CHAPTER 9

Finite Element Matching Methods*

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Abstract
We present here nonstandard finite element methods for solving an elliptic problem. The methods are nonstandard in the sense that they can use different finite element approximations in subdomains; the matching conditions on the interfaces are obtained with two Lagrange multipliers and a mortar element. A nonoverlapping Schwarz alternating method is then described.

1. Introduction.
Let $\Omega$ be a bounded open set in $\mathbb{R}^N$ $(N \leq 3)$. We consider a domain decomposition of $\Omega$ into an arbitrary number $E$ of subdomains:

\begin{equation}
\Omega = \bigcup_{e=1}^{E} \overline{\Omega}_e,
\end{equation}

where the $\Omega_e$ are disjoint connected open sets in $\mathbb{R}^N$ with piecewise smooth boundary $\partial\Omega_e$. So the domain decomposition is without overlapping. Let $\Gamma$ be the set of the internal boundaries:

\begin{equation}
\Gamma = \bigcup_{e=1}^{E} \Gamma_e,
\end{equation}

with

\begin{equation}
\Gamma_e = \partial\Omega_e \setminus (\partial\Omega_e \cap \partial\Omega), \quad \text{for all } e, \ 1 \leq e \leq E.
\end{equation}

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For $1 \leq e \neq d \leq E$, we shall introduce the notation:

$$\Gamma_{ed} = \partial \Omega_e \cap \partial \Omega_d = \Gamma_{de}$$

and $n_{ed}$ shall be unit outward normal to $\partial \Omega_e$ on $\Gamma_{ed}$: $n_{ed} = -n_{de}$. Then $\partial u / \partial n_{ed}$ is the $\Omega_e$-outward normal derivative:

$$\frac{\partial u}{\partial n_{ed}} = \text{grad } u \cdot n_{ed} = -\text{grad } u \cdot n_{de} = -\frac{\partial u}{\partial n_{de}} \quad \text{on } \Gamma_{ed} = \Gamma_{de},$$

if $u$ is sufficiently smooth in $\Omega_e \cup \Omega_d$.

The presentation of the methods is made on the model problem:

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,$$

where $f$ is given in $L^2(\Omega)$; so the unique solution of (5) belongs to the Sobolev space $H^1_0(\Omega)$.

Some approximation methods of the solution $u$ of (5) will be given with finite element approximations which are made independently on the interior of each subdomain $\Omega_e$. The matching is obtained with the use of two Lagrange multipliers on each interface $\Gamma_{ed}$ and of a mortar element. For solving the discrete problem, we shall study the possibility of developing a nonoverlapping Schwarz alternating method. So in Section 2, about the continuous model problem we introduce a nonoverlapping Schwarz alternating method analyzed by P.L. Lions. Then we recall the Glowinski-Le Tallec's interpretation of this method as a classical saddle-point algorithm. The numerical analysis of the similar saddle-point problem in the finite-dimensional case is done in Section 3. The corresponding saddle-point algorithm is given in Section 4.


For the model problem (5), a nonoverlapping Schwarz alternating method is to solve for $m \geq 1$:

$$-\Delta u^m_e = f \quad \text{in } \Omega_e, \quad \frac{\partial u^m_e}{\partial n_{ed}} + \rho u^m_e = -\frac{\partial u^{m-1}_d}{\partial n_{de}} + \rho u^{m-1}_d \quad \text{on } \Gamma_{ed} (e \neq d), \quad u^m_e = 0 \quad \text{on } \partial \Omega_e \cap \partial \Omega.$$

So at each iteration, on each interface $\Gamma_{ed} = \Gamma_{de}$ we have two Robin-Fourier's conditions: one for the solution of the boundary problem on $\Omega_e$ and one other for the solution of the boundary problem on $\Omega_d$. When the sequence $u^m$ is convergent in a convenient sense, this gives at the limit the two classical matching conditions: continuity of $u$ and continuity of the normal derivative $\partial u / \partial n$ at the interfaces.
The number $\rho$ is assumed to be a strictly positive parameter; in a little more
general context, P.L. Lions has proved ([8], Theorem 1, p. 208) that the sequence
$u^m_e$ converges weakly to $u|_{\Omega_e}$ in $H^1(\Omega_e)$ for all $e$, $1 \leq e \leq E$.

