functions which are linear on each simplex of the triangulation  $\Pi_i^h$ , and by  $\Delta_{\Pi_i^h}$  the approximation of a symmetric elliptic operator in  $W(\Pi_i^h)$ . According to Lemma 2.3, we need a continuous operator

$$R_{\Pi_i}: W(\Pi_i^h) \rightarrow W_{0,i} .$$

Let

$$R_{\Pi_i} = I_{\Omega_i} - t_h \cdot I_{\partial\Omega_i} ,$$

where  $I_{\Omega_i}$ ,  $I_{\partial\Omega_i}$  are the trace operators from  $\Pi_i^h$  onto  $\Omega_i^h$ ,  $\partial\Omega_i^h$ , respectively. The operator  $t_h$  is the extension operator which was described in section 3. From the continuity of  $I_{\Omega_i}$ ,  $I_{\partial\Omega_i}$  and  $t_h$ , we obtain the continuity of  $R_{\Pi_i}$ . We can define the operator

$$T_{\Pi_i}: W_{0,i} \rightarrow W(\Pi_i^h) ,$$

as an extension by zero. It is easy to see that the hypotheses of Lemma 2.3 are valid. Hence there exist positive constants  $c_1$  and  $c_2$ , independent of  $h$  and  $H$ , such that

$$c_1 \|u^h\|_{h^1(\Omega_i)}^2 \leq ((R_{\Pi_i}(-\Delta_{\Pi_i^h})^{-1} R_{\Pi_i}^*)^{-1} u, u) \leq c_2 \|u^h\|_{H^1(\Omega_i)}^2$$

for any  $u^h \in W_{0,i}$ .

Further, along with the mesh  $\Pi_i^h$  in the cube  $\Pi_i$  we will also consider an auxiliary uniform mesh  $Q_i^h$  with a mesh step  $h_0$ :  $c_3 h \leq h_0 \leq h_{\min}/\sqrt{3}$ . Here  $h_{\min}$  is the length of the minimal edge of simplices of the triangulation  $\Pi_i^h$ , and constant  $c_3$  is independent of  $h$  and  $H$ . Denote the nodes of the mesh  $Q_i^h$  by  $Z_{j,k,\ell} = (X_j, Y_k, Z_\ell)$ ,  $j, k, \ell = 0, 1, \dots, M_i$  and denote the cells of the mesh  $Q_i^h$  by

$$Q_{i,j,k,\ell} = \{(x, y, z) \mid X_j \leq x < X_{j+1}, Y_k \leq y < Y_{k+1}, Z_\ell \leq z < Z_{\ell+1}\} , \\ \ell, j, k = 0, 1, \dots, M_i - 1 .$$

On the mesh  $Q_i^h$ , we will consider the space  $W(Q_i^h)$  of mesh functions  $U(Z_{j,k,\ell})$ . Let us define the operator  $R_{Q_i}$ :

$$R_{Q_i}: W(Q_i^h) \rightarrow W(\Pi_i^h)$$

which introduce a correspondence of each function  $U(Z_{j,k,\ell}) \in W(Q_i^h)$  to a function  $u^h \in W(\Pi_i^h)$  in the following way. Let  $z_m$  be a node of the triangulation  $\Pi_i^h$  and let  $z_m \in Q_{j,k,\ell}$ . Let

$$u^h(z_m) = U(Z_{j,k,\ell}) .$$

Note that by the assumptions on  $h_0$  at most one node  $z_m$  of the triangulation  $\Pi_i^h$  can belong to the cell  $Q_{j,k,\ell}$ . The operator  $T_{Q_i}$ ,

$$T_{Q_i}: W(\Pi_i^h) \rightarrow W(Q_i^h)$$



is now defined as follows. If the cell  $Q_{j,k,\ell}$  contains a node  $z_m$  of the triangulation  $\Pi_i^h$ , we set

$$U(Z_{j,k,\ell}) = u^h(z_m) .$$

At the other nodes of the mesh  $Q_i^h$  the function  $U(z_{j,k,\ell})$  can be defined in a relatively arbitrary way, for instance, as follows. Let the node  $z_{j,k,\ell}$  belong to the simplex  $\tau_m$  of the triangulation  $\Pi_i^h$  with the vertices  $z_{m_1}, \dots, z_{m_4}$ . Set

$$U(z_{j,k,\ell}) = \frac{1}{4}(u^h(z_{m_1}) + \dots + u^h(z_{m_4}) , ) .$$

It can be easily shown that the operators  $R_{Q_i}, T_{Q_i}$  defined in this way satisfy the hypotheses of Lemma 2.3 and that the corresponding constants do not depend on  $h$  and  $H$ . Thus, there exist positive constants  $c_4, c_5$ , independent of  $h$  and  $H$ , such that

$$c_4 \|u^h\|_{H^1(\Omega_i)}^2 \leq ((R_{\Pi_i} R_{Q_i} (-\Delta_{Q_i})^{-1} R_{Q_i}^* R_{\Pi_i}^*)^{-1} u, u) \leq c_5 \|u^h\|_{H^1(\Omega_i)}^2$$

