CHAPTER 12

Heterogeneous Domain Decomposition: Principles, Algorithms, Applications

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Abstract

Heterogeneous domain decomposition is a field of research bridging (homogeneous) domain decomposition and mathematical modeling. The underlying mathematical assumption is that differential equations of different kind are coupled one another across interfaces of disjoined subdomains. In several circumstances such an approach allows a more flexible description of the physical problem at hand and fosters the use of different numerical methods within different zones of the computational domain. Besides, it may sometimes yield a remarkable simplification of the solution algorithm.

Here we review the basic issues of heterogeneous domain decomposition, discuss its main theoretical properties together with its algorithmical aspects, and present some applications to fluid dynamical problems for convective-dominated flows. Several numerical results sustain our theoretical conclusions.

1. Introduction

Heterogeneous domain decomposition arises whenever in the approximation of certain physical phenomena two *different kind* of (initial) boundary-value problems are assumed to hold within two disjoined subregions of the computational domain Ω, say Ω_1 and Ω_2.

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This is a generalization of the classical, homogeneous domain decomposition approach, in which the *same kind* of problem is faced upon each subdomain.

From a mathematical viewpoint, a crucial issue is the set up of matching equations across the interface separating the two subregions. The derivation of such interface conditions oughts to be carried out sticking as close as possible to the physics of the underlying problem. Most often, the inspiring criterium should be that the solution to the coupled problem (the one provided by the individual subproblems in Ω_1 and Ω_2, along with the interface conditions) is also the limit of solutions to a sequence of global variational problems set in the whole domain Ω.

In everyday scientific applications we may encounter problems that are heterogeneous "in nature". A family of examples is provided by the interaction between fluids and solids. For instance, this is the case of off-shore mechanics, or else of undersea pipelines, or again of underwater acoustic phenomena in shallow waters ([L]). Typically, the above problems can be modeled by the coupling between Navier-Stokes equations and the system of elasticity.

Another example is provided by the propagation of acoustic waves in heterogeneous media, whose mathematical modeling is accomplished by the Maxwell equations whose conductivity coefficient degenerates upon a subregion of the computational domain ([DL], [QV2]; see also [KN]).

The cohexistence of different flow regimes in the upper athmosphere, the molecular and the continuous one, can be modeled through the coupling between a Boltzmann kinetic model and the Navier-Stokes equations for viscous, compressible fluids ([C], [BLTQ]).

Still in aerodynamics applications, let us mention the subsonic-supersonic transition for potential flows (e.g., [G], [F]).

The crystal growth's process is another instance of multicomponent problem that is modeled by multiple differential equations mutually interacting one another ([BD]).

At some extent, most free-boundary problems can be regarded as different boundary value problems which are coupled throughout an interface whose location and shape are both unknowns.

Another class is provided by those problems which, although homogeneous in nature, can be faced in an heterogeneous fashion, after reducing the given problem to a simplified one in a subregion of Ω. We encounter situations of this type when Navier-Stokes equations for incompressible flows are matched with shallow-water equations in a bay or in the vicinity of a coast.

Another very important situation arises in fluid dynamics whenever convective dominated viscous flows yield internal and/or boundary layers. In aerodynamic simulations, neglecting the viscous effects far from sharp layers bring to the coupling of Euler and Navier-Stokes equations.

Similarly, neglecting the thermal conductivity coefficient in a thermodynamic problem, or else the viscous diffusivity in convection-diffusion-reaction equations, leads to the coupling of hyperbolic and parabolic equations.

The latter approach, that is based on the use of a fictitious interface between

the two inhomogeneous subregions, has provided the basis for many numerical algorithms since the early 80's (e.g., [CGVV], [DGPT], [GPT], [S], [HH], and the references therein).

In more recent years, this issue has been faced in a rigorous mathematical fashion. To start with, a general criterium to find out interface conditions has been devised. Then, sound numerical algorithms have been set up allowing the solution to the coupled problem to be achieved through a sequence of independent solves upon either subdomain.

With regard to convection-diffusion problems, the analysis for one-dimensional systems and two-dimensional equations is carried out in [GQ] and [GQS], respectively, whilst algorithmical aspects are discussed in [FPQ]. Another example, the so-called generalized Stokes problem, is faced in [QSV] and [CZ] in the framework of finite element and spectral approximations, respectively.

The potential interest behind this approach is manifold. Using two rather than one model problem allows higher flexibility in setting up the numerical method that fits better the nature of the physical phenomenon within each subregion. Most often, far from sharp layers, the expected solution is smooth and exhibits slow variations, hence very inexpensive numerical approximation on the reduced problem suffice to produce accurate results. Moreover, the coupled viscous-inviscid problem yields a couple of subproblems that might be solved sequentially one after the other, whenever the convective field has constant orientation at the interface. This splitting property allows the achievement of the solution of the coupled problem without any iteration between subdomains.

Along this paper we review the heterogeneous domain decomposition method for convection-diffusion problems and discuss several theoretical and algorithmical aspects of the fictitious interface method.

An outline is as follows. In Sect. 2 we introduce a linear convection-diffusion problem, and we formulate the reduced problem that couples the convection-diffusion equation with the convection one through proper interface conditions. We then present some test cases enlighting the effectiveness of the reduced model. In Sect. 3 we discuss the algorithmical aspects of our approach. In particular, we set up an iteration-by-subdomain procedure to solve the reduced problem, and discuss its rate of convergence as a function of the several parameters of our approximation. In Sect. 4 we analyze the issue of interface location. In Sect. 5 we show how to extend our approach to the case of non-linear problems, by analyzing at which extent the interface conditions can be devised for several kinds of time-advancing methods. In Sect. 6 we conclude mentioning other applications of the above approach.

2. Basic principles and problem statement

For the sake of exposition, we focus on a scalar equation of convection-diffusion type. We make use of the following notations: Ω is a bounded two-dimensional domain, $\partial\Omega$ denotes its boundary that we assume to be partitioned in the form

$\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, $\mathbf{n}$ will denote the unit normal vector on $\partial\Omega$, directed outward.

We consider the initial-boundary value problem: find $w = w(x,t)$ satisfying for any positive t

$$(2.1)\qquad \begin{cases} \frac{\partial w}{\partial t} - div(\nu\nabla w) + div(\mathbf{b}w) + c\,w = f\ , & x \in \Omega \\ w = w_0\ , & x \in \Omega\ ,\ t = 0 \\ w = d\ , & x \in \partial\Omega_D \\ \frac{\partial w}{\partial n} = 0\ , & x \in \partial\Omega_N\ , \end{cases}$$

where: $f(x,t)$, $c(x,t)$, $\mathbf{b}(x,t)$, $w_0(x)$, $d(x,t)$ are given functions; $\nu = \nu(x) > 0$ is the viscosity coefficient. It is well known that, under mild regularity assumption on the data, the above problem has a unique solution for all time (e.g., [HP], [LM]).