By a mathematical programming approach, Glowinski and Le Tallec have given
an interpretation of the above method as a classical saddle-point algorithm, (cf. [7]).
More precisely, they have obtained the same nonoverlapping Schwarz alternating
method in an augmented Lagrangian framework, [5], [6], with the following aug-
mented Lagrangian $L$:

\begin{equation}
L(v_e, q; \mu_e) = \sum_{e=1}^{E} \int_{\Omega_e} \left( \frac{1}{2} |\text{grad } v_e|^2 - f v_e \right) \, dx
- \sum_{e=1}^{E} \int_{\Gamma_e} \mu_e (v_e - q) \, d\gamma + \frac{\rho}{2} \sum_{e=1}^{E} \int_{\Gamma_e} |v_e - q|^2 \, d\gamma.
\end{equation}

The saddle-point solution is $u_e =$ restriction of $u$ to $\Omega_e$, $p =$ trace of $u$ on $\Gamma$,
$\lambda_e = \partial u / \partial n_e$ on $\Gamma_e$; in particular $\lambda_e + \lambda_d = 0$ on $\Gamma_{ed}$.

This approach with no a priori imposed matching condition has been found
more convenient for the generalization of this kind of method in finite-dimensional
approximations.

3. Nonstandard finite element methods.

For all $e = 1, \ldots, E$ let

\begin{equation}
X_{he} \text{ be a finite-dimensional subspace of } \{ v_e \in H^1(\Omega_e); \ v_e = 0 \text{ on } \partial \Omega \cap \partial \Omega_e \}
\end{equation}

and

\begin{equation}
\Lambda_{he} \text{ be a finite-dimensional subspace of } L^2(\Gamma_e).
\end{equation}

In practice the spaces $X_{he}$ are obtained using a conforming finite element approx-
imation of $H^1(\Omega_e)$, independently for each subdomain $\Omega_e$. The spaces $\Lambda_{he}$ are
obtained using a finite element approximation of $L^2(\Gamma_{ed})$. We denote $X_h$ and $\Lambda_h$
the product spaces:

\begin{equation}
X_h = \prod_{e=1}^{E} X_{he}
\end{equation}

and

\begin{equation}
\Lambda_h = \prod_{e=1}^{E} \Lambda_{he}.
\end{equation}
Moreover let
\begin{equation}
Y_h \text{ be a finite-dimensional subspace of } H^{1/2}_{00}(\Gamma),
\end{equation}
where $H^{1/2}_{00}(\Gamma)$ is the subspace of $L^2(\Gamma)$ constituted by traces on $\Gamma = \bigcup_{e=1}^{E} \Gamma_e$ of the functions which belong to $H^1_0(\Omega)$. Finally we introduce a subspace $W_h$ of $X_h \times Y_h$ and a subspace $M_h$ of $\Lambda_h$; the choices $W_h = X_h \times Y_h$ and/or $M_h = \Lambda_h$ should not be excluded.

