Domain Decomposition Algorithms for the Biharmonic Dirichlet Problem

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Abstract. We consider additive Schwarz methods for the biharmonic Dirichlet problem and show that the algorithms have optimal convergence properties for some conforming finite elements. Some multilevel methods are also discussed.

1. Introduction. We are interested in solving the following biharmonic Dirichlet problem in a plane region

$$\begin{cases} \Delta^2 u = f & \text{in} \quad \Omega, \\ u = g_0 & \text{on} \quad \partial\Omega, \\ \frac{\partial u}{\partial n} = g_1 & \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

It is convenient to work with the weak formulation: Find $u \in H_0^2(\Omega)$ such that

$$a(u,v) = f(v), \quad \forall v \in H_0^2(\Omega), \tag{2}$$

where f is a bounded linear functional on $H_0^2(\Omega)$ and $a(u,v)$ is a symmetric, continuous, H_0^2-elliptic bilinear form. Two examples of such bilinear forms are

$$a(u,v) = \int_\Omega \Delta u \Delta v \, dx, \tag{3}$$

and

$$a(u,v) = \int_\Omega \{\Delta u \Delta v + (1-\sigma)(2\frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1 \partial x_2} - \frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1^2})\} \, dx, \tag{4}$$

where $0 < \sigma < 1/2$ is *Poisson's coefficient* of the plate. The first one arises in *Fluid Dynamics*, and the second provides a variational formulation of the *Clamped Plate Problem.*

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The rest of the paper is organized as follows. In section 2, we introduce some standard conforming finite element approximations. In section 3, we study some additive Schwarz algorithms and establish optimal convergence properties of the algorithms. In section 4, we study a multilevel algorithm for the biharmonic equation. In this brief paper, we do not provide proofs and details of the algorithms; cf. Zhang [11] for further discussion of the algorithms and some numerical experiments.

2. Finite Elements for the Biharmonic Equation.

2.1. Formulation. We triangulate the domain Ω into non-overlapping regions called elements, generally triangles or rectangles. Let V^h be a space of piecewise polynomials with respect to the triangulation. The finite element solution $u_h \in V^h$ satisfies

$$a(u_h, \phi_h) = f(\phi_h), \quad \forall \phi_h \in V^h. \tag{5}$$

Let $\{\phi_i\}$ be the nodal basis for V^h. Then u_h can be represented as

$$u_h = \sum_i x_i \phi_i.$$

Thus, we obtain a linear system for x, the degrees of freedom of u_h,

$$K^h x = b,$$

where $K^h = \{a(\phi_i, \phi_j)\}$ and $b_i = f(\phi_i)$.

The stiffness matrix K^h is symmetric, positive definite. After a proper scaling, its condition number $\kappa(K^h) = O(h^{-4})$. Since the system are usually very large, and the condition number of K^h is also very large, solving the system can be very expensive. Many preconditioners have been designed for K^h. Among them, the additive Schwarz methods studied in this paper seem to be particularly successful and promising.

2.2. Some Conforming Elements. For biharmonic equation, the finite elements are all relatively complicated. In this paper, we restrict ourselves to some standard conforming elements. In particular, we consider the Argyris triangle V_A^h, the Bell triangle V_B^h and the bicubic element V_Q^h. These elements are complicated but among the simplest conforming elements for the biharmonic equation; cf. Ciarlet[5].

The Argyris element consists of continuous differentiable functions, the restriction of which to any element is in $\mathcal{P}_5$. The *degrees of freedom* for the Argyris element in a triangle with vertices $a_i, i = 1, 2, 3$, are given by

$$\{ \ \frac{\partial^\alpha}{\partial x^\alpha} p(a_i), |\alpha| \le 2, \ \frac{\partial}{\partial n_i} p(b_i) \ \},$$

where b_i is the midpoint of the edge $\overline{a_j a_k}$, and n_i is the outward normal of $\overline{a_j a_k}$. The number of the degrees of freedom for one triangle is 21.

