Finite Element Matching Methods*

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Abstract

We present here nonstandard finite element methods for solving an elliptic problem. The methods are nonstandard in the sense that they can use different finite element approximations in subdomains; the matching conditions on the interfaces are obtained with two Lagrange multipliers and a mortar element. A nonoverlapping Schwarz alternating method is then described.

1. Introduction.

Let Ω be a bounded open set in R^N ($N \leq 3$). We consider a domain decomposition of Ω into an arbitrary number E of subdomains:

$$\overline{\Omega} = \bigcup_{e=1}^{E} \overline{\Omega}_e, \tag{1}$$

where the Ω_e are disjoint connected open sets in R^N with piecewise smooth boundary $\partial\Omega_e$. So the domain decomposition is without overlapping. Let Γ be the set of the internal boundaries:

$$\Gamma = \bigcup_{e=1}^{E} \Gamma_e, \tag{2}$$

with

$$\Gamma_e = \partial\Omega_e \setminus \big(\partial\Omega_e \cap \partial\Omega\big), \qquad \text{for all } e,\ 1 \leq e \leq E. \tag{3}$$

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For $1 \leq e \neq d \leq E$, we shall introduce the notation:

$$\Gamma_{ed} = \partial\Omega_e \cap \partial\Omega_d = \Gamma_{de} \tag{4}$$

and $\mathbf{n}_{ed}$ shall be unit outward normal to $\partial\Omega_e$ on Γ_{ed} : $\mathbf{n}_{ed} = -\ \mathbf{n}_{de}$. Then $\partial u/\partial n_{ed}$ is the Ω_e-outward normal derivative:

$$\frac{\partial u}{\partial n_{ed}} = \mathbf{grad}\ u \cdot \mathbf{n}_{ed} = -\mathbf{grad}\ u \cdot \mathbf{n}_{de} = -\frac{\partial u}{\partial n_{de}} \qquad \text{on } \Gamma_{ed} = \Gamma_{de},$$

if u is sufficiently smooth in $\overline{\Omega}_e \cup \overline{\Omega}_d$.

The presentation of the methods is made on the model problem:

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{5}$$

where f is given in $L^2(\Omega)$; so the unique solution of (5) belongs to the Sobolev space $H_0^1(\Omega)$.

Some approximation methods of the solution u of (5) will be given with finite element approximations which are made independently on the interior of each subdomain Ω_e. The matching is obtain with the use of two Lagrange multipliers on each interface Γ_{ed} and of a mortar element. For solving the discrete problem, we shall study the possibility of developing a nonoverlapping Schwarz alternating method. So in Section **2**, about the continuous model problem we introduce a nonoverlapping Schwarz alternating method analyzed by P.L. Lions. Then we recall the Glowinski-Le Tallec's interpretation of this method as a classical saddle-point algorithm. The numerical analysis of the similar saddle-point problem in the finite-dimensional case is done in Section **3**. The corresponding saddle-point algorithm is given in Section **4**.

2. Nonoverlapping Schwarz alternating method : continuous case.

For the model problem (5), a nonoverlapping Schwarz alternating method is to solve for $m \geq 1$:

$$\begin{aligned} -\Delta u_e^m &= f \qquad \text{in } \Omega_e, \\ \frac{\partial u_e^m}{\partial n_{ed}} + \rho u_e^m &= -\frac{\partial u_d^{m-1}}{\partial n_{de}} + \rho u_d^{m-1} \qquad \text{on } \Gamma_{ed}\ (e \neq d), \\ u_e^m &= 0 \qquad \text{on } \partial\Omega_e \cap \partial\Omega. \end{aligned} \tag{6}$$

So at each iteration, on each interface $\Gamma_{ed} = \Gamma_{de}$ we have two Robin-Fourier's conditions: one for the solution of the boundary problem

on Ω_e and one other for the solution of the boundary problem on Ω_d. When the sequence u^m is convergent in a convenient sense, this gives at the limit the two classical matching conditions: continuity of u and continuity of the normal derivative $\partial u/\partial n$ at the interfaces.

