CHAPTER 42

Heterogeneous Domain Decomposition for Acoustic Bottom Interaction Problems

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Abstract. Underwater acoustic phenomena in shallow waters are highly dependent on the acoustic and topographic properties of the bottom. It is therefore important to include modelling of both sea and bottom in realistic models for underwater acoustics.

We describe a domain decomposition technique using spectral collocation to solve the equations for non-linear acoustic propagation in the sea and the equations of linear elasticity in the bottom. The interface conditions between the subdomains are expressed in terms of characteristic variables for the different set of PDEs.

Numerical results showing the time evolution of the velocity vector and the stress tensor for some types of acoustic sources are presented to show the validity of the interface conditions.

1. Introduction. Domain decomposition (DD) techniques using different sets of equations in different subdomains has been investigated recently, e.g. [ReiRod89] for the application to boundary layer problems, and [GaQuaSa90] for the application to coupling between hyperbolic and elliptic equations.

Few results have appeared on coupling different sets of hyperbolic equations, but a slightly related problem of shock fitting in spectral methods is discussed by Hussaini and Kopriva, [Hussain85]. Kopriva has in a series of papers and reports, [Kop86, Kop89a, Kop91], developed a homogeneous multidomain technique for solving hyperbolic PDEs by spectral methods. His approach of using so-called generalized Riemann variables will be generalized and used extensively here. Quarteroni [Quart90] has developed a method for homogeneous domain decomposition for hyperbolic systems based on subdomain iteration. The tools he uses to develop the interface procedures are similar to those of Kopriva.

Macaraeg and Streett [MacStr88] are using a multi-domain technique for conservation laws based on a flux balance principle. Their method could possibly be generalized to solve different equations in different subdomains if the physical boundary conditions match the flux expressions in the method. The spectral element method, see e.g. [Patera84], applied to hyperbolic PDEs uses the same interface principle as in the papers by Kopriva.

The theoretical analysis of multi-domain spectral methods for hyperbolic equations is still in a very early stage even if the stability and convergence analysis for a single domain

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is fairly complete, see [GoLiTa87a] and [GoLiTa87b]. Funaro, [Funaro90] analyzes the single model equation $u_t = u_x$ with penalty type of boundary treatment and obtains convergence estimates for both Legendre and Chebyshev methods. A convergence analysis for a linear constant coefficient system is presented in [Quart90] using techniques different from those used by Funaro.

Acoustics in a coupled ocean-bottom medium has been discussed extensively in the acoustics literature, see e.g. [Jensen84] [Jensen90]. Computations using finite difference and finite element methods are presented, e.g. in [Steph88] using the ordinary wave equation for the seawater. However, the numerical methods used in these computations does not use domain decomposition techniques. Fornberg, [Fornb87] applies the Fourier methods to the linear elastic and acoustic equations and hence circumvents all boundary and interface conditions for a finite domain.

In the following sections we will derive interface conditions for the sets of equations describing the acoustics in seawater and ocean bottom. We consider two-dimensional rectangular domains as shown in Fig. 1.1. The types of boundary conditions are indicated on the figure.

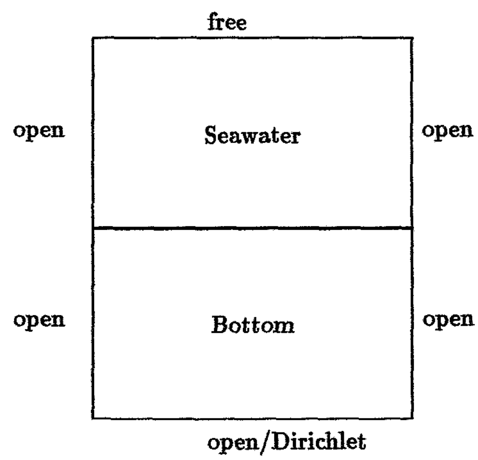

FIG. 1.1. *Seawater and ocean bottom domains with boundary conditions.*

2. The governing equations and characteristic boundary conditions.

2.1. The equations for seawater. In the seawater we use the equations for adiabatic wave motion in two space dimensions and include gravity to enable the model to take internal waves into account. The equations express the conservation of mass, momentum and entropy:

$$\rho_t + \rho u_x + \rho v_y + u\rho_x + v\rho_y = 0 \tag{2.1}$$

$$u_t + uu_x + vu_y + (1/\rho)p_x = 0 \tag{2.2}$$

$$v_t + uv_x + vv_y + (1/\rho)p_y = -g \tag{2.3}$$

$$p_t + up_x + vp_y - C^2(\rho_t + u\rho_x + v\rho_y) = 0 \tag{2.4}$$

where p is the pressure, ρ is the density, and u and v are the horizontal and vertical components of the velocity vector respectively.
C is the sound speed given by:

$$C^2 = \left(\frac{\partial p}{\partial \rho}\right)_S \tag{2.5}$$

In addition to the above equations we use the "International Equation of State for Seawater", see e.g. [Robert90], to determine C based on temperature and salinity information.

Since we have to use a bounded domain in the simulations we have to impose open boundary conditions to avoid unwanted reflections from waves leaving the domain. It is well known that one has to use characteristic boundary conditions for hyperbolic systems. The equations for acoustic propagation in water are essentially the same as the Euler equations in gas dynamics, and we can use the results on the characteristics of the Euler equations, see e.g. [Pulli82], for the development of the characteristic boundary conditions here. For the boundary conditions we use a locally one-dimensional approach, see e.g [AndLie91].

Consider first the left/right boundary, hence ignoring y-derivatives in the equations. The characteristic speeds are

$$\{u, u, u + C, u - C\}$$

with corresponding characteristic variables

$$\{\rho - p/C^2, v, p + \rho C u, p - \rho C u\}$$

The determination of the direction of the characteristics (incoming/outgoing) is done locally in time and space. For example, consider a section of the left boundary and assume that $u > 0$ there. Since we always assume that $u < C$ the characteristic corresponding to $u - C$ is outgoing and the rest of the characteristics are incoming. Characteristic variables corresponding to outgoing characteristics are computed from the variables from within the domain. Variables corresponding to incoming characteristics are set to the value of the background field. Slow modes (those having characteristic speed u) are extrapolated in space and time, for details see [AndWas89].

For the top/bottom boundaries the characteristic speeds for the locally one-dimensional system is:

$$\{v, v, v + C, v - C\}$$

with characteristic variables

$$\{\rho - p/C^2, u, p + \rho C v, p - \rho C v\}$$

and the same procedure, now based on the sign of v is used here to compute the boundary values.

Treatment of corner points is relatively tricky, and there is hardly a procedure which can be used in all cases. We have adopted the strategy used in [AndWas89] and also suggested by others. The strategy is based on defining a "normal to the corner" pointing inwards and bisecting the angle between the adjacent boundary lines. The determination of the direction of the characteristics is done with respect to this normal vector. The boundary treatment is then done in the coordinate system formed by the normal and the tangent to the corner, the final operation being the transformation back to the original coordinates.

2.2. The equations for the ocean bottom. The equations for the elastic medium are written as a first order hyperbolic system:

$$\rho_b u_t = \sigma_{11,x} + \sigma_{12,y} \tag{2.6}$$
$$\rho_b v_t = \sigma_{12,x} + \sigma_{22,y} + \rho_b g \tag{2.7}$$
$$\sigma_{11,t} = (\lambda + 2\mu)u_x + \lambda v_y \tag{2.8}$$
$$\sigma_{12,t} = \mu(v_x + u_y) \tag{2.9}$$
$$\sigma_{22,t} = \lambda u_x + (\lambda + 2\mu)v_y \tag{2.10}$$

where u and v are the velocity components and σ_{ij} are the components of the (symmetric) stress tensor. λ and μ are the Lamé coefficients, and ρ_b is the density. Both λ, μ and ρ_b can be space-dependent. The characteristic speeds for this set of equations are well known, see e.g. [Ziv69]:

$$\{0, \pm c_P, \pm c_S\}$$

where

$$c_P = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \qquad c_s = \sqrt{\frac{\mu}{\rho}}$$

are the speeds for P- and S-waves respectively.