We are interested in those physical applications in which the viscous (second order) term is dimensionally negligible upon a subregion, say Ω_1, of the domain Ω. This situation occurs, typically although not exclusively, in boundary layer simulations. From a numerical viewpoint, in any such application it is often useful to face a reduced problem that couples the original equation in $\Omega_2 = \Omega\backslash\Omega_1$ with the inviscid equation in Ω_1. The major problem in this respect is the set up of matching conditions between the viscous and inviscid solutions across the subdomain interface Γ.

As far as problem (2.1) is concerned, its *reduced* counterpart reads as follows. Find u and v such that for any positive t they satisfy:

$$(2.2)\qquad \frac{\partial u}{\partial t} + div(\mathbf{b}u) + c\,u = f\ ,\quad x \in \Omega_1$$

$$(2.3)\qquad \frac{\partial v}{\partial t} - div(\nu\nabla v) + div(\mathbf{b}v) + c\,v = f\ ,\quad x \in \Omega_2$$

$$(2.4)\qquad (\mathbf{b}_{|\Omega_1} \cdot \mathbf{n}_\Gamma)u = (\mathbf{b}_{|\Omega_2} \cdot \mathbf{n}_\Gamma)v\ ,\quad x \in \Gamma_{in}$$

$$(2.5)\qquad (\mathbf{b}_{|\Omega_1} \cdot \mathbf{n}_\Gamma)u = (\mathbf{b}_{|\Omega_2} \cdot \mathbf{n}_\Gamma)v - \nu\frac{\partial v}{\partial n_\Gamma}\ ,\quad x \in \Gamma$$

$$(2.6)\qquad u = d\ ,\quad x \in \partial\Omega_D \cap \partial\Omega_{1,in}$$

$$(2.7)\qquad v = d\ ,\quad x \in \partial\Omega_D \cap \partial\Omega_2$$

$$(2.8)\qquad \frac{\partial v}{\partial n} = 0\ ,\quad x \in \partial\Omega_N \cap \partial\Omega_2\ .$$

In addition, at $t = 0$ $u = w_0$ in Ω_1 and $v = w_0$ in Ω_2. Symbol explanation is as follows: $\mathbf{n}_\Gamma$ is the unit normal vector on Γ, directed toward Ω_2,

$$\Gamma_{in} = \{x \in \Gamma \,|\, (\mathbf{b}_{|\Omega_1} \cdot \mathbf{n}_\Gamma)(x) < 0\} \ , \ \partial\Omega_{1,in} = \{x \in \partial\Omega_1 \setminus \Gamma \,|\, (\mathbf{b} \cdot \mathbf{n})(x) < 0\}$$

(notice that these sets can depend on t). An example for a fixed time is described in Fig. 1.

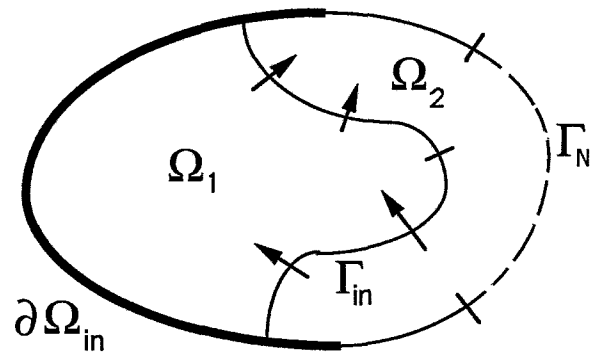

Fig. 1. The partition of the domain Ω into Ω_1 and Ω_2 at a given time t.

The matching conditions are given by (2.4), (2.5); the former states that continuity occurs only on Γ_{in} (provided $\mathbf{b} \cdot \mathbf{n}_\Gamma$ is continuous), while the latter enforces the flux balance on the whole Γ. For the case of a time-independent problem these interface conditions have been introduced and thoroughly analyzed in [GQS], where the leading principle was to obtain the coupled problem by a singular perturbation analysis. Precisely, the interface conditions of the reduced problem are the limit (as ϵ goes to zero) of those pertaining to a full viscous problem in which the viscosity coefficient ν was replaced by ϵ within Ω_1.

The continuity of the fluxes across Γ is a most desirable property that follows from the remark that the solution (u, v) of the reduced problem is actually the limit of the solution of a global variational problem in the whole Ω.

The apparently bad news is that the reduced solution itself can be discontinuous on $\Gamma_{out} := \Gamma \setminus \Gamma_{in}$, whereas the solution to the original problem (2.1) is continuous everywhere. However, the size of the jump is as small as the value of the viscosity on Γ. This issue is discussed in Sect. 4.

From the point of view of numerical approximation, solving the reduced problem requires the using of a spatial discretization based upon finite element, finite difference, spectral or other methods, then to advance in time, typically by a finite difference scheme. It goes by itself that the interface conditions (as well as the boundary conditions) need to be fulfilled at the new time level. In Sect. 5 we discuss the practical aspects of this issue for the case of nonlinear problems.

Here below we report some numerical results for a steady convection-diffusion problem. The spatial discretization is achieved by the spectral collocation method

(e.g., [GO], [CHQZ]), in which u (resp., v) is approximated by u_{N_1} (resp., v_{N_2}), an algebraic polynomial of degree less than or equal to N_1 (resp., N_2) in each variable. (For a detailed presentation of the spectral collocation method for this kind of problems we refer to [FPQ]).

The problem at hand is: find w satisfying

$$-div(\nu\nabla w) + div(\mathbf{b}w) + c\,w = f \quad \text{in} \quad \Omega = (-1,1)\times(-1,1) \quad , \tag{2.9}$$

with boundary conditions as in (2.1). Notice that any implicit time-advancing method on (2.1) yields at each time-level a problem like (2.9).

The reduced problem is: find u and v satisfying

$$div(\mathbf{b}u) + c\,u = f \quad \text{in} \quad \Omega_1 \tag{2.10}$$

$$-div(\nu\nabla v) + div(\mathbf{b}v) + c\,v = f \quad \text{in} \quad \Omega_2 \quad , \tag{2.11}$$

together with the interface conditions (2.4) and (2.5), and the boundary conditions (2.6)-(2.8).

