CHAPTER 3

Domain Decomposition Method with Nonsymmetric Interface Operator

Frédéric d'Hennezel*

Abstract. An iterative substructuring method for problems in structural analysis is presented. A new elimination procedure for the interior unknowns in each substructure is introduced resulting in a nonsymmetric interface operator; the GMRES algorithm leads to a method having the same convergence properties as the Neumann-Dirichlet algorithm as well as allowing for more general decompositions. Numerical tests performed on a CRAY 2 are presented, illustrating the efficiency of the method compared to a global Cholesky factorization or global preconditioned conjugate gradient using incomplete Cholesky factorization.

1. Introduction. We consider an elliptic boundary value problem posed on a domain Ω divided into non-overlapping subdomains. The elimination of the interior unknowns in each subdomain of Ω allows the global problem to be transformed into a problem on the interface between these subdomains. The introduction of an operator acting on the subdomains through boundary conditions on the interface —called interface operator— joined to an iterative method, allows for global problem to be solved by a sequence of independent problems on each subdomain. The choice for this interface operator determines the numerical efficiency of the resulting iterative substructuring method.

A three-dimensional linear elasticity problem is considered here. In such problems, displacement boundary conditions are most often considered on a small part Γ_0 of the boundary of the solid occupying the domain Ω. Thus, some of the subdomains might have boundaries distinct from Γ_0. We propose here a new approach for general applications which proves efficient and easy to implement in the present case. It generalizes the methods presented in Marini & Quarteroni [1988], Bramble, Pasciak & Schatz [1986], Bjorstad & Widlund [1986], all using a Neumann-Dirichlet approach. Along the interface, we consider Dirichlet boundary conditions in one subdomain and Neumann boundary conditions in the next subdomain. The difference here is that we have one "Neumann interface" *and* one "Dirichlet interface" for a same subdomain. Consequently, any local problem (i.e. on a subdomain) is well-posed, in particular when its boundary is distinct from Γ_0.

A formulation of the interface operator is given in §2 and its nonsymmetry is illustrated.

*Institut National de Recherche en Informatique et Automatique, Domaine de Voluceau, Rocquencourt, B.P. 105, 78153 Le Chesnay Cedex, France.

In §3 we show how the GMRES method is employed with an appropriate preconditioner, leading to a new Neumann-Dirichlet algorithm. Numerical tests on large three-dimensional elasticity problems will illustrate the efficiency of our approach in §4.

2.1 The global problem. The method described in this paper can be applied to any elliptic boundary value problem. In particular, let us consider three-dimensional linear elasticity problem applied to an homogeneous isotropic elastic body occupying a domain Ω.

Let Γ_0 be a part of the boundary of Ω and $\Gamma_1 = \partial\Omega \setminus \Gamma_0$. Let λ and μ be the Lamé coefficients of the elastic body; then $\mathcal{A}$, the elasticity tensor, is given by

$$\mathcal{A}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}).$$

Hereafter, $e_{ij}(v) = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$ represents the linearized strain tensor for a displacement vector field $v = (v_1, v_2, v_3)$; the linearized stress tensor is then given by Hooke's law

$$\sigma(v) = \mathcal{A} : e(v).$$

The unknown displacement field u is the solution of the following variational equation:

$$(1) \qquad \left\{ \begin{array}{l} \text{find } u \in V \\ \displaystyle\int_\Omega \sigma(u) : e(v)\, dx \;=\; \int_\Omega f.v\, dx + \int_{\Gamma_1} g.v\, d\Gamma \quad \forall v \in V, \end{array} \right.$$

where the space V is defined by

$$V = \left\{ v \in \left[H^1(\Omega)\right]^3 ; v/_{\Gamma_0} = 0 \right\}$$

and the applied forces $f \in [L^2(\Omega)]^3$ and $g \in [L^2(\Gamma_1)]^3$.

Thanks to Korn's inequality in $[H^1(\Omega)]^3$ space, combined with the assumed boundary condition of place, the bilinear form associated with equation (1) is coercive so that it has a unique solution $u \in V$.