The number ρ is assumed to be a strictly positive parameter; in a little more general context, P.L. Lions has proved ([8], Theorem 1, p. 208) that the sequence u_e^m converges weakly to $u|_{\Omega_e}$ in $H^1(\Omega_e)$ for all e, $1 \leq e \leq E$.

By a mathematical programming approach, Glowinski and Le Tallec have given an interpretation of the above method as a classical saddle-point algorithm, (cf. [7]). More precisely, they have obtained the same nonoverlapping Schwarz alternating method in an augmented Lagrangian framework, [5], [6], with the following augmented Lagrangian $\mathcal{L}$:

$$\mathcal{L}(v_e, q; \mu_e) = \sum_{e=1}^{E} \int_{\Omega_e} \left(\frac{1}{2}|\textbf{grad}\ v_e|^2 - f v_e\right) dx - \sum_{e=1}^{E} \int_{\Gamma_e} \mu_e (v_e - q)\, d\gamma + \frac{\rho}{2} \sum_{e=1}^{E} \int_{\Gamma_e} |v_e - q|^2\, d\gamma. \tag{7}$$

The saddle-point solution is u_e = restriction of u to Ω_e, p = trace of u on Γ, $\lambda_e = \partial u / \partial n_e$ on Γ_e; in particular $\lambda_e + \lambda_d = 0$ on Γ_{ed}.

This approach with no a priori imposed matching condition has been found more convenient for the generalization of this kind of method in finite-dimensional approximations.

3. Nonstandard finite element methods.

For all $e = 1, \ldots, E$ let

(8) X_{he} be a finite-dimensional subspace of $\{v_e \in H^1(\Omega_e);\ v_e = 0 \text{ on } \partial\Omega \cap \partial\Omega_e\}$

and

(9) Λ_{he} be a finite-dimensional subspace of $L^2(\Gamma_e)$.

In practice the spaces X_{he} are obtained using a conforming finite element approximation of $H^1(\Omega_e)$, independently for each subdomain Ω_e. The spaces Λ_{he} are obtained using a finite element approximation of $L^2(\Gamma_{ed})$. We denote X_h and Λ_h the product spaces:

$$X_h = \prod_{e=1}^{E} X_{he} \tag{10}$$

and

$$\Lambda_h = \prod_{e=1}^{E} \Lambda_{he}. \tag{11}$$

Moreover let

(12) Y_h be a finite-dimensional subspace of $H_{00}^{\frac{1}{2}}(\Gamma)$,

where $H_{00}^{\frac{1}{2}}(\Gamma)$ is the subspace of $L^2(\Gamma)$ constitued by traces on $\Gamma = \bigcup_{e=1}^{E} \Gamma_e$ of the functions which belong to $H_0^1(\Omega)$.
Finally we introduce a subspace W_h of $X_h \times Y_h$ and a subspace M_h of Λ_h ; the choices $W_h = X_h \times Y_h$ and/or $M_h = \Lambda_h$ should not be excluded.

Then we search saddle-points on $W_h \times M_h$ for the augmented Lagrangian $\mathcal{L}$. With the definition (7) for $\mathcal{L}$, it is straightforward to see that $(u_{he}, p_h; \lambda_{he})$ is a saddle-point of $\mathcal{L}$ if and only if $w_h = (u_{he}, p_h)$ and $\lambda_h = (\lambda_{he})$ are solution of:

(13)
$$\begin{aligned} & w_h \in W_h, \quad \lambda_h \in M_h, \\ & \forall z_h \in W_h, \quad a(w_h, z_h) + b(z_h, \lambda_h) = \int_\Omega f z_h \, dx, \\ & \forall \mu_h \in M_h, \quad b(w_h, \mu_h) = 0 \end{aligned}$$

with

(14)
$$a(w_h, z_h) = \sum_{e=1}^{E} \int_{\Omega_e} \mathbf{grad}\ u_{he} \cdot \mathbf{grad}\ v_{he} \, dx + \\ + \rho \sum_{e=1}^{E} \int_{\Gamma_e} (u_{he} - p_h)(v_{he} - q_h) \, d\gamma$$

and

(15)
$$b(z_h, \lambda_h) = -\sum_{e=1}^{E} \int_{\Gamma_e} \lambda_{he} (v_{he} - q_h) \, d\gamma,$$

for $w_h = (u_{he}, p_h)$, $z_h = (v_{he}, q_h)$, $\lambda_h = (\lambda_{he})$. Using the Babuška-Brezzi's theory (cf. for example [10]), it is now well known that the problem (13) has a unique solution if and only if the two following assumptions are satisfied:

(16)
$$\inf_{z_h \in V_h \setminus \{0\}} \sup_{w_h \in V_h} a(z_h, w_h) > 0,$$

where the subspace V_h is defined as the kernel of the bilinear form $b(.,.)$:

(17)
$$V_h = \{ z_h \in W_h;\ \forall \mu_h \in M_h,\ b(z_h, \mu_h) = 0 \}$$

and

(18)
$$\inf_{\mu_h \in M_h \setminus \{0\}} \sup_{z_h \in W_h} b(z_h, \mu_h) > 0.$$

Clearly if $a(z_h, z_h) = 0$ for some $z_h = (v_{he}, q_h)$ in W_h, then v_{he} is constant on Ω_e, $v_{he} = 0$ on $\partial\Omega_e \cap \partial\Omega$ and $q_h = v_{he}$ on Γ_e for all e, $1 \leq e \leq E$. Then the bilinear form $a(.,.)$ is W_h-elliptic and so (16) is always satisfied .
On the other hand (18) is satisfied if and only if

$$\text{(19)} \qquad \forall \mu_h \in M_h \text{ with } \mu_h \neq 0,\ \exists z_h = (v_{he}, q_h) \in W_h \text{ such that}$$

$$\sum_{e=1}^{E} \int_{\Gamma_e} \mu_{he}(v_{he} - q_h)\, d\gamma \neq 0.$$

We give some applications of the above; for the sake of simplicity only the case of plane geometry ($N = 2$) will be treated hereafter.

Example 1. We assume that $W_h = X_h \times Y_h$.
Then (19) is satisfied as soon as we have for all e, $1 \leq e \leq E$,

$$\text{(20)} \qquad \forall \mu_{he} \in \Lambda_{he} \text{ with } \mu_{he} \neq 0,\ \exists v_{he} \in X_{he} \text{ such that } \int_{\Gamma_e} \mu_{he} v_{he}\, d\gamma \neq 0.$$

When X_{he} is a space of continuous, piecewise polynomial functions on $\overline{\Omega}_e$, vanishing on $\Gamma_e \cap \partial\Omega$, (conforming finite element on Ω_e), the analysis of (20) is what is made in hybrid finite element method when a Dirichlet condition is dualized with the use of a boundary Lagrangian multiplier (cf. [2], [3], [10]).

Example 2. For a given integer k, let

$$\text{(21)} \qquad Y_h = \{q_h \in \mathcal{C}^0(\Gamma);\ q_h|_{\Gamma_{ed}} \in P_k(\Gamma_{ed}),\ 1 \leq e \neq d \leq E\}.$$

Let

$$\text{(22)} \qquad \Lambda_h = \prod_{e=1}^{E} \Lambda_{he}, \qquad \text{with } \Lambda_{he} = \prod_{d \neq e} P_{k-2}(\Gamma_{ed}),$$

(each Γ_{ed} is assumed to be a straight line; a function $\mu_{he} \in \Lambda_{he}$ is not defined at the vertices of $\partial\Omega_e$).
Take for a subspace M_h

$$\text{(23)} \qquad M_h = \{\mu_h \in \Lambda_h;\ \text{ if } e < d,\ \mu_{hd} = 0 \text{ on } \Gamma_{ed}\}.$$