The first example refers to the case in which:

$$\nu = 10^{-4} \quad , \quad \mathbf{b} = (1 + \frac{1}{3}cos\pi x_1, 1)^t \quad , \quad c = f = 0 \quad . \tag{2.12}$$

The boundary conditions for w are as follows: $w = 1$ on the sides $x_1 = -1$, $x_2 = -1$, $\frac{\partial w}{\partial x_2} = 0$ on $x_2 = 1$ and $w = 0$ on $x_1 = 1$ (notice that the latter condition yields a boundary layer of thickness ν in the vicinity of the vertical side $x_1 = 1$). We take the interface Γ equal to the vertical side $x_1 = 1 - \sqrt{\nu}$. In the current example the inviscid equation is therefore solved on a domain which is about 200 times thicker than that where the complete equation is faced. Fig. 2 refers to a calculation performed using $N_1 = N_2 = 28$ as polynomial degree of the spectral solutions.

This numerical solution is practically undistinguishable from the one achievable solving the fully elliptic problem (2.9) everywhere with the same partition as before and the same polynomial degrees. Moreover, one can play to reducing the number of degrees of freedom in Ω_1 without compromising the overall accuracy. In this regard, we report in Fig. 3 the results obtained on the same problem as before, using now $N_2 = 14$ (rather than 28). The maximum difference between the solutions of Figs. 2 and 3 is 0.0014.

An even more dramatic situation is reported in Fig. 4, which refers to the case where

$$\nu = 7\cdot 10^{-4} \quad , \quad \mathbf{b} = (0, -\frac{1}{2}(x_1 - 1))^t \quad , \quad c = f = 0 \quad , \tag{2.13}$$

and the boundary conditions are the same as before. (This example is also considered in [FFH]). The interface Γ is located at $x_1 = 0.5$, and $N_1 = 4$, $N_2 = 20$.

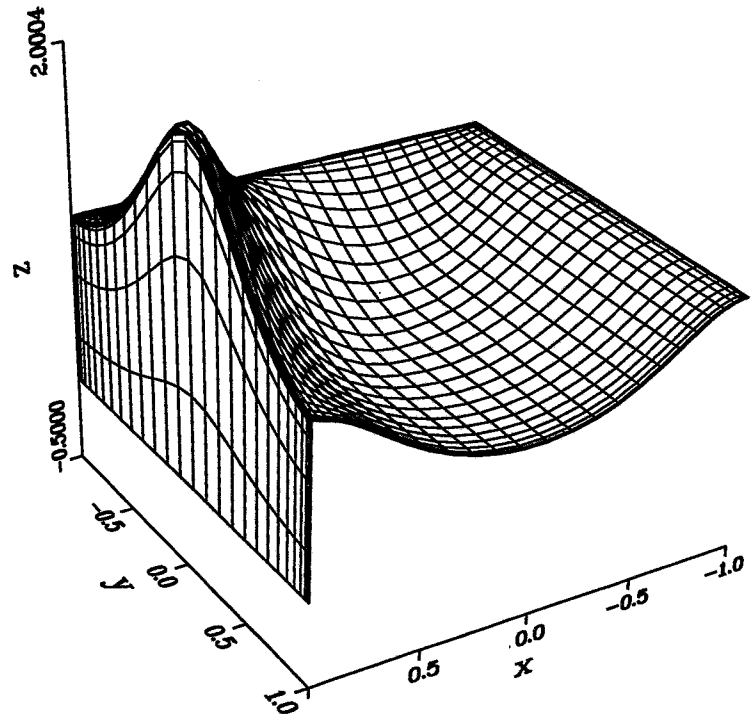

Fig. 2. Numerical simulation of boundary-layer problem (2.10)-(2.12).

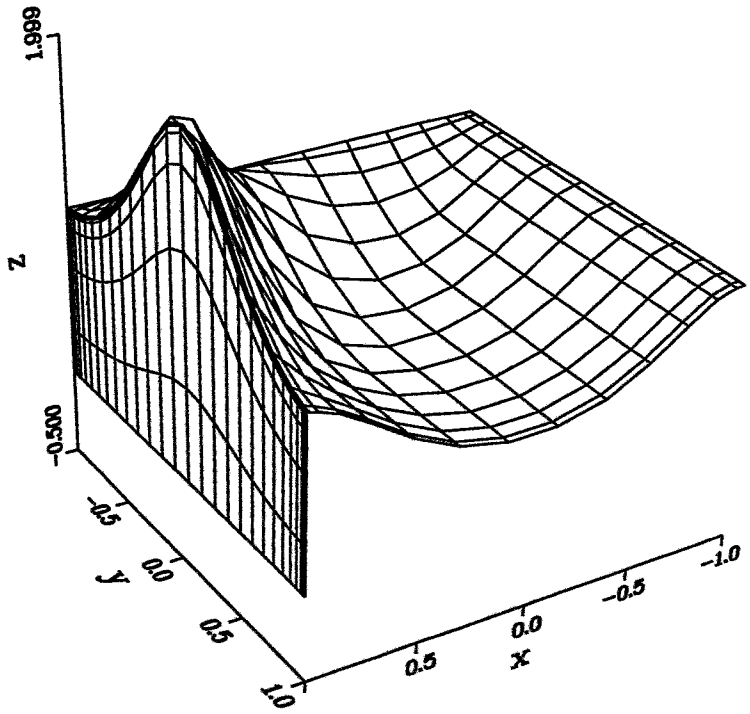

Fig. 3. Numerical simulation of problem (2.10)-(2.12) using a coarse grid in Ω_1.

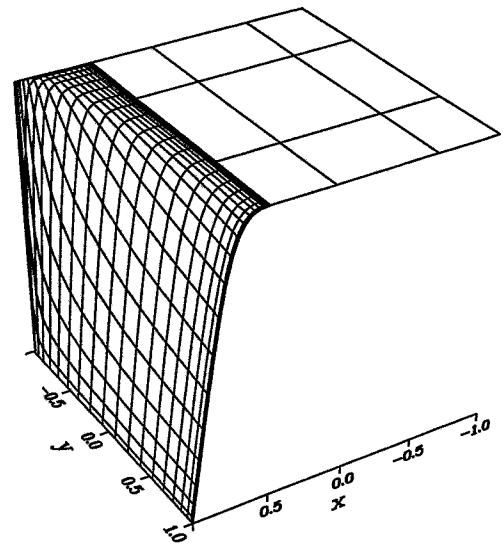

Fig. 4. Numerical simulation of the boundary layer problem (2.10), (2.11), (2.13).

Notice that in both cases reported in Figs. 3 and 4 the spectral approximation is of *non-conforming* type.