Consider a decomposition of the domain Ω into non-overlapping subdomains; as in the case of the Neumann-Dirichlet algorithm (see Bjorstad & Widlund [1986] or Bramble, Pasciak & Schatz [1986]) we suppose that we have a red-black ordering of the subdomains of Ω; the union of the red subdomains will be denoted Ω_1 and the union of the black subdomains Ω_2. For the sake of clarity, we describe the algorithm in the case of non-crossing interfaces (i.e. two different subdomains of Ω_1 or Ω_2 do not have a common vertex or edge). We also suppose in a first step that each subdomain has a part of its boundary on Γ_0. The general case is discussed in §3.3.

The interface between the subdomains is defined by $S = \partial\Omega_1 \cap \partial\Omega_2$. As shown in Fig. 1 for the 2-D case, each connected component of S is denoted by either S_N or S_D.

Consider a conforming finite element discretization of problem (1). All the subdomains are further divided into elements. The common assumption in finite element theory that all elements are shape regular is adopted. In the resulting finite dimensional subspace V^h of V, problem (1) becomes

$$(2) \qquad \left\{ \begin{array}{l} \text{find } u^h \in V^h \\ \displaystyle\int_\Omega \sigma(u^h) : e(v^h)\, dx \;=\; \int_\Omega f.v^h\, dx + \int_{\Gamma_1} g.v^h\, d\Gamma \quad \forall v \in V^h. \end{array} \right.$$

The trace operator on γ, where γ is any part of $\partial\Omega_i$, is denoted by tr^γ. For $i = 1, 2$, let us introduce the notations

$$V_i^h = \left\{ v = w/_{\Omega_i} ; w \in V^h \right\},$$

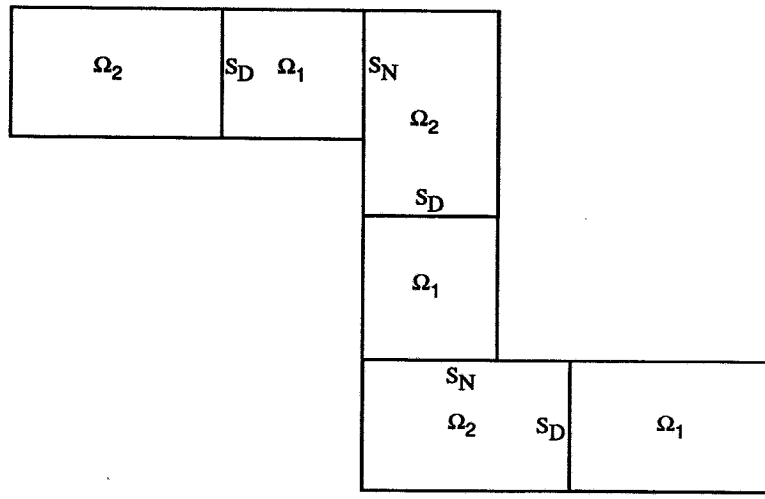

Figure 1: *Decomposition of Ω with non-crossing interfaces in 2-D.*

$$V_i^{hD} = \left\{ v \in V_i^h;\ tr^{S_D \cap \partial\Omega_i} v = 0 \right\}, \quad V_i^{hN} = \left\{ v \in V_i^h;\ tr^{S_N \cap \partial\Omega_i} v = 0 \right\}$$

and the spaces of discrete trace functions on S_D and S_N

$$V_{S_D} = \left\{ w = tr^{S_D} v;\ \ v \in V^h \right\}, V_{S_N} = \left\{ w = tr^{S_N} v;\ \ v \in V^h \right\}.$$

The basis functions of V_i^h that are non-zero on the interface S correspond to the *interface unknowns*, and the remaining ones will correspond to the *interior unknowns.*

2.2 Elimination of the interior unknowns. In order to solve problem (2) using a domain decomposition method, we first need to pose the problem on the interface S. Let $u_i^h = u^h/_{\Omega_i}$, $i = 1, 2$ be the solution restricted to Ω_i; we write

$$u_i^h = u_i^I + u_i^H, \tag{3}$$

where u_i^I is the solution of the following equation

$$\left\{ \begin{array}{l} u_i^I \in V_i^{hD} \\ \displaystyle\int_{\Omega_i} \sigma(u_i^I) : e(v)\, dx \ = \ \int_{\Omega_i} f.v\, dx + \int_{\Gamma_1 \cap \partial\Omega_i} g.v\, d\Gamma \quad \forall v \in V_i^{hD}. \end{array} \right. \tag{4}$$