for any  $u^h \in W_{0,i}$ . Here  $\Delta_{Q_i}$  is the mesh approximation of a symmetric elliptic operator in  $W(Q_i^h)$ . The preconditioning operator  $B_{0,i}^+$  for  $W_{0,i}$  can now be defined as an extension by zero of the operator  $R_{\Pi_i} R_{Q_i} (-\Delta_{Q_i})^{-1} R_{Q_i}^* R_{\Pi_i}^*$ . Set

$$B_0^+ = B_{0,1}^+ + \dots + B_{0,n}^+ .$$

We now define the preconditioning operator  $B_1^+$  for the subspace  $W_1$ . According to Lemma 2.2, we can define

$$B_1^+ = t_h \Sigma^{-1} t_h^*$$

where the operator  $\Sigma$  generates an equivalent norm in the trace space  $W_{1/2}$  and the extension operator  $t_h$  was described in section 3. To define the operator  $\Sigma$  we will use the decomposition (3.3). Let us first consider the coarse subspace  $V_0$ . Let  $W_0$  be a mesh space which corresponds to the values of the functions  $\phi^h \in V_0$  at the nodes of the triangles  $T_i$  from (3.3). Define, in the space  $W_0$ , the symmetric operator  $S_0$ :

$$(S_0 \phi, \phi) = \sum_{i=1}^k \left( \sum_{j=1}^3 (H^3(\phi_i^{(j)})^2 + H(\phi_i^{(j)} - \phi_i^{(j-1)})^2) \right) ,$$

where  $\phi_i^{(1)}, \phi_i^{(2)}, \phi_i^{(3)} = \phi_i^{(0)}$  are the values of the function  $\phi^h \in V_0$  at the nodes of the triangle  $T_i$ . The dimension of the operator  $S_0$  is equal to the number of the nodes which belong to  $S^h \setminus \Gamma_0$ . Let  $\tau_0$  be a piecewise linear interpolation operator:

$$\tau_0: W_0 \rightarrow W_{1/2} .$$

According to Lemma 2.2, we can set

$$\Sigma_0^+ = \tau_0 S_0^{-1} \tau_0^* .$$

Let us now consider the subspaces  $V_i, i = 1, \dots, k$ . From Lemmas 2.1, 2.2 and the definition of  $V_i$ , we can define

$$\Sigma_i^+ = \tau_i S_{i,N}^{-1} \tau_i^* + S_{i,D}^+ ,$$

where  $S_{i,N}$ ,  $S_{i,\mathcal{D}}$  generate norms in  $W_{1/2,i}$  and  $\tilde{W}_{1/2,i}$ , respectively. To define  $S_{i,N}$  and  $S_{i,\mathcal{D}}^+$  we will again use Lemma 2.3. Let  $T_i$  be embedded into the square  $\Pi_i$  and the triangulation of  $T_i$  be complemented to triangulation  $\Pi_i^h$  of the larger region. In the square  $\Pi_i$ , we will also consider the auxiliary square mesh  $Q_i^h$  with the step  $h_0$ :  $c_6 h \leq h_0 \leq h_{\min}/\sqrt{2}$ . Let  $W(Q_i^h)$  be a space of mesh functions which is defined on  $Q_i^h$  and let  $\Sigma_{Q_i}$  be an operator which generates an equivalent norm in the trace space  $W(Q_i^h)$ . According to Lemma 2.3, we can define

$$\begin{aligned} S_{i,N} &= R_{N,\Pi_i} R_{Q_i} \Sigma_{Q_i}^{-1} R_{Q_i}^* R_{N,\Pi_i}^* , \\ \tilde{S}_{i,\mathcal{D}} &= R_{\mathcal{D},\Pi_i} R_{Q_i} \Sigma_{Q_i}^{-1} R_{Q_i}^* R_{\mathcal{D},\Pi_i}^* , \end{aligned}$$

where

$$R_{N,\Pi_i} = P_i(I_i O_i) P_i^* .$$

Here  $I_i$  is the identity matrix which corresponds to nodes belonging to  $\bar{T}_i$ ,  $O_i$  the nullmatrix which corresponds to nodes belonging to  $\Pi_i \setminus \bar{T}_i$ , and  $P_i$  a permutation matrix. The operators  $R_{\mathcal{D},\Pi_i}$ ,  $R_{Q_i}$  are defined exactly as were those used when constructing  $B_{0,i}$  in the case of three dimensions. The operator  $S_{i,\mathcal{D}}$  is the extension of the operator  $\tilde{S}_{i,\mathcal{D}}$  by zero. We can thus define

$$\begin{aligned} \Sigma^{-1} &= \Sigma_0^+ + \Sigma_1^+ + \dots + \Sigma_k^+ , \\ B_1^+ &= t_h \Sigma^{-1} t_h^* , \\ B^{-1} &= B_0^+ + B_1^+ \end{aligned} \tag{4.1}$$

The following theorem is valid.

**Theorem.** *There exist positive constants  $c_7$ ,  $c_8$ , independent of  $h$  and  $H$ , such that*

$$c_7(Bu, u) \in (Au, u) \in c_8(Bu, u) .$$

Here  $A$  is defined by (1.3) and  $B$  is defined by (4.1).

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