We show here below that the issue of interface conditions for heterogeneous equations is a very delicate task, and that our theory in this respect is sharp. A fairly common approach has been to solve everywhere in Ω the purely convective problem, then to correct the inviscid solution by solving the complete convection-diffusion problem upon Ω_2 solely. For the sake of clarity, we can refer to this method as to *viscous-correction* approach. The key step in this method is how to provide boundary data on Γ for the viscous correction. The most natural approach in this respect is to satisfy either the continuity of the solution (Dirichlet condition) or else the balance of the fluxes (Neumann one) across the interface. Both approaches can reveal strongly unadequate, as shown in next examples. We still consider problem (2.9) where:

$$\mathbf{b} = (x_1(x_1 - 0.8), 0)^t \ , \quad c = 2 \ , \quad f = 0 \ , \tag{2.14}$$

with boundary conditions: $w = 1$ on the sides $x_1 = -1$, $x_1 = 1$; $w = 0$ on $x_2 = -1$ and $0.5 \leq x_1 < 1$; $\frac{\partial w}{\partial x_2} = 0$ on $x_2 = 1$, $x_2 = -1$ and $-1 < x_1 < 0.5$.

We first solve the convection equation (2.10) in the whole domain Ω. Then, setting $\Omega_2 = (0.5, 1) \times (-1, 1)$ and denoting by Γ the vertical side $x_1 = 0.5$, we solve the complete problem (2.11) in Ω_2, using as boundary data on Γ either the continuity condition $v = u$ (see Fig. 5) or else the flux balance condition $-\nu \frac{\partial v}{\partial n_\Gamma} + (\mathbf{b}_{|\Omega_2} \cdot \mathbf{n}_\Gamma) v = (\mathbf{b}_{|\Omega_1} \cdot \mathbf{n}_\Gamma) u$ (see Fig. 6).

In either case, the computed solution is dramatically different from the one obtained by the coupling method based on (2.10), (2.11), (2.4), (2.5) using the same degrees of freedom (see Fig. 7). Indeed, $\Gamma = \Gamma_{in}$ for the problem at hand, and therefore according to (2.4) and (2.5) we should enforce on Γ ***both*** continuity and flux balance. Figs. 5, 6, 7 all refer to the case $\nu = 10^{-2}$.

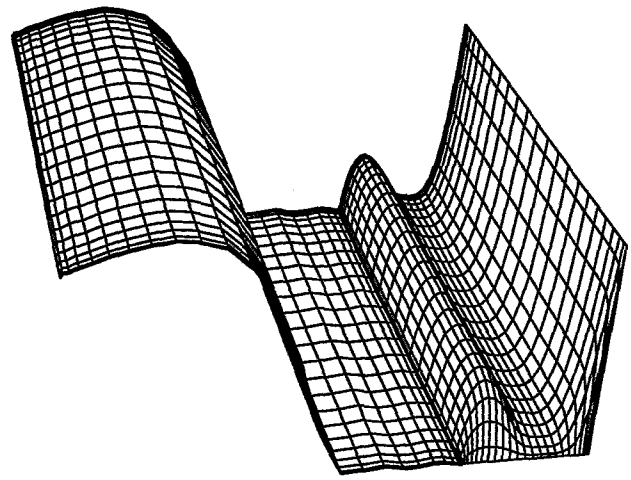

Fig. 5. Numerical simulation of problem (2.10), (2.11), (2.14) with the viscous-correction approach: continuity (but not flux balance) is fulfilled at the interface.

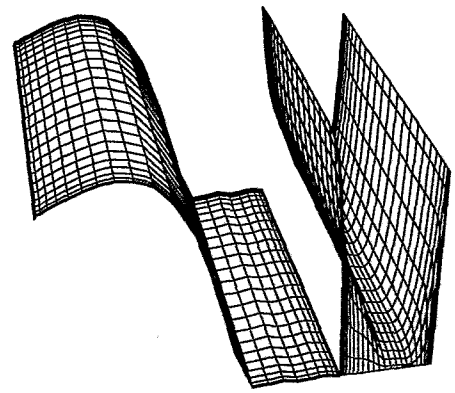

Fig. 6. Numerical simulation of problem (2.10), (2.11), (2.14) with the viscous-correction approach: flux balance (but not continuity) is fulfilled at the interface.

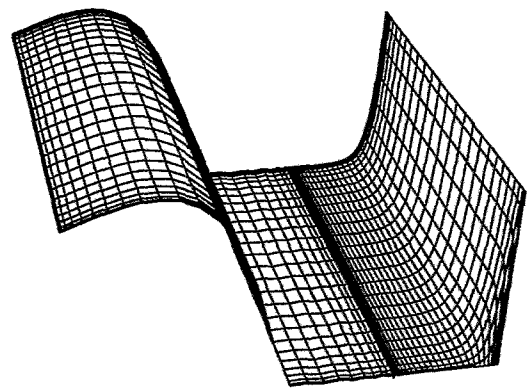

Fig. 7. Numerical simulation of problem (2.10), (2.11), (2.14) with the correct coupling method: both continuity and flux balance are fulfilled at the interface.

Finally, we show in Fig. 8 the maximum error on Γ between each of the solutions of Figs. 5, 6, 7 and the solution to the global viscous problem (2.9), for different values of the viscosity coefficient ν, and the same number of degrees of freedom.

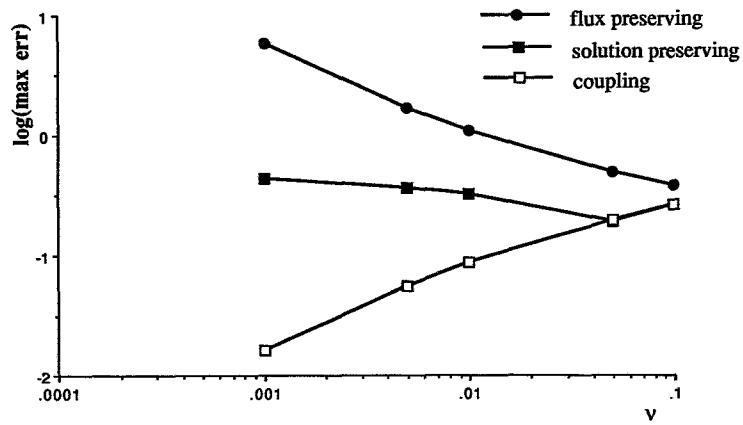

Fig. 8. Difference between the solutions displayed in Figs. 5, 6, 7 and the one to the global viscous problem (2.9).

3. Algorithmical aspects

Let us focus now on the practical resolution of the reduced problem (2.2)-(2.8).

A particularly simple situation occurs if, for a fixed time, the sign of $\beta_1 \equiv \mathbf{b}_{|\Omega_1} \cdot \mathbf{n}_\Gamma$ is constant on the whole interface Γ (this yields $\Gamma = \Gamma_{in}$ if $\beta_1 < 0$, whereas $\Gamma = \Gamma_{out}$ if $\beta_1 \geq 0$), as in this case the problems in Ω_1 and Ω_2 are decoupled one another.