We note $s^N = tr^{S_N} u_1^I - tr^{S_N} u_2^I \in V_{S_N}$ and $q^D \in V_{S_D}^*$ the linear form

$$q^D : \left\{ \begin{array}{ccl} V_{S_D} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \displaystyle -\sum_{i=1}^{2} \left[\int_{\Omega_i} \sigma(u_i^I) : e(R_i t)\, dx - \int_{\Omega} f.R_i t\, dx - \int_{\Gamma_1} g.R_i t\, d\Gamma \right] \end{array} \right.$$

where R_i is a lifting operator from V_{S_D} onto V_i^h. From the definition of u_i^I, we can see that q^D does not depend on the chosen lifting operators R_i.

It now remains to compute $u_i^H \in V_i^h$ such that $u_i^I + u_i^H$, $i = 1,2$ are the solutions of (2) on Ω_i, with

$$tr^{S_N \cup S_D}(u_1^I + u_1^H) = tr^{S_N \cup S_D}(u_2^I + u_2^H).$$

From the definitions of s^N and q^D above, it means that for all $v \in V^h$ with $v_i = v/_{\Omega_i}$, the functions $u_i^H \in V_i^h$, $i = 1,2$ have to be solutions of

$$\sum_{i=1}^{2} \int_{\Omega_i} \sigma(u_i^H) : e(v_i)\, dx = < q^D, tr^{S_D} v >_{S_D}, \tag{5}$$

with

$$\begin{cases} tr^{S_N} u_1^H &= tr^{S_N} u_2^H - s^N \\ tr^{S_D} u_1^H &= tr^{S_D} u_2^H, \end{cases} \tag{6}$$

where $< .,. >_{S_N}$, denotes the dual product between $V_{S_N}^*$ and V_{S_N}.

From (5)-(6) one can see clearly that functions u_i^H depend only on q^D and s^N which belong respectively to $V_{S_D}^*$ and V_{S_N}; we thus have an *interface problem.*

2.3 Computation of u_i^H, $i = 1,2$ as the solution of an interface operator. For any $(r,p) \in V_{S_D} \times V_{S_N}^*$, let u_i, $i = 1,2$ be the solution to the following problem:

$$\begin{cases} u_i \in V_i^h \\ \displaystyle\int_{\Omega_i} \sigma(u_i) : e(v)\, dx &= < p, tr^{S_N} v >_{S_N} \quad \forall v \in V_i^{hD} \\ tr^{S_D} u_i = r. \end{cases} \tag{7}$$

We note $s_i = tr^{S_N} u_i \in V_{S_N}$ and $q_i \in V_{S_D}^*$ the linear form

$$q_i : \begin{cases} V_{S_D} &\longrightarrow \mathbb{R} \\ t &\longmapsto \left[\displaystyle\int_{\Omega_i} \sigma(u_i) : e(R_i t)\, dx - < p, tr^{S_N} R_i t >_{S_N}\right]. \end{cases}$$

The *interface operators* A_i are then given by:

$$A_i : \begin{cases} V_{S_D} \times V_{S_N}^* &\longrightarrow V_{S_N} \times V_{S_D}^* \\ (r,p) &\longmapsto (s_i, q_i) \end{cases} \tag{8}$$

with (r,p) and (s_i,q_i) defined above.

We also need to define the following operators to take into account the change of the sign of the outer unit normal vector between the subdomains of Ω_1 and Ω_2,

$$I_{S_N} : \begin{cases} V_{S_D} \times V_{S_N}^* &\longrightarrow V_{S_D} \times V_{S_N}^* \\ (r,p) &\longmapsto (r,-p) \end{cases} \tag{9}$$

and

$$I_{S_D} : \begin{cases} V_{S_N} \times V_{S_D}^* &\longrightarrow V_{S_N} \times V_{S_D}^* \\ (r,p) &\longmapsto (r,-p). \end{cases} \tag{10}$$

From these definitions we can formulate the global problem (2) in terms of the interface unknowns:

Lemma 2.1 *u_i^H, $i = 1,2$ are the solutions of (7) if, and only if, $(r,p) \in V_{S_D} \times V_{S_N}^*$ is the solution of*

$$(A_2 - I_{S_D} \circ A_1 \circ I_{S_N})(r,p) = (s^N, q^D). \tag{11}$$

Proof. From the definitions of A_i, I_{S_D} and I_{S_N} we have

$$(A_2 - I_{S_D} \circ A_1 \circ I_{S_N})(r,p) = (s^N, q^D) \Longleftrightarrow \left\{ \begin{array}{l} s_2 - s_1 = s^N \\ q_1 + q_2 = q^D \end{array} \right.$$

so that the corresponding solutions u_i of (7) are solutions of the interface problem (5)-(6). □

2.4 Properties of the operators A_i. For all $(r,p) \in V_{S_D} \times V^*_{S_N}$ we note

$$(s_i^r, q_i^r) = A_i(r,0) \in V_{S_N} \times V^*_{S_D} \quad \text{and} \quad (s_i^p, q_i^p) = A_i(0,p) \in V_{S_N} \times V^*_{S_D}.$$

Using the above notation we can write A_i in terms of a two by two block matrix

$$\left(\begin{array}{cc} A_i^{DD} & A_i^{ND} \\ A_i^{DN} & A_i^{NN} \end{array} \right) \tag{12}$$

with

$$A_i^{DD} r = q_i^r \quad A_i^{ND} p = q_i^p \quad A_i^{DN} r = s_i^r \quad A_i^{NN} p = s_i^p.$$

Lemma 2.2 *The operator A_i defined in (8) has the following properties:*
(i) *A_i is invertible and its inverse is defined by interchanging S_D and S_N,*
(ii) ${}^t A_i^{ND} = -A_i^{DN}$,
(iii) *The symmetric part of A_i is given by*

$$S_{A_i} = \left(\begin{array}{cc} A_i^{DD} & 0 \\ 0 & A_i^{NN} \end{array} \right), \tag{13}$$

where S_{A_i} is positive definite.

proof. Let $(s_i, q_i) = A_i(r,p)$ and $\tilde{u}_i$ be the solution of the following problem

$$\left\{ \begin{array}{rcl} \tilde{u}_i \in V_i^h & & \\ \displaystyle\int_{\Omega_i} \sigma(\tilde{u}_i) : e(v)\, dx & = & < q_i, tr^{S_D} v >_{S_D} \quad \forall v \in V_i^{hN} \\ tr^{S_N} \tilde{u}_i = s_i. & & \end{array} \right.$$

Then, from the definition (8) of (s_i, q_i) and the corresponding solution u_i of (7) we have

$$\left\{ \begin{array}{rcl} \tilde{u}_i - u_i \in V_i^{hN} & & \\ \displaystyle\int_{\Omega_i} \sigma(\tilde{u}_i - u_i) : e(v)\, dx & = & 0 \quad \forall v \in V_i^{hN}, \end{array} \right.$$

and thus $\tilde{u}_i = u_i$ so that assertion (i) is proved.

In order to prove (ii), let us first define u_i^p and u_i^r which are solutions of

$$\left\{ \begin{array}{rcl} u_i^p \in V_i^{hD} & & \\ \displaystyle\int_{\Omega_i} \sigma(u_i^p) : e(v)\, dx & = & < p, tr^{S_N} v >_{S_N} \quad \forall v \in V_i^{hD}, \end{array} \right. \tag{14}$$

and

$$\left\{ \begin{array}{l} u_i^r \in V_i^h \\ \displaystyle\int_{\Omega_i} \sigma(u_i^r) : e(v)\, dx \;=\; 0 \quad \forall v \in V_i^{hD} \\ tr^{S_D} u_i^r = r. \end{array} \right. \tag{15}$$

Then we have

$$< A_i^{ND}(p), r >_{S_D} = \int_{\Omega_i} \sigma(u_i^p) : e(R_i r)\, dx - < p, tr^{S_N}(R_i r) >_{S_N} .$$

If we choose $R_i r = u_i^r$

$$< A_i^{ND}(p), r >_{S_D} = - < p, tr^{S_N}(u_i^r) >_{S_N} = - < p, A_i^{DN}(r) >_{S_N},$$

property (ii) is proved.