It is easy to see that, in general, the normal derivatives of an Argyris element is a polynomial of degree 4. Let $\mathcal{P}_B$ denote the subspace of $\mathcal{P}_5$ formed by those polynomials of $\mathcal{P}_5$ whose normal derivatives along each side of a triangle are polynomials of degree 3 in t, t being the abscissa along an axis containing the side. We note that $\mathcal{P}_4 \subset \mathcal{P}_B \subset \mathcal{P}_5$. The Bell element consists of C^1 functions whose restrictions to a triangle are in $\mathcal{P}_B$. The degrees of freedom for the Bell element are given by

$$\{ \ \frac{\partial^\alpha}{\partial x^\alpha} p(a_i), |\alpha| \le 2 \ \}.$$

If the domain is built from rectangles, then we can also use the bicubic element, known as the Bogner-Fox-Schmit rectangle in the engineering literature. It is a space of C^1 functions whose restrictions to a rectangular element are in $Q_3 = \text{span}\{x^i y^j, 0 \le i, j \le 3\}$. The degrees of freedom of bicubic element are given by

$$\{\ p(a_i),\ \frac{\partial p}{\partial x_1}(a_i),\ \frac{\partial p}{\partial x_2}(a_i),\ \frac{\partial^2 p}{\partial x_1 \partial x_2}(a_i)\ \}.$$

3. Additive Schwarz Methods for the Biharmonic Problem. In this section, we study the additive Schwarz method for the biharmonic problem. We consider the bicubic element, the Argyris element and the Bell element.

The additive Schwarz schemes were designed by Dryja and Widlund [6] and Matsokin and Nepomnyaschikh [9]. The optimal convergence properties of the algorithm were established for second order self-adjoint elliptic problems, see Dryja and Widlund [6,7,8]. Generalizations to the nonsymmetric or indefinite cases have been made by Cai and Widlund; cf. [2,3,4]. We show in this paper that the condition number of certain additive Schwarz methods for the biharmonic equation is uniformly bounded.

Suppose that the finite element space V can be written as a sum of subspaces,

$$V = V_0 + V_1 + \cdots + V_N.$$

Instead of solving the original finite element equation, in the additive Schwarz scheme, we solve

$$Pu_h = (P_{V_0} + P_{V_1} + \cdots + P_{V_N})u_h = g_h$$

for some g_h. Here $P_{V_i} : V \to V_i$, is a projection defined by

$$a(P_{V_i}u, \phi) = a(u, \phi), \quad \forall \phi \in V_i. \tag{6}$$

The natural question is how to find decompositions of V^h, and what properties of the decomposition give optimal algorithms.

As for the second order cases, a coarse problem is crucial in the algorithms. In the second order case, an obvious candidate for the coarse subspace is the space V^H associated with a coarse triangulation $\mathcal{T}^H$. However, for the biharmonic case, when the Argyris and Bell elements are used, the coarse finite element space V_B^H (V_A^H) is not a subspace of V_B^h (V_A^h). Therefore, we cannot use V_B^H or V_A^H as our coarse subspaces and a new coarse subspace has to be found.

In the case of the iterative substructuring methods, the situation is even worse. We recall that these are domain decomposition algorithms which use nonoverlapping subregions; cf. Widlund [7]. When we use the bicubic element and thus $V_Q^H \subset V_Q^h$ can be used as coarse subspace, the direct generalization of certain algorithms designed for second order problems results in algorithms with condition numbers which grow at least like $1/H^2$. Better algorithms are obtained by adding certain vertex spaces to the space-decomposition. We will not discuss the iterative substructuring method further in this paper.

The difficulty of proving the optimality of the algorithms are due to the presence of the high order derivatives in the elements. The tools that work for second order equations and linear element cannot be used here.

3.1. The bicubic Element. We first triangulate the domain Ω into nonoverlapping rectangles $\Omega_i, i = 1, \cdots, N$, to obtain a coarse triangulation $\mathcal{T}^H = \{\Omega_i\}_1^N$. Then each rectangle Ω_i is further divided into smaller rectangles τ_i to obtain the fine triangulation $\mathcal{T}^h = \{\tau_i\}$.