If we again use a locally one-dimensional approach for boundary treatment, we get the following characteristic variables for the left/right boundaries:

$$\{-\lambda\sigma_{11} + (\lambda + 2\mu)\sigma_{22}, \sigma_{11} \mp \rho_b c_P u, \sigma_{12} \mp \rho_b c_S v\}$$

Since we use a linear hyperbolic system the characteristic speeds have fixed sign, so the direction of the characteristics is independent of space and time.

For the top/bottom boundaries we have the following characteristic variables:

$$\{(\lambda + 2\mu)\sigma_{11} - \lambda\sigma_{22}, \sigma_{22} \mp \rho_b c_P v, \sigma_{12} \mp \rho_b c_S u\}$$

The open boundary treatment goes exactly as for the seawater case: The outflow variables are computed from within the domain and the inflow variables are given the values of the background. The corner points are treated in the same way as described for the seawater, the only difference being that the stress tensor has to be transformed (rotated) to get the correct expressions for the characteristic variables in the local coordinate system for the corners. The transformed stress tensor σ' is computed as follows:

$$\sigma'_{ij} = a_{ik}a_{jl}\sigma_{kl}$$

where $\{a_{ij}\}$ is the transformation matrix. See [AndLie91] for details.

3. Interface conditions. We consider now the interface between a seawater domain and a bottom domain (see figure 3.1)) and we will derive interface conditions based on the characteristic variables for the two sets of equations derived in the previous section. The number of possible types of waves in such a configuration is large, see [Brekh60]. The two main classes are the volume waves in the two media and the surface waves. However, all the waves will propagate along the characteristics for the two sets of equations. This implies that the interface conditions based on characteristic variables will include the surface waves. For a discussion on open boundary conditions for surface waves, see [Bamberg88]

The interface conditions between the two media will have to be based on the physical boundary conditions at the interface. These are as follows:

a) Continuity in normal velocity:

$$(\mathbf{u}-\mathbf{v})\cdot\mathbf{n}=0$$

where **u** and **v** are the velocity vectors for seawater and bottom respectively, and **n** is the normal vector of the interface.

b) Tangential velocity condition. This depends on the physical properties of the interface.

c) Continuity in stress:

$$\bar{\sigma}_{ij}n_j-\sigma_{ij}n_j=\gamma(\frac{1}{R_1}+\frac{1}{R_2})n_i$$

where $\bar{\sigma}_{ij}$ and σ_{ij} are the stress tensor components in water and elastic medium respectively. The term on the right hand side is the surface tension which can be neglected in our case. Obviously, $\bar{\sigma}_{ij}=p\delta_{ij}$ so the conditions reduces to $pn_i-\sigma_{ij}n_j=0$. For a horizontal interface we then have the physical boundary conditions:

$$p-\sigma_{22}=0\ ,\qquad \sigma_{12}=0\ ,\qquad u_2-v_2=0$$

and a slip condition determining the horizontal velocity. In addition, the interface is usually assumed to be stress free, see e.g. [Brekh60], thus $\sigma_{11}=0$.

Consider now the characteristics for the two sets of equations at the (horizontal) interface. In water we have $p\pm\rho Cv$ as the acoustic modes and in the bottom we have $\sigma_{22}\mp\rho_b c_P v_2$ and $\sigma_{12}\mp\rho_b c_S v_1$ as the pressure and shear modes. We see that the corresponding characteristics $p\pm\rho Cv$ and $\sigma_{22}\mp\rho_b c_p v_2$ have the same structure and that the essential variables (p,v,σ_{22},v_2) are continuous at the interface for the two pairs of characteristics. This fact will be utilized in the construction of the interface conditions.

We will now describe a modification of the interface procedure described by Kopriva [Kop91], for construction of the interface conditions. The basic procedure is described in detail in [Kop91], and we refer to this report for the background for the procedure. Kopriva's terminology will be used throughout.