(i) Suppose that $\Gamma = \Gamma_{in}$: using (2.4) one can rewrite (2.5) as

$$\frac{\partial v}{\partial n_\Gamma} = 0 \quad \text{on } \Gamma \,,$$

hence the problem in Ω_2 can be solved prior to that in Ω_1. The obtained value of $(\mathbf{b}_{|\Omega_2} \cdot \mathbf{n}_\Gamma)v$ on Γ is then used as inflow data to compute the solution u in Ω_1. Thus the solution of the reduced problem is obtained by only one sweep on Ω_2 and Ω_1.

(ii) A similar situation occurs when $\Gamma = \Gamma_{out}$. In this case condition (2.4) disappears, and the inviscid problem in Ω_1 can be solved prior to that in Ω_2. The value of $(\mathbf{b}_{|\Omega_1} \cdot \mathbf{n}_\Gamma)u$ that we obtain on Γ is then used as a known data for the flux of v, allowing the calculation of v in Ω_2.

The same conclusion holds for steady problems. In this respect, let us notice that both Figs. 2 and 4 refer to a situation of type (ii), whereas Fig. 7 refers to one of type (i).

These examples shed light on a further advantage enjoyed by the coupling approach, namely the possibility (in many physical applications) of achieving the global solution by solving only once two subproblems, one in Ω_1 and the other in Ω_2. (A domain decomposition method for the full viscous problem (2.9) would instead require iterating between Ω_1 and Ω_2). Of course, this remark about subdomain iterations doesn't concern time-dependent problems with *explicit* time-stepping.

(iii) In a general case, the solution to the reduced problem (at any given time) can be achieved as limit of solutions of the two subproblems (2.10) and (2.11). Indeed, owing to (2.4) we can restate the interface condition (2.5) as follows:

$$(2.5)_1 \qquad \frac{\partial v}{\partial n_\Gamma} = 0 \quad \text{on } \Gamma_{in}$$

$$(2.5)_2 \qquad (\mathbf{b}_{|\Omega_2} \cdot \mathbf{n}_\Gamma) v - \nu \frac{\partial v}{\partial n_\Gamma} = (\mathbf{b}_{|\Omega_1} \cdot \mathbf{n}_\Gamma) u \quad \text{on } \Gamma_{out} \ .$$

At each iteration, we can therefore solve problem (2.2) in Ω_1 with boundary conditions (2.6) and (2.4) (with the right hand side computed from previous iteration through a relaxation procedure). Then we solve problem (2.3) in Ω_2 with boundary conditions (2.7), (2.8), $(2.5)_1$ and $(2.5)_2$ (for the latter equation, the right hand side is taken from the preceding calculation in Ω_1). Same iterative procedure applies for the steady problem as well.

This iterative method has been introduced and thoroughly analyzed in [GQS]. For the sake of notation, we will name it the Dirichlet/Neumann (D/N) method. Details on its implementation are discussed in [FPQ]. It is shown in [GQS] that its rate of convergence is practically independent of all problem's data, including the number of degrees of freedom involved by the numerical approximation. We provide an example by considering the steady problem (2.10), (2.11) where

$$(3.1) \qquad \mathbf{b} = \beta(-x_2, x_1)^t \ \ (\beta > 0) \ , \ \ c = 1 \ , \ \ f = 0 \ ,$$

with Dirichlet conditions $u = 0$ on $\{(1, x_2) \,|\, -1 < x_2 < 0\}$ and $u = 1$ on the subsets $\{(1, x_2) \,|\, 0 \le x_2 \le 1\}$, $\{(x_1, 1) \,|\, -1 \le x_1 \le 0\}$, $\{(-1, x_2) \,|\, -1 \le x_2 \le 0\}$ and $\{(x_1, -1) \,|\, 0 \le x_1 \le 1\}$, and homogeneous Neumann condition on the remaining part of the boundary. We use the decomposition $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 = (-1, 1 - \sqrt{\nu}) \times (-1, 1)$ and $\Omega_2 = (1 - \sqrt{\nu}, 1) \times (-1, 1)$; then $\mathbf{b} \cdot \mathbf{n}_\Gamma$ is positive on the lowest half of the interface Γ, whereas it is negative on the upper half of Γ. The number of subdomain iterations needed to achieve convergence up to a tolerance of 10^{-8} is included between 5 and 12 for a broad range of data, namely

$$10^{-3} \le \nu \le 10^{-1} \ , \ \ 1 \le \beta \le 15 \ , \ \ 20 \le M \le 800 \ ,$$

where M denotes the total number of gridpoints of the spectral collocation problem.

In [QV1] it has been proven that the D/N method can be interpreted as a gradient method (with a suitable preconditioning) for the solution of the interface Steklov-Poincaré problem

$$\mathcal{S}_{in}\lambda_{in} = \phi_{in} \quad \text{on } \Gamma_{in} \ , \tag{3.2}$$

where λ_{in} is the value of v on Γ_{in}, ϕ_{in} is a known function (depending on the data of (2.9)), while $\mathcal{S}_{in}$ is a suitable interface operator. Precisely, (3.2) is obtained from the Steklov-Poincaré equation

$$\mathcal{S}\lambda = \phi \quad \text{on } \Gamma \tag{3.3}$$

(where now $\lambda = v$ on Γ) by formally eliminating $\lambda_{out} = \lambda_{|\Gamma_{out}}$.

At the algebraic level, $\mathcal{S}$ is the Schur complement (with respect to λ) of the matrix associated with the reduced problem, whilst $\mathcal{S}_{in}$ is the Schur complement of $\mathcal{S}$ with respect to λ_{in}.

The latter problem could be faced directly by a preconditioned gradient method. Splitting $\mathcal{S}$ as $\mathcal{S}_1 + \mathcal{S}_2$, where the subindex refers to the subdomain, the preconditioner we take is $\mathcal{S}_2$ (a spectrally optimal one for the full viscous case whenever the convective field $\mathbf{b}$ vanishes).

We have tried both the Richardson method with minimum residual (e.g., [Y]) and the so-called CGSTAB method ([V]) (the latter is likely among the best versions of conjugate gradient methods nowadays available for non-symmetric systems). In Fig. 9 we compare their performances with the one of the D/N method, for several values of N (the degree of polynomial solution within each subdomain). The test problem is still that relative to the data (3.1), with $\beta = 1$ and $\nu = 10^{-2}$.