We assume that Ω can be decomposed into overlapping subdomains $\Omega = \cup_{i=1}^N \hat{\Omega}_i$ and the decomposition satisfies

ASSUMPTION 3.1. *The decomposition* $\Omega = \cup_{i=1}^N \hat{\Omega}_i$ *satisfies*

- $\partial\hat{\Omega}_i$ *aligns with the boundaries of fine elements, i.e.* $\hat{\Omega}_i$ *is the union of some elements* τ_j
- $\{\hat{\Omega}\}_{i=1}^N$ *forms a finite covering of* Ω *with a covering constant* N_c, *i.e. we can color* $\{\hat{\Omega}\}_{i=1}^N$ *using at most* N_c *colors in such way that the subdomains of the same color are disjoint.*
- *There exists a partition of unity* $\{\theta_i\}$ *satisfying*

$$\sum \theta_i = 1, \quad \text{with } \theta_i \in C_0^\infty(\hat{\Omega}_i), 0 \le \theta_i \le 1 \text{ and } |\nabla\theta_i| \le C/H.$$

One way of constructing $\{\hat{\Omega}_i\}$ satisfying the above assumption is described in Dryja and Widlund [6]. We extend each Ω_i to a larger region $\hat{\Omega}_i$ so that $C_1 H_i \le \text{dist}(\partial\hat{\Omega}_i, \partial\Omega_i) \le C_2 H_i$. We cut off the part of $\hat{\Omega}_i$ that is outside Ω. Another way of constructing $\{\hat{\Omega}_i\}$ is described in section 4.

Let $V_0 = V_Q^H$ and V_Q^h be the bicubic elements associated with the triangulations $\mathcal{T}^H$ and $\mathcal{T}^h$, respectively. Let $V_i = V_Q^h(\hat{\Omega}_i) = V_Q^h \cap H_0^2(\hat{\Omega}_i)$, and $P_{V_i} : H_0^2(\Omega) \to V_i$, be the orthogonal projection, and let

$$P = \sum_{i=0}^{N} P_{V_i}.$$

We have the following additive Schwarz algorithm

ALGORITHM 3.1. *Find* $u_h \in V^h$ *such that*

$$Pu_h = g_h, \tag{7}$$

with $g_h = \sum_i g_i$, *where* g_i *is given by the solutions for the following finite element problems*

$$a(g_i, \phi_h) = a(P_{V_i}u, \phi_h) = f(\phi_h), \quad \forall\phi_h \in V_i. \tag{8}$$

To find u_h, we first find the right hand side g_h by solving (8) and we then use the conjugate gradient method to solve the system. In each iteration, we need to compute Pv_h for some element $v_h \in V^h$. This is done in the following steps

- Compute $P_0 v_h$ by solving the finite element problems $K^H x = b$ in subspace V_Q^H. Here K^H is the stiffness matrix and $\dim(V_Q^H) = 4N$, where N is the number of interior coarse grid points.
- Compute $P_i v_h$ for each subdomain $\hat{\Omega}_i$ by solving the finite element problems $K_i^h x = b$ in the subdomain $\hat{\Omega}_i$. Here K_i^h is the stiffness matrix and $\dim(V_i) = 4n_i$, where n_i is the number of fine grid points inside $\hat{\Omega}_i$.

THEOREM 3.1. *The iteration operator* P *of the additive Schwarz scheme satisfies*

$$C_1 a(u, u) \le a(Pu, u) \le C_2 a(u, u), \quad \forall u \in V_Q^h.$$

It is of course desirable to reduce the work in each iteration without decreasing the rate of convergence. Let $\Lambda^h(D)$ and $\Lambda^H(D)$ be the sets of fine and coarse grid points in the set D, respectively. Let ϕ_k^α and Φ_k^α be the nodal basis functions of V_Q^h and V_Q^H, respectively. To describe an alternative algorithm, we also need to introduce the following new subspaces.

- The vertex space $V_{x_k} = \text{span}\{\phi_k^{xy}(x)\}$, for each vertex x_k. For $\phi \in V_{x_k}$, $\tau(x_k) = \text{supp}(\phi)$ is a polygon of diameter $O(h)$. We note that $\dim(V_{x_k}) = 1$.
- $\tilde{V}_i = \text{span}\{\phi_k^\alpha(x), |\alpha| \leq 1, x_k \in \Lambda^h(\Omega_i)\}$. $\tilde{V}_i$ is a subspace of V_i consisting of functions with vanishing second derivative ($\frac{\partial^2 u_h}{\partial x \partial y}$) at all the vertices of the elements. The stiffness matrix for $\tilde{V}_i$ is a principal minor of the stiffness matrix for V_i and $\dim(\tilde{V}_i) = \frac{3}{4}\dim(V_i)$.
- $\tilde{V}_Q^H = \text{span}\{\Phi_i^\alpha, |\alpha| \leq 1, i \in \Lambda^H\}$. $\tilde{V}_Q^H$ is a subspace of V_Q^H consisting of functions with vanishing second derivative ($\frac{\partial^2 u_h}{\partial x \partial y}$) at all the vertices of the substructures. The stiffness matrix for $\tilde{V}_Q^H$ is a principal minor of the stiffness matrix for V_Q^H and $\dim(\tilde{V}_Q^H) = \frac{3}{4}\dim(V_Q^H)$.