Assertion (iii) is a consequence of (ii), and from the fact that A_i^{DD} and A_i^{NN} are symmetric positive definite operators. Indeed, A_i^{NN} is the Poincaré-Steklov operator in terms of Agoshkov [1988], associated with S_N, and A_i^{DD} is the inverse of that same operator associated with S_D. □

Lemma 2.2 allow us to define a scalar product in $V_{S_D} \times V_{S_N}^*$ associated with the symmetric part S_{A_i} of A_i; for any (r,p) and $(s,q) \in V_{S_D} \times V_{S_N}^*$ we note

$$[(r,p),(s,q)]_i = < A_i^{DD} r, s >_{S_D} + < A_i^{NN} p, q >_{S_N} . \tag{16}$$

The following lemma generalizes the theorem proved in Bramble, Pasciak & Schatz [1986] or Bjorstad & Widlund [1986].

Lemma 2.3. *There exists two positive constants independent of h such that for all $(r,p) \in V_{S_D} \times V_{S_N}^*$*

$$C_1[(r,p);(r,p)]_2 \leq [(r,p);(r,p)]_1 \leq C_2[(r,p);(r,p)]_2. \tag{17}$$

The proof of this result uses the continuity of the trace operators on S_N and S_D and a priori estimates for the solutions of problems (14) and (15) with constants independent of h. Using the same arguments, we can show that

Lemma 2.4. *The operators A_i^{ND}, $i = 1,2$ are bounded with constants C_i independent of h for the norm associated with $[.,.]_i$*

$$\forall (r,p) \in V_{S_D} \times V_{S_N}^* \quad | < A_i^{ND}(p), r >_{S_D} |^2 \leq C_i[(r,0),(r,0)]_i[(0,p),(0,p)]_i. \tag{18}$$

3.1. Domain decomposition algorithm. In order to solve problem (11) efficiently with an iterative method, we need to define a preconditioner. Suppose that we set $S_N = \emptyset$ and $S_D = S = \partial\Omega_1 \cap \partial\Omega_2$. Then $(A_2 - I_{S_D} \circ A_1 \circ I_{S_N})$ becomes $(A_2 - I_{S_D} \circ A_1) = (A_2 + A_1)$ which is the Schur complement matrix. A preconditioned conjugate gradient algorithm can be applied to $(A_2)^{-1}(A_2 + A_1)$ and leads to the classical Neumann-Dirichlet algorithm.

In the general case we have $S_N \neq \emptyset$, but we can apply the same idea and use A_2^{-1} as a preconditioner (see lemma 2.2 (i)). Problem (11) then becomes: find $(r,p) \in V_{S_D} \times V_{S_N}^*$ solution of

$$\left[Id - A_2^{-1} \circ I_{S_D} \circ A_1 \circ I_{S_N}\right](r,p) = A_2^{-1}(s^N, q^D). \tag{19}$$

As we have a product of nonsymmetric operators, we will solve (19) using the GMRES method. This method, introduced by Saad & Schultz [1986], has been shown in practice to be powerful for a large class of problems. Using the notation $B = Id - A_2^{-1} \circ I_{S_D} \circ A_1 \circ I_{S_N}$ and $b = A_2^{-1}(s^N, q^D)$ the GMRES method for solving the linear problem $B(r,p) = b$ can be described as follows:

$[.,.]$ denotes a scalar product over $V_{S_D} \times V_{S_N}^*$ and $\|.\| = \sqrt{[.,.]}$ the corresponding norm. For an initial approximate solution $x^0 = (r^0, p^0)$ we have the initial residual $res^0 = b - B(r^0, p^0)$. At the k^{th} iteration, if x^k minimizes $\|res^0 - B(x)\|$ over the Krylov subspace

$$\mathcal{K}(x^0) = span\{x^0, Bx^0, \cdots, B^{k-1}x^0\},$$

then $x^0 + x^k$ is the k^{th} iterate with a residual $res^k = \|res^0 - Bx^k\|$. The method converges when $\|res^k\|/\|res^0\| \leq \varepsilon$ for a given precision ε.

An orthonormal basis of the Krylov subspace is constructed by an Arnoldi process, thus needing the storage of k vectors to reach the k^{th} iteration. This is a handicap for our method in comparison with symmetric formulations using preconditioned conjugate gradient methods.

However, in practice we take advantage of the fact that the unknowns are associated only with the interface. For example in a three-dimensional problem, if the global number of unknowns is proportional to N^3, then the number of unknowns associated with the interface problem is proportional to N^2. For this reason, GMRES method can be performed at a reasonable cost.

