CHAPTER 20

Domain Decomposition and PDE with Periodic Boundary Conditions

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Abstract: Problems with periodic boundary conditions arise naturally whenever we deal with repeating structures. At the same time, periodic boundary conditions yield some complications for standard solvers. The reason is that the bandwidth of the stiffness matrix of the system with enhanced boundary conditions may increase two times as compared with that without boundary conditions.

Using a geometrical approach developed earlier, we describe a variant of domain decomposition algorithm for solution of such problems. To assess the performance of the algorithm, we report our results of model problem analysis and of the solution of a mining engineering problem.

1. Introduction. Though most of the current research in the Domain Decomposition Methods has been motivated by the development of parallel computing, it seems that the method offers also some alternative to common sequential algorithms for the solution of systems of linear equations arising from the discretization of partial differential equations.

In this paper, we show that even the basic Schur complement algorithm may be more efficient than the standard elimination in the solution of problems with periodic boundary conditions. In particular, it turns out that the idea to reduce iterations to some interface is useful also in our case, where only one region is actually present in the computation. However, we belive that we are still entitled to speak about domain decomposition as an infinite number of regions is involved in formulation of such problems. To give a simple geometric insight into the algorithm, we describe the algorithm in terms of preconditioning by conjugate projector [1]. The efficiency is studied on a model problem and on the solution of a mining engineering problem.

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2. Algorithm. Suppose we are given a system of n linear equations

$$A_0 x = b \tag{1}$$

arising from the discretization of a system of elliptic partial differential equations in bounded region Ω in R^S with the boundary $\delta\Omega$, possibly with enhanced Dirichlet boundary conditions. We are to find the solution of (1) which satisfies

$$x_i = x_{m+i}, \quad i = 1, \cdots, p, \qquad p = n - m. \tag{2}$$

The unknows involved in (2) may correspond to nodes on $\delta\Omega$ which are affected by the periodic boundary conditions.

For any k natural, let I_k denote the identity matrix of the order k. Consider a decomposition of the set of indices into three sets $\{1, \cdots, p\}, \{p+1, \cdots, m\}$ and $\{m+1, \cdots, n\}$, which induces a block structure on $A_0, A_0 = (A_{ij}), \quad i, j = 1, 2, 3.$ Observe that the solution of (1, 2) is given by $x = U_0 y$, where

$$U_0 = \begin{pmatrix} I_p & O \\ O & I_{m-p} \\ I_p & O \end{pmatrix},$$

and y satisfies the system

$$U_0^T A_0 U_0 y = U_0^T b$$

whose matrix generally does not have the band structure of A_0. For simplicity, let us suppose that A_{22} is invetible and put $A = U_0^T A_0 U_0, b = U_0^T b_0$.

A: *Definition of Auxiliary Subspace and Initial Correction*

(i) $U = \begin{pmatrix} O \\ I_{m-p} \end{pmatrix}$, where O is the $p \times (m-p)$ zero matrix.

(ii) Find the factorization
$LL^T = U^T U_0^T A_0 U_0 U = U^T A U = A_{22}.$

(iii) $y_0 = UL^{-T}L^{-1}U^T b.$

Notice that $P = UL^{-T}L^{-1}U^T A$ is an A - conjugate projector and that $y_0 = PA^{-1}b$. For convenience, put $Q = I - P$ and denote by $\mathcal{V}$ the range of Q.

B: *Iterations*

(iv) $i = 0, \ \ p_0 = r_0 = b - Ay_0.$

(v) If $r_i = 0$, then put $y = y_i$ and stop.

(vi) $q_i = Qp_i.$

(vii) $\alpha_i = r_i^T q_i / q_i^T A q_i.$

(viii) $y_{i+1} = y_i + \alpha_i q_i.$

(ix) $r_{i+1} = r_i - \alpha_i A q_i \quad (= b - Ay_{i+1}).$

(x) $\beta_i = r_{i+1}Aq_i/q_i^T Aq_i$.

(xi) $p_{i+1} = r_{i+1} - \beta_i p_i$.

(xii) Put $i = i + 1$ and return to (v).