Using these subspaces, we have

ALGORITHM 3.2. *Find $u_h \in V_Q^h$ by solving*

$$P^{(2)}u_h \equiv (P_{\tilde{V}_Q} + \sum_i P_{\tilde{V}_i} + \sum_{k\in\Lambda^h} P_{V_{x_k}})u_h = g_h \tag{9}$$

with the appropriate right hand side g_h.

THEOREM 3.2. *The iteration operator P for Algorithm 3.2 satisfies*

$$C_1 a(u,u) \leq a(P^{(2)}u, u) \leq C_2 a(u,u), \quad \forall u \in V_Q^h$$

3.2. Argyris and Bell Elements. As in the bicubic element case, we define two triangulations, the coarse triangulation $\mathcal{T}^H = \{\Omega_i\}$ and the fine triangulation $\mathcal{T}^h = \{\tau_j\}$, using triangular elements. The subregions $\hat{\Omega}_i$ are defined similarly. We assume that all the substructures and elements are shape regular in the usual sense. We present our algorithms for the Bell element. The algorithms for the Argyris element are similar.

Let V_B^h and V_B^H be the space of Bell elements with respect to $\mathcal{T}^h$ and $\mathcal{T}^H$, respectively. V_A^h and V_A^H are similarly defined.

In general, the second derivatives of $\Phi \in V_B^H$ at the edge nodes x_i have two values except at the vertices of the substructures. Therefore, $V_B^H \not\subset V_B^h$. Thus, a new coarse space has to be found. An easy way of modifying V_B^H to achieve this goal is by replacing the basis functions of V_B^H. Note that, in a substructure Ω_j, a basis function Φ of V_B^H can be represented by the basis of V_B^h:

$$\Phi(x) = \sum_{|\alpha|\leq 2}\sum_{x_i\in\Lambda^h(\Omega_j)} \Phi_\alpha(x_i)\phi_i^\alpha(x) + \sum_{|\alpha|\leq 2}\sum_{x_i\in\Lambda^h(\partial\Omega_j)} \Phi_\alpha(x_i)\phi_i^\alpha(x) \quad x \in \overline{\Omega}_j,$$

which is now replaced by

$$\Psi(x) = \sum_{|\alpha|\leq 2}\sum_{x_i\in\Lambda^h(\Omega_j)} \Phi_\alpha(x_i)\phi_i^\alpha(x) + \sum_{|\alpha|< 2}\sum_{i\in\Lambda^h(\partial\Omega_j)} \Phi_\alpha(x_i)\phi_i^\alpha(x).$$

Note that the second derivatives of Ψ vanish at the nodes on $\partial\Omega_i$. We define the coarse space U_B^H as

$$U_B^H = \text{span}\{\Psi_i^\alpha, |\alpha| \le 2,\ i \in \Lambda^H\}.$$

It is easy to see that we have the inclusion $U_B^H \subset V_B^h$. Let $V_0 = U_B^H$, $V_i = V_B^h \cap H_0^2(\hat{\Omega}_i)$ and we obtain a space decomposition

$$V^h = \sum_{i=0}^{N} V_i$$

and

ALGORITHM 3.3. *Find $u_h \in V_B^h$ such that*

$$Pu_h = (P_{V_0} + P_{V_1} + \cdots + P_{V_N})u_h = g_h \tag{10}$$

with an appropriate g_h.