We can also point out that when a domain decomposition algorithm is applied to ill-conditioned three-dimensional elasticity problems (see for example Roux [1990] or Le Tallec, De Roeck & Vidrascu [1990]), a reorthogonalization procedure is used with conjugate gradient to ensure a faster convergence of the method. This reorthogonalization, of course, also requires the storage of a Krylov subspace basis.

3.2 Convergence. If B is positive definite with respect to $[.,.]$, according to the of Saad & Schultz [1986] theory, the rate of convergence of the GMRES method can be characterized by

$$\|res^k - B(x^k)\| \leq \left(1 - \frac{\lambda_B^2}{\Lambda_B^2}\right)^{k/2} \|res^0 - B(x^0)\|, \quad \forall k \leq 1,$$

where

$$\lambda_B = \inf_{u=(r,p)\neq 0} \frac{[Bu,u]}{[u,u]} \text{ and } \Lambda_B = \sup_{u=(r,p)\neq 0} \frac{\|Bu\|}{\|u\|}.$$

This convergence result does not seem easy to employ in the general case. But if each subdomain of Ω_2 has, either its boundary distinct of S_D, or distinct of S_N, we have $A_2^{ND} = 0$. In this particular case $A_2 = S_{A_2}$. If we consider the GMRES method with respect to the scalar product $[.,.]_2$ defined in (16), from lemma 2.3 and 2.4 we can show that λ_B and Λ_B are positive constants independent of h.

In the general case, we could use S_{A_2} as preconditioner for problem (11), and it would also be possible to show the existence of λ_B and Λ_B using the same technique. However, it leads to another algorithm with the additional computation of $A_2(r,p)$ for a given $(r,p) \in V_{S_D} \times V_{S_N}^*$ at each iteration, and is therefore not interesting in practice.

3.3 The use of the nonsymmetric formulation with general decompositions. For the case where only one or few of the subdomains have a part of their boundaries on Γ_0, it is not possible to use the classical Neumann-Dirichlet algorithm. In our appraoch, if the interfaces S_N and S_D are chosen carefully, the computation of $B(r,p)$, for a given (r,p), at each iteration of the Arnoldi process is possible. Indeed, if we suppose that each subdomain of Ω_1, whose boundary is distinct from Γ_0, has a part of its boundary on S_D the problem corresponding to the computation of $(s,q) = A_1 \circ I_{S_N}(r,p)$ is coercive due to the assumed boundary conditions of place. In the same way, if each subdomain of Ω_2, whose boundary is distinct from Γ_0, has a part of its boundary on S_N, the problem corresponding to the computation of $A_2^{-1} \circ I_{S_D}(s,q)$ is also coercive (see lemma 2.2 (i)). Then, for a given decomposition the user has to define a "coloring" of the interface S in S_D and S_N.

For the theoretical aspects of the method in the general case, it is possible to set the operator A_i on the quotient spaces $V_\gamma/R(\gamma)$ ($\gamma = S_D$ or S_N), where $R(\gamma)$ is the space of rigid displacement on γ. In these spaces, lemmas 2.1 through 2.4 remain valid (see D'Hennezel [1991]). As a Krylov subspace method is used, the algorithm converges in the image space of B that contains the domain of definition of A_2.

It is possible to formulate the same method in the case of interior subdomains and crossing interfaces. The remaining preconditioner is still local (i.e. it only exchanges information between neighbouring subdomains). The idea in this case is to have two unknowns in the interface problem for each degree of freedom corresponding to a mesh node that lies on an edge or vertex of a subdomain; one unknown is associated with V_{S_D} and the other with V_{S_N} (see D'Hennezel [1991]). It leads to an easy-to-implement algorithm and gives good results as shown by the following numerical tests.

4. Numerical tests. The numerical implementation of the algorithm presented in §3 was done within the Modulef finite element library (cf. Bernadou et al. [1985]), in multiproblem and multielement framework. The local problems corresponding to boundary value problems (7) are solved with a Cholesky method. The following tests are performed on a CRAY 2 computer without multitasking.