Consider the generalized Riemann variables for the water equations. These are, from [Kop91]:

$$R^{\pm}=p\pm\rho Cu\ ,\qquad S^{\pm}=p\pm\rho Cv$$

corresponding to 4 chosen bicharacteristics, see figure 3.1. Note that these Riemann variables are identical to the characteristic variables derived in the previous section. $R^{\pm}$ and $S^{\pm}$ are computed from spectral approximations within the subdomains and the interface procedure consists of a correction technique for the interface variables.

Once this correction has been computed we use it for both domains according to the physical boundary conditions.
To illustrate the interface procedure, consider a point of outflow from water, see Fig. 3.1: Note that in this case the bicharacteristic S^+ originates from the bottom while

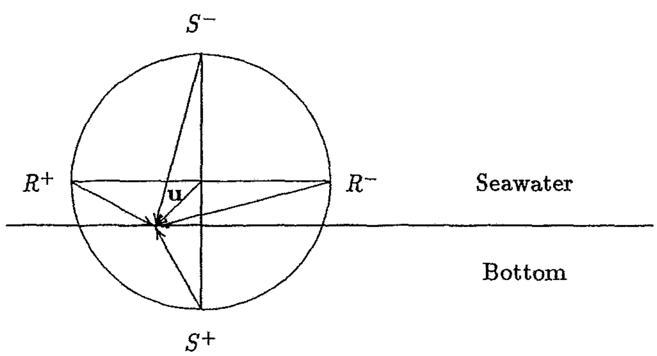

FIG. 3.1. *Correction procedure for outflow point from water*

the rest originates from the water.
From the expression for $R^\pm$ we immediately get the corrected u:

$$u_{corr} = \frac{1}{2\rho C}(R^+ - R^-)$$

which of course is identical to the value computed in the water. The correction procedure then consists of correcting the interface value for the bottom depending on the slip conditions used.
To compute the vertical velocity we use the following procedure: We use the S^--characteristic $p - \rho C v$, and assume that we have an unknown inflow characteristic of the same type: $\hat{S}^+ = \hat{p} + \rho C \hat{v}$. The variables in this characteristic will depend on the corresponding variables both in water and elastic medium because of reflection and transmission of waves at the interface.
Similarly for the elastic medium, we use $S^+ = \sigma_{22} - \rho_b c_P v_2$ and the unknown inflow characteristic $\hat{S}^- = \hat{\sigma}_{22} + \rho_b c_P \hat{v}_2$. The corrected vertical velocity in water is

$$v_{corr} = \frac{1}{2\rho C}(\hat{S}^+ - S^-)$$

and in the elastic medium:

$$v_{2,corr} = \frac{1}{2\rho_b c_P}(\hat{S}^- - S^+)$$

From the physical boundary condition we know that these velocities must be equal and hence we obtain the relation

$$\hat{S}^+ - S^- = \frac{\rho C}{\rho_b c_P}(\hat{S}^- - S^+) = \gamma(\hat{S}^- - S^+)$$

From the continuity of the vertical stress we get another relation:

$$\hat{S}^+ + S^- = S^+ + \hat{S}^-$$

From these equations we can compute the unknown characteristic variables:

$$\hat{S}^+ = \frac{-2\gamma}{1-\gamma}S^+ + \frac{1+\gamma}{1-\gamma}S^- \tag{3.1}$$

$$\hat{S}^- = \frac{2}{1+\gamma}S^- - \frac{1+\gamma}{1-\gamma}S^+ \tag{3.2}$$

In the monograph of Brekhovskikh [Brekh60], plane wave reflection and transmission in layered media is studied in detail, and the reflection and transmission coefficients obtained there (for waves at normal incidence) are just $\frac{1-\gamma}{1+\gamma}$ and $\frac{2\gamma}{1+\gamma}$, so we can give the expressions for $\hat{S}^+$ and $\hat{S}^-$ a physical meaning.
We then get for the corrected vertical velocity and pressure:

$$v_{corr} = \frac{1}{2\rho C}\left(-\frac{2\gamma}{1-\gamma}S^+ + \frac{1+\gamma}{1-\gamma}S^- - S^-\right) = \frac{1}{\rho C}\frac{\gamma}{1-\gamma}(S^- - S^+) \tag{3.3}$$

$$p_{corr} = \frac{1}{2}\left(-\frac{2\gamma}{1-\gamma}S^+ + \frac{1+\gamma}{1-\gamma}S^- + S^-\right) = -\frac{\gamma}{1-\gamma}S^+ + \frac{1}{1-\gamma}S^- \tag{3.4}$$