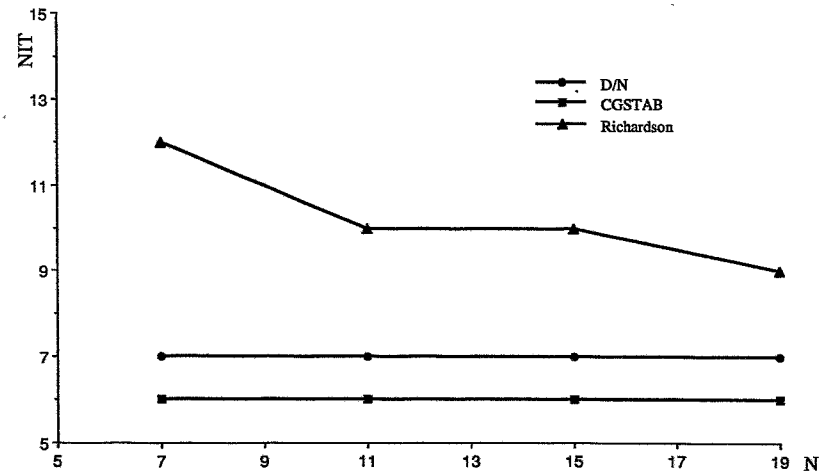

Fig. 9. Comparison between D/N iterations for system (3.2) and CGSTAB and Richardson iterations for system (3.3).

The D/N method needs about the same number of iterations as the CGSTAB method, however we should keep in mind that, for each iteration, the latter method is three times more expensive than the former one. Furthermore, the D/N method is carried out on problem (3.2), which is of smaller size than problem (3.3).

4. Interface location and sensitivity analysis

According to our theory, from the interface conditions (2.4), (2.5) it turns out that the solution to the reduced problem may be discontinuous on Γ_{out}. The jump, however, is comparable with the magnitude of the viscosity ν whenever the viscous domain Ω_2 is large enough to embody the boundary or internal layers (if any). Despite a rigorous proof is available for one-dimensional problems only ([GQ]), a broad numerical experience supports this conclusion.

We present two elementary examples of steady problems. The former is the one-dimensional equation

$$-\nu w_{xx} + w_x = 1 \ , \quad 0 < x < 1 \ , \quad w(0) = w(1) = 0 \ , \tag{4.1}$$

whose solution has a boundary layer of thickness ν around $x = 1$. Denoting with α the abscissa of the interface point, the jump between the solution v in $\Omega_2 = (\alpha, 1)$ and u in $\Omega_1 = (0, \alpha)$ is given by $j(\alpha) = (1 + \nu)exp[-\nu^{-1}(1 - \alpha)] - \nu$. The function $\nu^{-1}j(\alpha)$ is plotted in Fig. 10 for several values of ν. We see that $j(\alpha) \sim \nu$ as soon as $meas(\Omega_2)$ is asymptotically larger than ν (e.g., $meas(\Omega_2) = \sqrt{\nu}$).

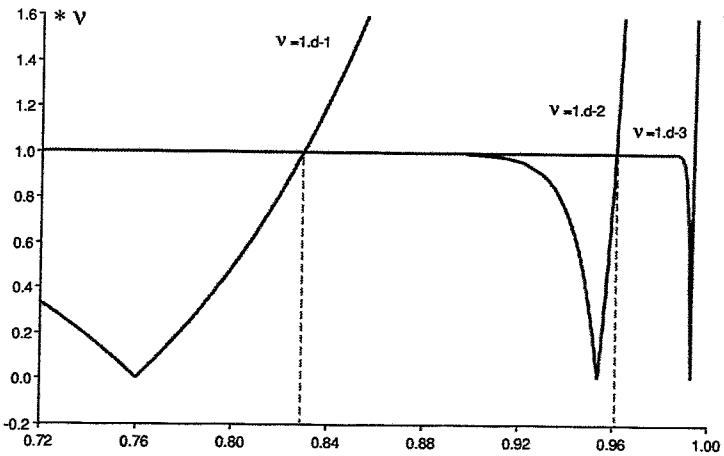

Fig. 10. Interface jump for the coupled approximation to problem (4.1).

The second example is provided by the equation

$$(4.2) \qquad \begin{cases} -\nu\Delta w + w_{x_1} = 0 \ \text{ in } \ \Omega = (0,1)^2 \\ w = 0 \ \text{ on the side } x_1 = 0 \\ w = cos\pi x_2 \ \text{ on the side } x_1 = 1 \\ w_{x_2} = 0 \ \text{ on the sides } x_2 = 0 \text{ and } x_2 = 1 \ . \end{cases}$$

Again, there is a boundary layer of thickness ν by the side $x_1 = 1$. If we take $\Omega_1 = (0,\alpha) \times (0,1)$, $\Omega_2 = (\alpha,1) \times (0,1)$ for $0 < \alpha < 1$, and replace (4.2) by a reduced viscous-inviscid problem like (2.10), (2.11), the jump between the two solutions u and v across Γ is

$$j(\alpha) = \nu p\, exp(\lambda^+\alpha)[(1-\nu\lambda_-)exp(\lambda^+) - (1-\nu\lambda^+)exp(\lambda_- + \alpha p)]^{-1} \ ,$$

where

$$\lambda^+_- = 1/(2\nu) \pm p/2 \ , \quad p = 4\pi^2 - \nu^{-2} \ .$$

We plot $\nu^{-1}j(\alpha)$ in Fig. 11 for several values of ν. The same conclusion as before holds for this case as well (with an even better behaviour).

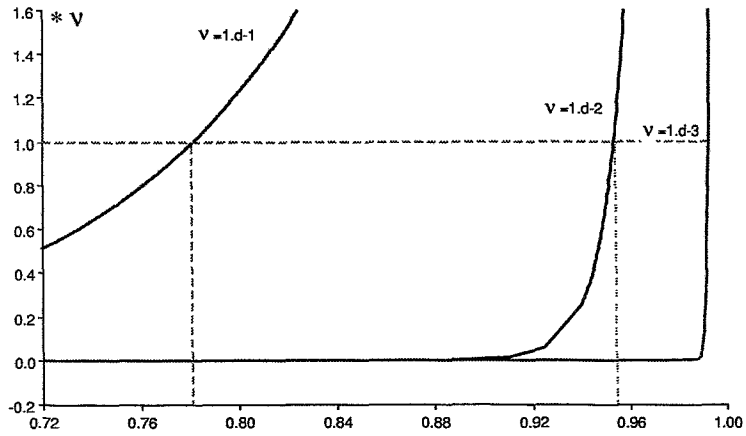

Fig. 11. Interface jump for the coupled approximation to problem (4.2).

Based on available theory and on our own experience, our conclusion is that the solution to the reduced problem is quite insensitive to interface's location and shape, whenever the interface itself is placed *outside* sharp layers.