Then we search saddle-points on $W_h \times M_h$ for the augmented Lagrangian $L$. With the definition (7) for $L$, it is straightforward to see that $(u_{he}, p_h; \lambda_{he})$ is a saddle-point of $L$ if and only if $w_h = (u_{he}, p_h)$ and $\lambda_h = (\lambda_{he})$ are solution of:
\begin{equation}
w_h \in W_h, \quad \lambda_h \in M_h,
\end{equation}
\begin{equation}
\forall z_h \in W_h, \quad a(w_h, z_h) + b(z_h, \lambda_h) = \int_{\Omega} f z_h \, dx,
\end{equation}
\begin{equation}
\forall \mu_h \in M_h, \quad b(w_h, \mu_h) = 0
\end{equation}
with
\begin{equation}
a(w_h, z_h) = \sum_{e=1}^{E} \int_{\Gamma_e} \text{grad } u_{he} \cdot \text{grad } v_{he} \, dx + \rho \sum_{e=1}^{E} \int_{\Gamma_e} (u_{he} - p_h)(v_{he} - q_h) \, d\gamma
\end{equation}
and
\begin{equation}
b(z_h, \lambda_h) = -\sum_{e=1}^{E} \int_{\Gamma_e} \lambda_{he}(v_{he} - q_h) \, d\gamma,
\end{equation}
for $w_h = (u_{he}, p_h)$, $z_h = (v_{he}, q_h)$, $\lambda_h = (\lambda_{he})$. Using the Babuška-Brezzi's theory (cf. for example [10]), it is now well known that the problem (13) has a unique solution if and only if the two following assumptions are satisfied:
\begin{equation}
\inf_{z_h \in V_h \setminus \{0\}} \sup_{w_h \in V_h} a(z_h, w_h) > 0,
\end{equation}
where the subspace $V_h$ is defined as the kernel of the bilinear form $b(\cdot, \cdot)$:
\begin{equation}
V_h = \{ z_h \in W_h; \forall \mu_h \in M_h, \quad b(z_h, \mu_h) = 0 \}
\end{equation}
and
\begin{equation}
\inf_{\mu_h \in M_h \setminus \{0\}} \sup_{z_h \in V_h} b(z_h, \mu_h) > 0.
\end{equation}
Clearly if \( a(z_h, z_h) = 0 \) for some \( z_h = (v_{he}, q_h) \) in \( W_h \), then \( v_{he} \) is constant on \( \Omega_e \), \( v_{he} = 0 \) on \( \partial \Omega_e \cap \partial \Omega \) and \( q_h = v_{he} \) on \( \Gamma_e \) for all \( e \), \( 1 \leq e \leq E \). Then the bilinear form \( a(\cdot, \cdot) \) is \( W_h \)-elliptic and so (16) is always satisfied.

On the other hand (18) is satisfied if and only if

\[
\forall \mu_h \in M_h \text{ with } \mu_h \neq 0, \exists z_h = (v_{he}, q_h) \in W_h \text{ such that }
\sum_{e=1}^{E} \int_{\Gamma_e} \mu_{he}(v_{he} - q_h) \, d\gamma \neq 0.
\]

We give some applications of the above; for the sake of simplicity only the case of plane geometry (\( N = 2 \)) will be treated hereafter.

**Example 1.** We assume that \( W_h = X_h \times Y_h \).

Then (19) is satisfied as soon as we have for all \( e \), \( 1 \leq e \leq E \),

\[
\forall \mu_{he} \in \Lambda_{he} \text{ with } \mu_{he} \neq 0, \exists v_{he} \in X_{he} \text{ such that } \int_{\Gamma_e} \mu_{he} v_{he} \, d\gamma \neq 0.
\]

When \( X_{he} \) is a space of continuous, piecewise polynomial functions on \( \Omega_e \), vanishing on \( \Gamma_e \cap \partial \Omega \), (conforming finite element on \( \Omega_e \)), the analysis of (20) is what is made in hybrid finite element method when a Dirichlet condition is dualized with the use of a boundary Lagrangian multiplier (cf. [2], [3], [10]).

**Example 2.** For a given integer \( k \), let

\[
Y_h = \{ q_h \in C^0(\Gamma); q_h|_{\Gamma_{ed}} \in P_k(\Gamma_{ed}), 1 \leq e \neq d \leq E \}.
\]

Let

\[
\Lambda_h = \prod_{e=1}^{E} \Lambda_{he}, \quad \text{with } \Lambda_{he} = \prod_{d \neq e} P_{k-2}(\Gamma_{ed}),
\]

(each \( \Gamma_{ed} \) is assumed to be a straight line; a function \( \mu_{he} \in \Lambda_{he} \) is not defined at the vertices of \( \partial \Omega_e \)).

