CHAPTER 19

Domain Decomposition Method and Slow Passage Through a Hopf Bifurcation

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Abstract. Slow passage through a Hopf bifurcation gives rise to a delay in the appearance of the bifurcation. We use normal form to reveal the essence of this phenomenon. Moreover, numerically it is found that the solution is influenced by the numerical methods, such as the routines used, the tolerance settings, and by machine precision. To overcome these numerical method dependent deficiencies, we suggest a domain decomposition method for the calculation of a slow passage through a Hopf bifurcation, by coupling a local normal form transformation and its associated analytic representation in terms of quadratures with ODE solvers away from a Hopf bifurcation.

1. Introduction. In conventional bifurcation theory, the control parameters are usually assumed independent of time. In many applications, however, it is more natural to consider that the parameters vary slowly with time. For examples, in hydraulic devices, fluid properties such as viscosity and density change due to the temperature variation during the entire operating time; in epidemiology, the transmission rate of an infectious disease can vary seasonally.

Erneux and Reiss (1988) study the delay of transition to the asymptotically stable limit cycle of a supercritical Hopf bifurcation caused by slowly varying parameter. Baer, Erneux and Rinzel (1989) use the FitzHugh-Nagumo equations as a model to describe the mathematical and qualitative features of the slow passage through a Hopf bifurcation. They use the WKB method to analyze the model and point out a delay in the appearance of the bifurcation. In this paper, we use the normal form to show the essence of the delay effect in the slow passage through a Hopf bifurcation. It is found that in certain examples the numerical solution is very sensitive to the routines used, the tolerance settings as well as the machine

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precisions. To overcome these numerical method dependent deficiencies, we have developed a domain decomposition method for the calculation of slow passage through a Hopf bifurcation, by coupling a local normal form transformation and its analytic representation in terms of quadrature with ODE solvers away from a Hopf bifurcation.

2. Normal form and delay in the appearance of bifurcation. The normal form near a Hopf bifurcation of a 2-dimensional system can be written as (see, e.g., Guckenheimer and Holmes, 1983; for convenience, our a bears an opposite sign)

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} d\mu & -(\omega + c\mu) \\ \omega + c\mu & d\mu \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (x_1^2 + x_2^2)\begin{pmatrix} -a & -b \\ b & -a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad (1)$$

or in the polar coordinate system

$$\begin{aligned} \frac{dr}{dt} &= r(d\mu - ar^2), \\ \frac{d\theta}{dt} &= \omega + c\mu + br^2, \end{aligned} \quad (2)$$

where μ is the bifurcation parameter. When one calculates the normal form of a system in a practical problem, the coefficients a, b and c may depend on the bifurcation parameter μ (or its multiple $d\mu$) prior to further simplifications. Let us consider the system

$$\begin{aligned} \frac{dr}{dt} &= r(s(t) - a(s(t))r^2), \\ \frac{d\theta}{dt} &= g(s(t)) + b(s(t))r^2, \end{aligned} \quad (3)$$

where $s(t)$ is the slowly varying parameter. Without loss of generality, we assume that the Hopf bifurcation occurs at $t = \bar{t}$ with $s(\bar{t}) = 0$, $g(0) = \omega \neq 0$; and $a(0) \neq 0$, otherwise, the Hopf bifurcation is degenerate. We further assume the Hopf bifurcation is supercritical so that $a(s(t)) > 0$. Note that the first equation in (3) is decomposed from the second and is a Bernoulli equation. If the initial conditions are $r(0) = r_0 > 0$, and $\theta(0) = \theta_0$, the square of the solution, $r(t)$, can be written as

$$[r(t)]^2 = \frac{r_0^2 \exp(2\int_0^t s(\tau)d\tau)}{1 + 2r_0^2 \int_0^t a(s(\xi)) \exp(2\int_0^\xi s(\tau)d\tau)d\xi}. \quad (4)$$