THEOREM 3.3. *The iteration operator P for Algorithm 3.3 satisfies*

$$C_1 a(u,u) \le a(P^{(2)}u, u) \le C_2 a(u,u), \quad \forall u \in V_Q^h$$

REMARK 3.1. In computing $P_0 v_h$, we need to solve the coarse problem $K_{U_B^H} x = b$, where $K_{U_B^H} = \{a(\Psi_i, \Psi_j)\}$ and Ψ_i, Ψ_j the modified basis functions. Thus, we need to compute the matrix $K_{U_B^H} = \{a(\Psi_i, \Psi_j)\}$. A standard way is to use numerical integration. An alternative way is to replace $K_{U_B^H}$ by $K_{V_B^H}$. This is equivalent to using an inexact solver P_0' to replace P_0. It can be shown that $\{a(\Phi_i, \Phi_j)\}$ and $\{a(\Psi_i, \Psi_j)\}$ are spectrally equivalent. Thus we still have an algorithm with uniformly bounded condition number.

There are also simplified algorithms for the Argyris and Bell elements which are quite similar to those for the bicubic element.

4. Multilevel Methods for the Biharmonic Problem. In this section, we consider multilevel additive Schwarz methods for the biharmonic equation. Although all the two level algorithms in this paper can be generalized to more than two levels, we only consider a special case, which in matrix form corresponds to a *multilevel block diagonal scaling* (MBDS). For simplicity, we use the bicubic elements.

We define a sequence of nested rectangular triangulations $\{\mathcal{T}^l\}_{l=1}^L$. We start with a coarse triangulation $\mathcal{T}^1 = \{\tau_i^1\}_{i=1}^{N_1}$, where τ_i^1 represents an individual rectangle. The successively finer triangulations $\mathcal{T}^l = \{\tau_i^l\}_{i=1}^{N_l}$ are defined by dividing the rectangles of the triangulation $\mathcal{T}^{l-1}$ into four rectangles. Let $h_i^l = \text{diameter}(\tau_i^l), h_l = \max_i h_i^l$, and $h = h_L$. The level l grid points are denoted by Λ^l, and the basis functions by $\phi_{i,\alpha}^l, i \in \Lambda^l$. Here α represents the order of derivatives.

Let $V^l = V_Q^{h_l}$ be the bicubic elements associated with $\mathcal{T}^l$. Let $\Omega_i^l = \text{supp}\{\phi_{i,\alpha}^l\}$ be the support of an individual basis function, and let $V_i^l = \text{span}\{\phi_{i,\alpha}^l\}$ be the span of the level l basis functions at the grid point x_i. We note that for the bicubic element $\dim\{V_i^l\} = 4$. On each level, we have an overlapping decomposition of the domain

$$\Omega = \cup_i \Omega_i^l.$$

This decomposition satisfies Assumption 3.1. We have the space decomposition

$$V^h = V^L = V^H + \sum_{l=1}^{L}\sum_{i=1}^{N_l} V_i^l.$$

The operator of the L-level additive Schwarz algorithm is given by

$$P = \sum_{l=1}^{L}\sum_{i=1}^{N_l} P_i^l \stackrel{\text{def}}{=} \sum_{l=1}^{L}\sum_{i=1}^{N_l} P_{V_i^l},$$

where $P_{V_i^l}$ is the $a(\cdot,\cdot)$-orthogonal projection from V^h onto V_i^l.

ALGORITHM 4.1 (MBDS ALGORITHM). *Find $u_h \in V^L$ by solving*

$$Pu_h = g_h \tag{11}$$

with an appropriate right hand side g_h.

In the matrix form, equation (11) can be written as:

$$B^{-1}K_L x = B^{-1}b$$

where $B^{-1} = D_L^{-1} + \Pi_{L-1}D_{L-1}^{-1}\Pi_{L-1}^t + \cdots + \Pi_1 K_1^{-1}\Pi_1^t$. Here K_l is the stiffness matrix associated with $\mathcal{T}^l$ and $D_l = \text{diag}\{K_l\}$, Π_l a prolongation operator, and Π_l^t a restriction operator. The **MBDS** algorithm is a natural generalization of a block diagonal scaling method.

THEOREM 4.1. *The multilevel additive Schwarz operator P satisfies*

$$C_1 L^{-1} a(u,u) \leq a(Pu,u) \leq CLa(u,u).$$

Thus

$$\kappa(B^{-1}K) \leq CL^2$$

REMARK 4.1. For second order problems, the corresponding algorithm is a multilevel diagonal scaling, which is equivalent to the BPX algorithm of Bramble, Pasciak and Xu [1]. We note that in the second order case, we have that the operator P has a constant upper bound; cf. Zhang [10]. It is also possible to strengthen the result in theorem 4.1.

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