Take for a subspace \( M_h \)

\[
M_h = \{ \mu_h \in \Lambda_h; \text{ if } e < d, \mu_{hd} = 0 \text{ on } \Gamma_{ed} \}.
\]

Take for a subspace \( W_h \)

\[
W_h = \{ w_h = (v_{he}, q_h) \in X_h \times Y_h; v_{he}(a) = q_h(a) = v_{hd}(a) \text{ for each vertex } a \in \Gamma_{ed}; \text{ if } e < d, \text{ } v_{hd} = q_h \text{ on } \Gamma_{ed} \}.
\]

Let us assume that

\[
\forall t \in P_k(\Gamma_{ed}) \cap H^1_0(\Gamma_{ed}), \exists v_{he} \in X_{he} \text{ such that } v_{he} = t \text{ on } \Gamma_{ed} \text{ for all } e < d.
\]
So (19) is satisfied. We recovered here a variant of the mortar finite element method (cf. [9], [11]).


When the space \( W_h \) is the product space \( X_h \times Y_h \) as in Example 1., an algorithm for solving the saddle-point of \( \mathcal{L} \) over \( W_h \times M_h \) is:

Given \( p_h^0 \) and \( \lambda_h^0 \),
then for \( m \geq 0 \), \( p_h^m \) and \( \lambda_h^m = (\lambda_{he}^m) \) known; solve successively

1° \( u_h^{m+\frac{1}{2}} \in X_h \), for all \( v_h \in X_h \)

\[
\sum_{e=1}^{E} \left\{ \int_{\Gamma_e^h} \nabla u_{he}^{m+\frac{1}{2}} \cdot \nabla v_{he} \, dx + \rho \int_{\Gamma_e^h} u_{he}^{m+\frac{1}{2}} v_{he} \, d\gamma \right\} = \\
= \sum_{e=1}^{E} \left\{ \int_{\Gamma_e} f u_{he} \, dx + \int_{\Gamma_e^h} (\lambda_{he}^m + \rho p_h^m) v_{he} \, d\gamma \right\},
\]

2° \( \lambda_h^{m+\frac{1}{2}} \in M_h \), for all \( \mu_h \in M_h \)

\[
\sum_{e=1}^{E} \int_{\Gamma_e} \lambda_{he}^{m+\frac{1}{2}} \mu_{he} \, d\gamma = \sum_{e=1}^{E} \int_{\Gamma_e^h} \left\{ \lambda_{he}^m \mu_{he} + \rho (u_{he}^{m+\frac{1}{2}} - p_h^m) \mu_{he} \right\} \, d\gamma,
\]

3° \( p_h^{m+1} \in Y_h \), for all \( q_h \in Y_h \)

\[
\int_{\Gamma} p_h^{m+1} q_h \, d\gamma = \sum_{e=1}^{E} \int_{\Gamma_e} (u_{he}^{m+\frac{1}{2}} + \frac{1}{\rho} \lambda_{he}^{m+\frac{1}{2}}) q_h \, d\gamma,
\]

4° \( \lambda_h^{m+1} \in M_h \), for all \( \mu_h \in M_h \)

\[
\sum_{e=1}^{E} \int_{\Gamma_e} \lambda_{he}^{m+\frac{1}{2}} \mu_{he} \, d\gamma = \sum_{e=1}^{E} \int_{\Gamma_e^h} \left\{ \lambda_{he}^{m+\frac{1}{2}} \mu_{he} + \rho (u_{he}^{m+\frac{1}{2}} - p_h^{m+1}) \mu_{he} \right\} \, d\gamma.
\]

The step 1° may be computed in parallel. In fact, we are solving in this step the finite element approximation of \( E \) independent Fourier-Robin's problems. Moreover when \( M_h = \Lambda_h \), the steps 2° and 4° are also parallelizable. But the step 3° may not be reduced to parallel computations on the \( \Gamma_e \). It is the synchronization step in the algorithm.

Remark. We cannot develop such algorithm when the space \( W_h \) is a proper subspace of \( X_h \times Y_h \) as in Example 2. In the situation of this Example 2., it is convenient to eliminate directly the Lagrange multipliers and then to solve the elliptic problem in \( V_h \) by a preconditioned conjugate gradient method. It can be computed parallely with the method of Bramble et al., [4].
REFERENCES