Thus $\theta(t)$ is a quadrature:

$$\theta(t) = \theta_0 + \int_0^t g(s(\tau))d\tau + \int_0^t b(s(\tau))r^2(\tau)d\tau. \quad (5)$$

Note that under the assumption that $a(s(t)) > 0$, one can obtain an upper bound for $r(t)$:

$$r(t) \leq r_0 e^{\int_0^t s(\tau)d\tau}. \quad (6)$$

If the slowly varying parameter $s(t)$ is a linear function of time t:

$$s(t) = s_0 + \epsilon t, \quad s_0 < 0, \quad 0 < \epsilon << 1, \tag{7}$$

then

$$\int_0^t s(\tau) d\tau = s_0 t + \frac{1}{2}\epsilon t^2, \tag{8}$$

which is negative for $t \in (0, 2\bar{t})$, where $\bar{t} = -s_0/\epsilon$ is the time when the parameter $s(t)$ reaches the Hopf bifurcation value 0. Thus $|r(t)| \le r_0$ until $t > 2\bar{t}$. If the initial condition r_0 is very small, we can not observe the oscillation of the system approaching the asymptotically stable limit cycle until the time becomes double of that required to reach the usual Hopf bifurcation value.

Notice that the solution (4) and the upper bound (6) are generally true, while the conclusion of double bifurcation time is a result of the special assumption of the linearly varying parameter (7).

3. Asymptotical behavior. Now that the regular limit cycles are not immediately observed when the slowly varying parameter $s(t)$ passes through the bifurcation value 0, a very natural question is whether they will be observed when the parameter becomes large. Assume the slowly varying parameter $s(t)$ is monotonically increasing from a negative initial value s_0, and $\int_0^t s(\tau)d\tau$ tends to positive infinity as t approaches positive infinity. Moreover, it is reasonable to expect that the other parameters will change slower than $s(t)$, e.g.,

$$\lim_{t \to +\infty} \frac{a(s(t))}{s(t)} = 0, \tag{9}$$

$$\lim_{t \to +\infty} \frac{d}{dt}\left[\frac{a(s(t))}{s(t)}\right] = 0. \tag{10}$$

Under these conditions, we can show that $r(t)$ is asymptotic to $\sqrt{s(t)/a(s(t))}$ — the "usual" amplitude of the oscillation. Let t_1 be a value of t such that $s(t_1) > 0$. According to (4), for $t > t_1$

$$[r(t)]^2 = \frac{r_0^2 e^{2\int_0^{t_1} s(\tau)d\tau} e^{2\int_{t_1}^{t} s(\tau)d\tau}}{1 + 2r_0^2 \int_0^{t_1} a(s(\xi)) e^{2\int_0^{\xi} s(\tau)d\tau} d\xi + 2r_0^2 \int_{t_1}^{t} a(s(\xi)) e^{2\int_0^{\xi} s(\tau)d\tau} d\xi}$$

$$= \frac{r_0^2 e^{2\int_0^{t_1} s(\tau)d\tau} e^{2\int_{t_1}^{t} s(\tau)d\tau}}{1 + 2r_0^2 \int_0^{t_1} a(s(\xi)) e^{2\int_0^{\xi} s(\tau)d\tau} d\xi + 2r_0^2 \int_{t_1}^{t} a(s(\xi)) e^{2\int_0^{t_1} s(\tau)d\tau} e^{2\int_{t_1}^{\xi} s(\tau)d\tau} d\xi}.$$

Since $a(s(t))$ is bounded away from zero, the denominator of the above expression is dominated by the last term for large t. Therefore $[r(t)]^{-2}$ is asymptotic to

$$\frac{2\int_{t_1}^{t} a(s(\xi)) e^{2\int_{t_1}^{\xi} s(\tau)d\tau} d\xi}{e^{2\int_{t_1}^{t} s(\tau)d\tau}}. \tag{11}$$