We see again that the reflection and transmission coefficients appear in the expressions.
Note that the characteristic variable $\rho - p/C^2$ has no equivalent characteristic variable in the elastic medium because ρ is not a variable in the elastic equations we consider. We suggest the following method for calculating the corrected density ρ_{corr}:

$$\rho_{corr} - p_{corr}/C^2 = \rho - p/C^2$$

where the unsubscripted variables comes from the computations in the water and p_{corr} is computed from the formula above. Hence the density difference $\rho_{corr} - \rho$ is directly proportional to the pressure difference $p_{corr} - p$ which is fairly natural for both inflow and outflow situations.
The shear stress at the interface is set to zero due to the continuity condition for stress. Since we assume that the interface is stress-free in the horizontal direction, we let $\sigma_{11} = 0$.

The interface corner points are treated with a technique combining open boundary and interface treatment. Consider the left interface point and assume the $v < 0$, see Fig. 3.2. As before we get for the horizontal velocities in the two media:

$$u_{corr} = \frac{1}{2\rho C}(p_0 - p + \rho C v)$$

$$u_{2,corr} = \frac{1}{2\rho_b c_P}(\sigma_{11} + \rho_b c_P u_2 - \sigma_{11,0})$$

where p_0 and $\sigma_{11,0}$ are values of the background field for pressure and horizontal normal stress.

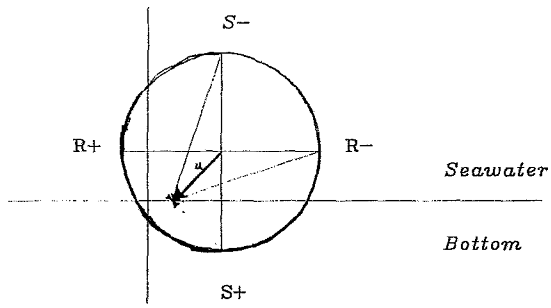

FIG. 3.2. *Left interface point*

To obtain the expressions for the vertical velocity and pressure we use the same procedure as described above for the interface line. We use the following set of characteristics

$$\begin{aligned} S^- &= p - \rho C v \\ \hat{S}^+ &= \hat{p} + \rho C \hat{v} \\ R^{\pm} &= p \pm \rho C u \\ S^+ &= \sigma_{22} - \rho_b c_P v_2 \\ \hat{S}^- &= \hat{\sigma}_{22} + \rho_b c_P \hat{v}_2 \end{aligned}$$

From the continuity of the vertical velocity we again have:

$$\frac{1}{2\rho C}(\hat{S}^+ - S^-) = \frac{1}{2\rho_b c_P}(\hat{S}^- - S^+)$$

or

$$\hat{S}^+ - S^- = \gamma(\hat{S}^- - S^+)$$

For the pressure (in the water) we use a formula of the type proposed by Kopriva:

$$(3.5)\quad p_{corr} = \frac{1}{2}(S^- + \hat{S}^+ + R^+ + R^- - 2p) = \frac{1}{2}(S^- + \hat{S}^+) + \frac{1}{2}(R^+ + R^-) - p$$

Note that this formula (like the one used by Kopriva) contains a non-characteristic quantity, namely the pressure p. However, Kopriva reports success with this type of formula in his series of tests despite the lack of argument behind it.
From the continuity condition for vertical stress we get the relation:

$$S^+ + \hat{S}^- = S^- + \hat{S}^+$$

and as before we have

$$\begin{aligned} \hat{S}^- &= \frac{2}{1-\gamma}S^- - \frac{1+\gamma}{1-\gamma}S^+ \\ \hat{S}^+ &= -\frac{2\gamma}{1-\gamma}S^+ + \frac{1+\gamma}{1-\gamma}S^- \end{aligned}$$

from which we get the corrected vertical velocity and pressure:

$$v_{corr} = \frac{1}{\rho C}\frac{\gamma}{1-\gamma}(S^- - S^+) \tag{3.6}$$

$$p_{corr} = -\frac{\gamma}{1-\gamma}S^+ + \frac{1}{1-\gamma}S^- + \frac{1}{2}(R^+ + R^- - 2p) \tag{3.7}$$

(3.8)

We see that if we ignore the contribution from the horizontal characteristics, we get the same expressions as we derived for the interface line.