For those applications in which no reliable guess is possible about either the location and the shape of boundary and internal layers, an adaptive procedure oughts to be devised in order to design the partition of Ω into Ω_1 and Ω_2. Along this direction, a non-linear (free-boundary like) approch is pursued in [BCR], [AC]. Alternatively, a control problem for interface shape optimization can be set up as follows. For the sake of simplicity, let us refer to a geometrical situation like that in Fig. 12, and to the steady problem (2.9).

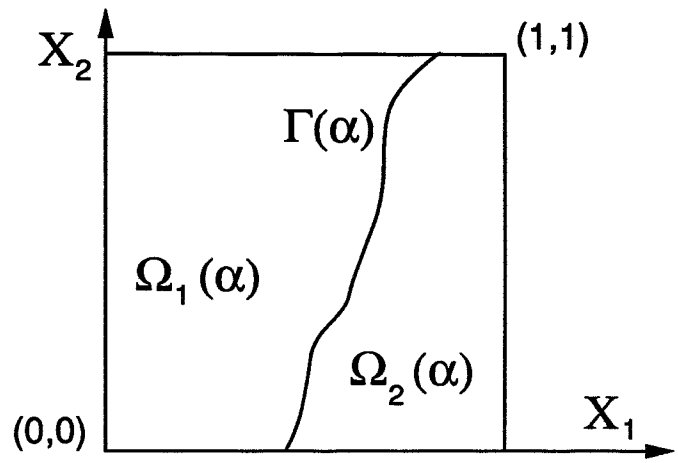

Fig. 12. Subdomain partition for interface shape optimization.

We look for

$$\Gamma(\alpha) = \{x_2 \in [0,1] \,|\, x_1 = \alpha(x_2)\} \quad , \; \alpha \in \mathcal{U}_{ad} \quad ,$$

where, e.g., $\mathcal{U}_{ad}$ is the space of Lipschitz equicontinuous admissible functions, such that the following cost functional is minimized:

$$\mathcal{I}(u(\alpha), v(\alpha)) := \int_{\Gamma(\alpha)} |\mathbf{b}_{|\Omega_1(\alpha)} \cdot \mathbf{n}_{\Gamma(\alpha)} u(\alpha) - \mathbf{b}_{|\Omega_2(\alpha)} \cdot \mathbf{n}_{\Gamma(\alpha)} v(\alpha)|^2 d\gamma \quad . \tag{4.3}$$

Here, the state equations for $u = u(\alpha)$ and $v = v(\alpha)$ are given by (2.10), (2.11), (2.4)-(2.8) (now with Ω_1, Ω_2 and Γ replaced by $\Omega_1(\alpha)$, $\Omega_2(\alpha)$ and $\Gamma(\alpha)$, respectively). The cost functional monitors the jump between viscous and inviscid solution across $\Gamma_{out}(\alpha)$.

Within any gradient (or quasi-gradient) procedure to solve the above minimization problem, a stopping criterium could be $\mathcal{I} \sim |\nu|^2$, according to our previous remark.

Despite our framework doesn't differ very much from more classical situations (e.g., [HN]), an existence, uniqueness and convergence analysis for this problem is not yet available to date.

5. Nonlinear problems

In this section we generalize our previuos approach to non-linear problems. We stick with a working example provided by the convection-diffusion-reaction equation:

$$\frac{\partial w}{\partial t} - div(\nu \nabla w) + div[\mathbf{f}(x,w)] + r(x,w) = 0 \quad , \; x \in \Omega \quad , \quad t > 0 \, , \tag{5.1}$$

supplemented with the initial condition $(2.1)_2$ and boundary conditions that we can still assume of the form $(2.1)_3$, $(2.1)_4$. The convective term $\mathbf{f}$ is a C^1-vector function that depends non-linearly on w, while the reaction term r may depend linearly or non-linearly on w.

We are not concerned here with the issue of stating sharp conditions on ν, f and r that ensure the non-singularity of the above problem (for this, see, e.g., [Sm]). Rather, we assume that along a time interval $0 < t \leq T$ there exists a unique solution, and deal with the reduction of (5.1) to a coupled problem. Then, at a given time t, Ω is partitioned into two disjoint subdomains Ω_1 and Ω_2, whose common interface is again denoted by Γ.

We look for the solution to the reduced problem:

$$\frac{\partial u}{\partial t} + div[\mathbf{f}(x,u)] + r(x,u) = 0 \quad , \; x \in \Omega_1 \quad , t > 0 \tag{5.2}$$

$$\frac{\partial v}{\partial t} - div(\nu \nabla v) + div[\mathbf{f}(x,v)] + r(x,v) = 0 \quad , \; x \in \Omega_2 \quad , t > 0 \; . \tag{5.3}$$

A theoretical analysis on interface conditions for this non-linear problem is not available as yet. However, by analogy with the linear case we suggest the following matching across Γ:

$$\mathbf{f}(x,u) \cdot \mathbf{n}_\Gamma = \mathbf{f}(x,v) \cdot \mathbf{n}_\Gamma \quad \text{on } \Gamma_{in} \equiv \{x \in \Gamma \,|\, \mathbf{f}_w(x,u) \cdot \mathbf{n}_\Gamma < 0\} \tag{5.4}$$

$$\nu \frac{\partial v}{\partial n_\Gamma} - \mathbf{f}(x,v) \cdot \mathbf{n}_\Gamma = -\mathbf{f}(x,u) \cdot \mathbf{n}_\Gamma \quad \text{on } \Gamma \; . \tag{5.5}$$

Here $\mathbf{f}_w$ denotes differentiation of $\mathbf{f}(x,\cdot)$ with respect to its second argument. Notice that Γ_{in} itself depends on the solution u.

The coupled problem (5.2)-(5.5) can be advanced in time either by an implicit or by an explicit method. For the sake of exposition, let us consider the first order Euler method. Denoting by t^k and $t^{k+1} = t^k + \Delta t$ two subsequent time-levels, and by ϕ^k the value of any function ϕ at the time level t^k, the *explicit method* transform (5.2)-(5.3) into:

$$\frac{u^{k+1} - u^k}{\Delta t} + div[\mathbf{f}(x,u^k)] + r(x,u^k) = 0 \quad , x \in \Omega_1 \tag{5.6}$$

$$\frac{v^{k+1} - v^k}{\Delta t} - div(\nu \nabla v^k) + div[\mathbf{f}(x,v^k)] + r(x,v^k) = 0 \quad , x \in \Omega_2 \quad , \tag{5.7}$$

while interface conditions need to be fulfilled at the *new* time-level t^{k+1} and read:

$$\mathbf{f}(x,u^{k+1}) \cdot \mathbf{n}_\Gamma = \mathbf{f}(x,v^{k+1}) \cdot \mathbf{n}_\Gamma \quad \text{on } \Gamma_{in}^k = \{x \in \Gamma \,|\, \mathbf{f}_w(x,u^k) \cdot \mathbf{n}_\Gamma < 0\} \tag{5.8}$$

$$\nu \frac{\partial v^{k+1}}{\partial n_\Gamma} - \mathbf{f}(x, v^{k+1}) \cdot \mathbf{n}_\Gamma = -\mathbf{f}(x, u^{k+1}) \cdot \mathbf{n}_\Gamma \quad \text{on } \Gamma \ . \tag{5.9}$$

Thus (5.6) and (5.7) can be advanced first at all internal points of Ω_1 and Ω_2, respectively, then at each grid point on Γ a non-linear algebraic equation remains to be solved. (Equation (5.8) oughts to be enforced on Γ_{in}^{k+1}, which for the sake of simplicity is approximated by Γ_{in}^{k}).