Denoting

$$\phi(t) = \int_{t_1}^{t} a(s(\xi)) e^{2\int_{t_1}^{\xi} s(\tau)d\tau} d\xi, \tag{12}$$

and

$$\psi(t) = e^{2\int_{t_1}^{t} s(\tau)d\tau}, \tag{13}$$

we can show that $\phi'(t) = a(s(t))\psi(t)$, $\psi'(t) = 2s(t)\psi(t)$, and $[r(t)]^{-2}$ is asymptotic to $2a(s(t))\phi(t)/\phi'(t)$. Integration by parts gives

$$\begin{aligned}\phi(t) &= \int_{t_1}^{t} \frac{a(s(\xi))}{2s(\xi)} d\psi(\xi)\\ &= \frac{a(s(\xi))}{2s(\xi)}\psi(\xi)\Bigg|_{\xi=t_1}^{\xi=t} - \frac{1}{2}\int_{t_1}^{t} \psi(\xi) d[a(s(\xi))/s(\xi)].\end{aligned} \tag{14}$$

The first term is dominated by $\psi(t)a(s(t))/(2s(t))$. Hence

$$\lim_{t\to+\infty} \frac{a(s(t))/s(t) - 1/r^2(t)}{a(s(t))/s(t)} = \lim_{t\to+\infty} \frac{\int_{t_1}^{t} \psi(\xi) d[a(s(\xi))/s(\xi)]}{\psi(t)a(s(t))/s(t)}. \tag{15}$$

According to l'Hospital's rule, the above limit is equal to that of

$$\frac{\psi(t)[a(s(t))/s(t)]'}{\psi(t)[a(s(t))/s(t)]' + \psi'(t)a(s(t))/s(t)} = \frac{[a(s(t))/s(t)]'}{[a(s(t))/s(t)]' + 2a(s(t))}. \tag{16}$$

Form (10) and the assumption $a(s(t))$ is bounded away from zero, we can see that the above expression approaches zero. Therefore, $r(t)$ is asymptotic to $\sqrt{s(t)/a(s(t))}$.

4. Numerical calculations with standard ODE solvers. Let us consider a simple example:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} s & -\omega \\ \omega & s \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (x_1^2 + x_2^2)\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{17}$$

which has the polar coordinate form

$$\begin{aligned}\frac{dr}{dt} &= r(s - ar^2),\\ \frac{d\theta}{dt} &= \omega,\end{aligned} \tag{18}$$

where a and ω are positive. Obviously, when $s > 0$ there is a stable limit cycle represented by $r = \sqrt{s/a}$. If s is a slowly varying parameter of the form of equation (7), and if the initial amplitude r_0 is small, then the amplitude of oscillation cannot be larger than r_0 until $t > 2\bar{t} = -2s_0/\epsilon$. However, if we use standard ODE solver, e.g., IMSL routines, we find that the result is very sensitive to the tolerance for error control in certain region. On a VAX 8500, we use the IVPRK which is an initial-value problem solver using the fifth- and six-order Runge-Kutta-Verner method. Parameters are set as $\omega = 0.06$, $a = 1.0$, $s_0 = -1.0$, $\epsilon = 0.001$, time changes from 0 to 4000 and initial conditions are $x_{10} = 0.02$, $x_{20} = 0$. If s were an ordinary parameter, orbits should approach stable limit cycles when $t > \bar{t} = 1000$. Since s is a slowly varying parameter, the oscillation with amplitude larger than 0.02 should be observed for $t > 2\bar{t} = 2000$.

However, if the tolerance is set less than or equal to 0.8×10^{-4}, no oscillation is observed (Fig. 1a). If the tolerance is set larger than or equal to 0.24×10^{-3} the large amplitude oscillation can start to be observed for t around 1750 (Fig. 1b), which is obviously inconsistent with the conclusion obtained from the exact solution of the equation.