The formula $\rho_{corr} - p_{corr}/C^2 = \rho - p/C^2$ is used to compute the corrected density for the water. From the assumption of the interface being a stress-free surface we get $\sigma_{12} = 0$ and $\sigma_{11} = 0$.

The right interface point is treated in a similar way, and the formulas are not presented here.

4. Numerical experiments. Below we show some results from the ocean/bottom acoustic model with interface conditions as described in section 3. We have used a 32×32 grid in both domains, and the boundary conditions are as shown in Fig. 1.1 and described in section 2. The physical dimensions of the domains are 600×300 meters. The experiments have been run on a Cray X-MP.
In the first experiment an acoustic pulse is generated at $t = 0$ in location $(-0.5, -0.5)$ in computational space. Figures 4.1 and 4.2 show the pressure (σ_{22} in the bottom) and velocity fields after 2 (scaled) time units. Each domain is in fact quadratic, the plots have been scaled down in the y-direction to make the figures smaller. The upper wavefront in seawater is the direct wave from the pulse propagating upwards. The lower wavefront is a reflection from the interface. In the bottom we see a downwards propagating P-wave which is detached, and surface waves propagating horizontally.From the velocity plot we see that the direct wave in the seawater and the leftmost surface waves are about to leave the domains, thus indicating the correctness of the open boundary conditions. Fig. 4.3 shows the shear stress in the ocean bottom at the same point in time.
In the second experiment an acoustic pulse is generated at the center of the ocean bottom domain. The boundary conditions at the bottom of the ocean bottom is now zero normal velocity. The pressure field after 1.5 time units is shown in Fig. 4.4. The upper wavefront in the ocean bottom is a reflection from the interface, thus propagating downwards. The lower wavefront is a reflection from the lower boundary (Dirichlet conditions) and is propagating upwards. An upward propagating wave transmitted from the bottom has just been generated in the seawater. At 0.5 time units later the two waves in the ocean bottom have interacted and formed a complex pattern as shown in Fig. 4.5.

5. Conclusion. We have developed a domain decomposition technique for coupling the equations of acoustic propagation in water with the equations of linear elasticity via the characteristic variables for the two sets of equations. This technique makes it possible to study ocean/bottom acoustic interaction in detail for low frequencies. The technique can also be extended to the case of arbitrary bottom