Test 1. Let us consider the three-dimensional elasticity problem described in §2 corresponding to the geometry in Fig. 2. The domain is divided into 16 subdomains. One extremity of the beam remains fixed, the other extremity is submitted to the load $g = (0, g_y, 0)$, where g_y is constant (the y axis being orthogonal to the axis of the beam). Young's modulus $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ and Poisson's ratio $\nu = \frac{\lambda}{2(\lambda+\mu)}$ are set to $E = 2.10^6$ et $\nu = 0.3$.

On Ω the space V^h is composed of Q2 hexahedra . The finite element mesh contains 108 elements and 679 nodes in each subdomain (2037 degrees of freedom). There are 5985 degrees of freedom on the interface.

Nonsymmetric domain decomposition method.
Number of iterations (residual $< 10^{-6}$): 47
CPU time for the factorization of the 16 matrices: 58 s
Storage of the 16 matrices: 8,635,856 words
CPU time for the GMRES method: 32 s.

Global Cholesky method.
CPU time for the factorization: 129 s.
Matrix storage: 14,568,077 words

For this decomposition neither the classical Neumann-Dirichlet method or a global preconditioner are convenient. On the other hand, we could have used the algorithm introduced by Bourgat, Glowinski, Le Tallec & Vidrascu [1988]. It has the advantage of being an al-

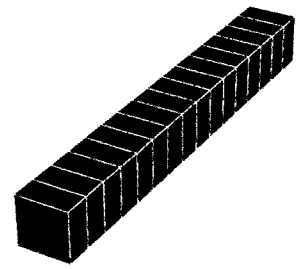

Figure 2: *The domain of test 1 its decomposition.*

gorithm of general application; it does not require a red-black ordering of the subdomains (that is of course not always possible), neither a choice of interfaces S_N and S_D. However, they need to solve two different boundary value problems on each subdomain at every iteration; one with Dirichlet boundary conditions and then the other corresponding to the preconditioning step with Neumann boundary conditions. Thus, there are two matrices to store and factorize in each subdomain.

Test 2. The three-dimensional elasticity problem is now applied to a domain Ω divided into 32 subdomains with crossing interfaces, as shown in Fig. 3. Here again, one extremity of the beam remains fixed. The loading is achieved with $f = (0, f_y, 0)$ and f_y constant; we keep $E = 2.10^6$ and $\nu = 0.3$.

In Ω, the space V^h is composed of Q2 hexahedra. The finite element mesh contains 125 elements and 756 nodes in each subdomain (2268 degrees of freedom). There are 14820 degrees of freedom on the interface.

Nonsymmetric domain decomposition method.

Number of iterations (residual $< 10^{-6}$): 137

CPU time for the Cholesky factorization of the 32 matrices: 190 s

Storage of the 32 matrices: 25,095,808 words

CPU time for the GMRES method: 217 s.

Global Cholesky method.

CPU time for a global Cholesky factorization: 1059 s.

Matrix storage: 75,645,101 words

Preconditioned conjugate gradient (ICCG).

Number of iterations (residual $< 10^{-6}$): 138

CPU time (incomplete factorization + iterations): 720 s

Storage of the global matrix and of its preconditioner: 17,844,000 words.

Here the number of iterations for the GMRES method is high, but the corresponding CPU times are still good. This implies that for three-dimensional problems, giving rise

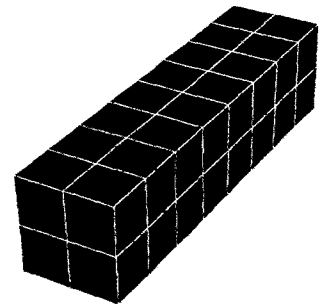

Figure 3: *The domain of test 2 its decomposition.*

to matrices with large band-widths, our method which uses a local preconditioner can be efficient with a relatively large number of subdomain. For a given global problem, the number of subdomains has to be chosen so that the number of unknowns on the interface remains small in comparison with the total number of degrees of freedom. In this test, the storage of Krylov subspace basis for the GMRES method requires 137 iterations $\times$ 14,820 unknowns = 2,030,340 words.

5. Conclusion. The efficiency of a domain decomposition method relies on both the numerical efficiency of the proposed algorithm and on its parallel implementation. Though tests 1 and 2 are only sequential, they show that a careful parallel implementation should lead to a powerful method on a multiprocessor machine.

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