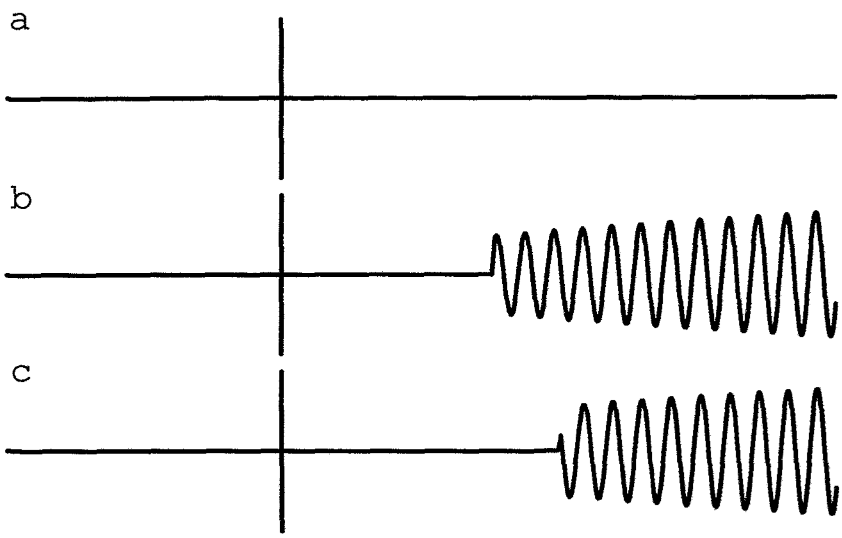

Figure 1. Numerical solution to system (17) and (7), where $a = 1$, $\omega = 0.06$, $s_0 = -1$, $\epsilon = 0.001$, $x_{10} = 0.02$, and $x_{20} = 0$. Horizontal axis represents the slowly varying parameter s (from -1 to 2) or time t (from 0 to 3000). Vertical axis represents x_1. The intersection of the vertical line and the horizontal axis (s=0) represents the Hopf bifurcation parameter value when s is not time dependent. For slowly varying s, theoretical analysis in the text shows that large amplitude oscillation should occur when $s > 1$ or $t > 2000$. (a) Use a standard ODE solver in IMSL. When the tolerance is too small, usually no significant oscillation is observed. (b) Use a standard ODE solver in IMSL. For some tolerance values, the large amplitude oscillation appears earlier than the theoretical delay time. (c) Use domain decomposition method given in the text. Delay time is consistent with theoretical value.

It is not hard to explain these results. When tolerance is too small, the intermediate result can hit the zero machine number because the origin is an asymptotically stable point for $s < 0$. Even though the origin becomes unstable for $s > 0$, if there is no further perturbation, the orbit can not leave this unstable equilibrium point. When the tolerance is too large, there will be significant intermediate errors, and they can play the roles of perturbations at the time later than the initial time 0, (with an effect of reducing the absolute value of s_0), and this can make the delay effect shorter. Thus the large amplitude oscillation appears earlier than the theoretically predicted time.

An interesting phenomenon is that when the tolerance is between 0.16×10^{-3} and 0.23×10^{-3} no large amplitude oscillations are observed (Fig. 1a); when the tolerance is between 0.9×10^{-4} and 0.15×10^{-3} the oscillation appears again around $t = 1750$.

We realize that this kind of phenomena is sensitive to the floating point system, the routines used, the tolerance settings (the above results are obtained in terms of absolute error) and some other features in the computation.