Notice that the iterations generate the same residuals as the conjugate gradient method for the solution of system $AQy = r_0$ with $y_0 = 0$, where only the positive definite restriction $AQ \mid A\mathcal{V}$ of the symmetric matrix AQ (considered here as a linear mapping) to $A\mathcal{V}$ takes part in the process of solution. Hence the error of iterations may be estimated by the spectral condition number $\kappa(AQ \mid A\mathcal{V})$ of $AQ \mid A\mathcal{V}$. Some other results concerning this algorithm may be found in [1].

3. Performance of the Algorithm. Consider the problem

$$-\Delta v = b \text{ in } \Omega, \tag{3}$$

$$v = 0 \text{ on } \Gamma_o, \tag{4}$$

$$v(x,0) = v(x,1) \text{ on } \Gamma_p, \tag{5}$$

where $\Omega = (0,1) \times (0,1), \Gamma_0 = (0,1) \times (0,1)$, and $\Gamma_p = (0,1) \times (0,1)$.

First we define a square grid for discretization of (3). Put $h = 1/p, p$ a natural number, $x_i = ih, y_i = ih, i = 0, 1, \cdots, p$. We shall use a horizontal meshline ordering.

The central difference approximation with enhancing of (4) yields the system $A_0 x = b_0$, where

$$A_0 = h^{-2}\begin{pmatrix} B-I & E & O \\ E^T & C & D \\ O & D^T & B-I \end{pmatrix}, \quad C = \text{tridiag}(-I_{p-1}, B, -I_{p-1}),$$

$$D^T = (0, \cdots, 0, -I_{p-1}), \qquad E = (-I_{p-1}, 0, \cdots, 0),$$
$$B = \text{tridiag}(-1, 4, -1).$$

The matrix B is of the order $p-1$; the matrices C, D, E have $p-1$ block rows, each block with $p-1$ rows.

Arising problem may be analysed in detail. In particular, it may be shown that the Schur complement involved in the iterations of the previous section is given by

$$\hat{A} = 2B - 2I - (E^T + D)C^{-1}(E + D^T)$$

and that

$$\kappa(\hat{A}) \leq 1.55h^{-1}.$$

Details may be found in [2].

Now we are able to assess the performance of the method. In particular, for $p = 100$ we get $\kappa(A) \leq 155$, which, substituted into the simplified Chebyschev error bound for the eucliden norm of the residua

$$\mid r_i \mid \leq 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^i \mid r_0 \mid, \tag{6}$$

implies that for reduction of $| r_0 |$ by the factor of 10^{-3} one does not need more than 48 steps of the above algorithm. In this case the execution of the step A requires less then $100^2 \times 100^2 = 10^8$ flops for the factorization and about $100^2 \times 2 \times 100 = 2 \times 10^6$ flops for the initial correction while 48 steps of iterations of the part B of the algorithm do not require more then $48 \times 100^2 \times 220 \cong 10^8$ flops. The finite solution of the problem requires about 4×10^8 flops, two times more than the presented algorithm. Our experience shows even better results, probably because the bound (6) is a bit pessimistic and does not reflect a nice self–accelerating property of the conjugate gradient algorithm. We have tested the convergence of the algorithm also on a mining engineering problem – cavern in a field of caverns of Figure 1 discretized by irregular 30×30 grid. Only own weight has been considered. The number of iterations to reduce the residuum by 10^{-3} was equal to 36, which required about the same time as initial decomposition. The solution required about half the time required by the standard solver.

We conclude that there are problems for which the algorithm presented is competetive.

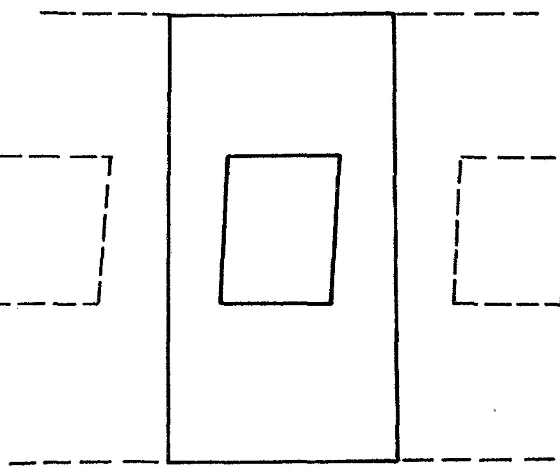

Figure 1. A cavern in a field of caverns.

REFERENCES

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