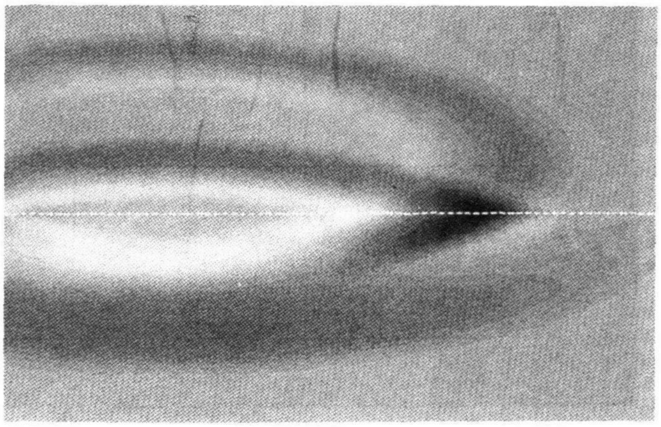

FIG. 4.1. *Pressure field in first experiment at t=2.0*

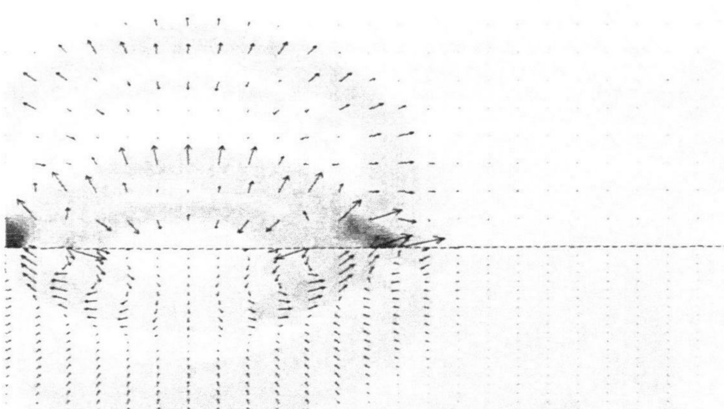

FIG. 4.2. *Velocity field in first experiment at t=2.0*

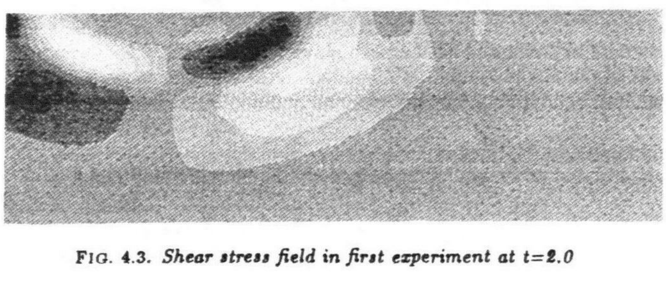

FIG. 4.3. *Shear stress field in first experiment at t=2.0*

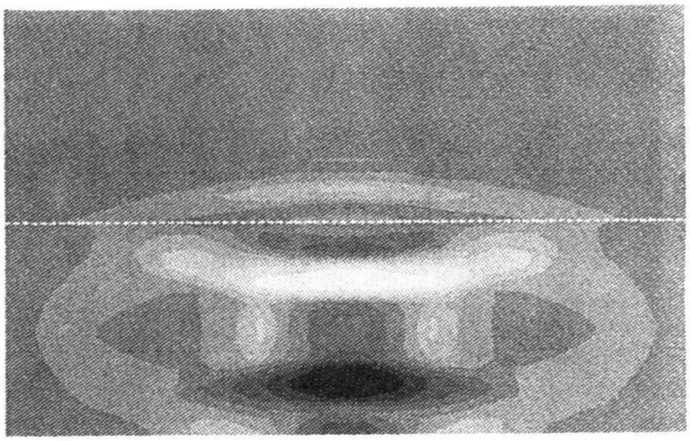

FIG. 4.4. *Pressure field in second experiment at t=1.5*

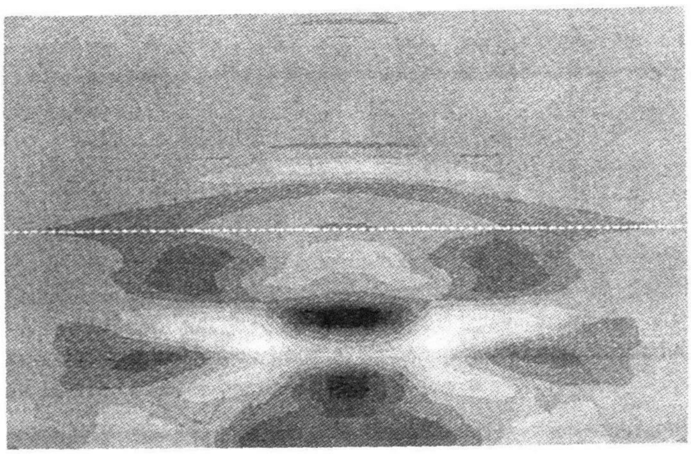

FIG. 4.5. *Pressure field in second experiment at t=2.0*

profile thus making it possible to study specific acoustic phenomena, e.g. sloping bottom, see [Lie91].
The numerical experiments show that the interface conditions works very well for a number of test cases.

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