Take for a subspace W_h

$$\text{(24)} \qquad W_h = \{w_h = (v_{he}, q_h) \in X_h \times Y_h;\ v_{he}(a) = q_h(a) = v_{hd}(a)$$
$$\text{for each vertex } a \in \Gamma_{ed};\ \text{ if } e < d,\ v_{hd} = q_h \text{ on } \Gamma_{ed}\}.$$

Let us assume that

$$\text{(25)} \quad \forall t \in P_k(\Gamma_{ed}) \cap H_0^1(\Gamma_{ed}),\ \exists v_{he} \in X_{he} \text{ such that } v_{he} = t \text{ on } \Gamma_{ed} \text{ for all } e < d.$$

So (19) is satisfied. We recovered here a variant of the mortar finite element method (cf. [9], [1]).

4. Nonoverlapping Schwarz alternating method: discrete case.

When the space W_h is the product space $X_h \times Y_h$ as in Example 1., an algorithm for solving the saddle-point of $\mathcal{L}$ over $W_h \times M_h$ is:

Given p_h^0 and λ_h^0,
then for $m \geq 0$, p_h^m and $\lambda_h^m = (\lambda_{he}^m)$ known; solve successively

1° $u_h^{m+\frac{1}{2}} \in X_h$, for all $v_h \in X_h$

$$\sum_{e=1}^{E} \left\{ \int_{\Omega_e} \mathbf{grad}\ u_{he}^{m+\frac{1}{2}} \cdot \mathbf{grad}\ v_{he}\, dx + \rho \int_{\Gamma_e} u_{he}^{m+\frac{1}{2}} v_{he}\, d\gamma \right\} =$$

$$= \sum_{e=1}^{E} \left\{ \int_{\Omega_e} f v_{he}\, dx + \int_{\Gamma_e} (\lambda_{he}^m + \rho\, p_h^m) v_{he}\, d\gamma \right\},$$

2° $\lambda_h^{m+\frac{1}{2}} \in M_h$, for all $\mu_h \in M_h$

$$\sum_{e=1}^{E} \int_{\Gamma_e} \lambda_{he}^{m+\frac{1}{2}} \mu_{he}\, d\gamma = \sum_{e=1}^{E} \int_{\Gamma_e} \left\{ \lambda_{he}^m \mu_{he} + \rho (u_{he}^{m+\frac{1}{2}} - p_h^m) \mu_{he} \right\} d\gamma,$$

3° $p_h^{m+1} \in Y_h$, for all $q_h \in Y_h$

$$\int_{\Gamma} p_h^{m+1} q_h\, d\gamma = \sum_{e=1}^{E} \int_{\Gamma_e} (u_{he}^{m+\frac{1}{2}} + \frac{1}{\rho} \lambda_{he}^{m+\frac{1}{2}}) q_h\, d\gamma,$$

4° $\lambda_h^{m+1} \in M_h$, for all $\mu_h \in M_h$

$$\sum_{e=1}^{E} \int_{\Gamma_e} \lambda_{he}^{m+1} \mu_{he}\, d\gamma = \sum_{e=1}^{E} \int_{\Gamma_e} \left\{ \lambda_{he}^{m+\frac{1}{2}} \mu_{he} + \rho (u_{he}^{m+\frac{1}{2}} - p_h^{m+1}) \mu_{he} \right\} d\gamma.$$

The step 1° may be computed in parallel . In fact, we are solving in this step the finite element approximation of E independent Fourier-Robin's problems. Moreover when $M_h = \Lambda_h$, the steps 2° and 4° are also parallelizable. But the step 3° may not be reduced to parallel computations on the Γ_e. It is the synchronization step in the algorithm.

Remark. We cannot develop such algorithm when the space W_h is a proper subspace of $X_h \times Y_h$ as in Example 2. In the situation of this Example 2., it is convenient to eliminate directly the Lagrange multipliers and then to solve the elliptic problem in V_h by a preconditioned conjugate gradient method. It can be computed parallelly with the method of Bramble et al., [4].

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