On VAX 8500, we also use the fourth-order Runge-Kutta routine ODEINT (which calls RKQC and RK4) in Numerical Recipes by Press, et al. (1986). Parameters are set the same as before. The ODEINT subroutine is called for each period of 20 time units within which the step size is adjusted by the error control tolerance. When tolerance is less than 0.61×10^{-5}, we don't see any large amplitude oscillation (Fig. 1a); when tolerance is more than 0.62×10^{-4}, we see large amplitude oscillations and the starting time ranges from 1720 to 2030. For tolerance between 0.61×10^{-5} and 0.62×10^{-4}, the asymptotic behavior of the system is quite strange. The set of tolerance values where no significant oscillations are observed is somehow like a fractal. We can find many fine structures. For example, between 0.62×10^{-5} and 0.39×10^{-4} most numerical calculations show the existence of large amplitude oscillations with starting time ranging from $t = 1720$ to $t = 2090$; however, we observe no oscillation for most tolerance values between 0.63×10^{-5} to 0.9×10^{-5}; but for 0.7×10^{-5} we observe oscillation again. We observe some other bands with no significant oscillations, they are 0.269×10^{-4} to 0.27×10^{-4}, 0.399×10^{-4} to 0.400×10^{-4}, 0.50×10^{-4} to 0.51×10^{-4}, 0.59×10^{-4} to 0.61×10^{-4}. We consider it as spurious chaos-like behavior in numerical calculations. As we run the same program on other machines, similar fine structures can be seen but at different tolerance values.

5. Domain decomposition method. In order to avoid the sensitivity of standard ODE solver on tolerance for this special kind of problems we suggest to make use of the exact solutions (4) and (5) to the normal form in the slow passage through Hopf bifurcation region and use standard ODE solver away from this region. We can integrate the numerator and denominator of (4) in terms of trapezoidal rule for t between 0 and $2\bar{t} = -2s_0/\epsilon$ and then switch to the standard ODE solver for $t > 2\bar{t}$. The integrands are very close to zero when the integral of s, (8), remains negative and away from zero, that is, when t is in a neighborhood of $\bar{t}$. Therefore we can use large steps for t close to $\bar{t}$ and use small steps for t close to 0 and $2\bar{t}$. Fig. 1c is the result obtained by using trapezoidal method for t between 0 and $2\bar{t} = 2000$ and then using routine IVPRK in IMSL. Only 11 mesh points (0, 62.5, 125, 250, 500, 1000, 1500, 1750, 1875, 1937.5 and 2000) are used for the trapezoidal method, therefore the speed is fast and there is neither spurious short delay effect nor spurious infinitely long delay effect.

6. Transformation to normal forms. In order to extend this method to other problems, we must overcome the tedious computation associated with the normal form transformation. To alleviate the tedium of hand computation, Rand and Keith (1985) and Rand and Armbruster (1987) have developed interactive programs for center manifold and normal form calculations based on Taylor series expansions using the symbolic algebraic manipulator MACSYMA. For complicated Hopf bifurcation problems such as FitzHugh-Nagumo neuron model (Baer, Erneux and Rinzel, 1989), and infectious disease model with non-bilinear transmission rate (Hethcote and Levin, 1989; Liu, Hethcote, and Levin, 1987; Liu, Levin, and Iwasa, 1986), the "usual" Taylor series expansion method even with the aid of symbolic algebra is extremely time-consuming and, furthermore, it should often be interfaced with numerical calculation. We have developed instead a matrix formulation of the normal form transformation to calculate straight forwardly the coefficients, a, b, c of (1) as well as those of the near identity transformations (Liu and Chin, 1991). This method circumvents the use of Taylor expansion for each particular problem and is ideally suited to numerical computation. The core of the calculation involves a series of matrix multiplications. In the case of a slow passage through a Hopf bifurcation, the matrix method is the method of choice as the solution of the normal form equations in polar coordinates (3) involves integrals with slowly varying integrands, for example, $\int_0^t s(\tau)d\tau$ and $\int_0^t a(s(\xi))\exp(2\int_0^\xi s(\tau)d\tau)d\xi$. They can be evaluated by quadrature methods, e.g.,

$$\int_0^t s(\tau)d\tau = \sum_{k=1}^{m} \lambda_k s(\tau_k).$$

Since $s(\tau)$ is a slowly varying function of τ, only a relatively small number of points is needed for an accurate evaluation of the integrals.

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