Using instead the *implicit method* would lead to the equations:

$$\frac{u^{k+1} - u^k}{\Delta t} + div[\mathbf{f}(x, u^{k+1})] + r(x, u^{k+1}) = 0 \quad , x \in \Omega_1 \tag{5.10}$$

$$\frac{v^{k+1} - v^k}{\Delta t} - div(\nu \nabla v^{k+1}) + div[\mathbf{f}(x, v^{k+1})] + r(x, v^{k+1}) = 0 \quad , x \in \Omega_2 \ , \tag{5.11}$$

still supplemented with the interface conditions (5.8), (5.9).

A possible drawback intrinsic with the approach (5.2)-(5.5) is that the reduced equation (5.2) might exhibit discontinuous solutions in Ω_1, whereas the original problem doesn't. The alternative approach is to resort to a reduced problem *only after linearizing* (in some way) the original equation in the whole domain. The linearization procedure yields at the time-level t^{k+1} a linear convection-diffusion equation (plus a reaction term) to which the sound theory of Sect. 2 can be applied straightforwardly.

A first example of linearization is provided by any suitable *semi-implicit time marching scheme* for (5.1). For the sake of clarity, let us consider a very simple first order one, that reads as:

$$\frac{w^{k+1} - w^k}{\Delta t} - div(\nu \nabla w^{k+1}) + \mathbf{f}_w(x, w^k) \cdot \nabla w^{k+1} = -R^{k,k+1} \quad , k \geq 0 \ . \tag{5.12}$$

The term $R^{k,k+1}(w)$ accounts for the reaction term $r(x, w)$ as well as for $\sum_j \frac{\partial f_j}{\partial x_j}$. The interface treatment for (5.12) is not influenced by the nature of the zero order term $R^{k,k+1}$. We therefore don't mind whether this term has been dealt with explicitly, implicitly or else semi-implicitly.

The left hand side of equation (5.12) fits into the general form of the linear, steady equation (5.9). A coupled approach is therefore in order, and the reduced problem reads as follows (with obvious notations):

$$\mathbf{f}_w(x, u^k) \cdot \nabla u^{k+1} + S^{k,k+1}(u) = 0 \ \text{ in } \Omega_1 \tag{5.13}$$

$$-div(\nu \nabla v^{k+1}) + \mathbf{f}_w(x, v^k) \cdot \nabla v^{k+1} + S^{k,k+1}(v) = 0 \ \text{ in } \Omega_2 \ , \tag{5.14}$$

with the linear interface conditions

$$\mathbf{f}_w(x, u^k) \cdot \mathbf{n}_\Gamma u^{k+1} = \mathbf{f}_w(x, v^k) \cdot \mathbf{n}_\Gamma v^{k+1} \quad \text{on } \Gamma_{in}^k \tag{5.15}$$

$$\nu \frac{\partial v^{k+1}}{\partial n_\Gamma} - \mathbf{f}_w(x, v^k) \cdot \mathbf{n}_\Gamma v^{k+1} = -\mathbf{f}_w(x, u^k) \cdot \mathbf{n}_\Gamma u^{k+1} \quad \text{on } \Gamma \ . \tag{5.16}$$

The solution algorithm of Sect. 3 applies straightforwardly, as well as the convergence theory that we have carried out on the linear problem.

Another linearization procedure can be accomplished as follows. We start with a fully implicit scheme to advance (5.1) from t^k to t^{k+1}, then the resulting equation is linearized by a *Newton method.* If we start with the Euler method, this process yields

$$F'(W_p)(W_{p+1} - W_p) = -F(W_p) \ , \ p \geq 1 \ ,$$

with $W_0 = w^k$ and $lim_p W_p = w^{k+1}$, where

$$F(z) := \frac{z - w^k}{\Delta t} - div(\nu \nabla z) + div[\mathbf{f}(x, z)] + r(x, z)$$

and $F'(W_p)$ denotes the Jacobian of F computed at the point W_p.

Each Newton iterate yields therefore a problem that can still be cast under the general form (2.9), thus the reduced approch (2.10), (2.11) with (2.4)-(2.8) applies straightforwardly.

From the computational viewpoint, in several circumstances the fully explicit method (5.6)-(5.9) is the cheapest one. The semi-implicit method (5.13)-(5.16) is however a viable alternative, that moreover lies on solid mathematical background.

A numerical simulation for a problem like (5.1) in a two-dimensional domain, based on the semi-implicit (coupled) procedure (5.13)-(5.16), is presented in [FPQ]. Here we show another application to the one-dimensional Burgers problem:

$$\begin{cases} w_t - \nu w_{xx} + w w_x = 0 \ , \ -1 < x < 1 \ , \ t > 0 \\ w(x, 0) = arctg[5(x - 0.9)] \ , \ -1 < x < 1 \\ w(1, t) = (1 + t) arctg(0.5) \ , \ t > 0 \\ w_x(0, t) = 0 \ , \ t > 0 \ . \end{cases} \tag{5.17}$$

At each $t > 0$ the space interval is partitioned into $\Omega_1 = (-1, \alpha)$, $\Omega_2 = (\alpha, 1)$, and the space discretization is accomplished through a spectral collocation method using N nodes in either domain. We compute at time $t = 10$ the solution w_N obtained solving in both domains the viscous Burgers equation (5.17), and with w_N^* the one achieved by coupling the viscous equation in Ω_2 with the reduced inviscid equation in Ω_1. We have chosen $N = 45$ and the interface α is set to 0.7. Notice that for all time $0 \leq t \leq 10$, $x = 0.7$ is an inflow boundary for the subdomain Ω_1.

We report in Fig. 13 the logarithm of the absolute value of the difference between w_N and w_N^* correspondingly to the explicit time-stepping (5.6)-(5.9), with $\Delta t = 10^{-5}$ and two values of ν. For both cases, the difference is at most equal to ν.

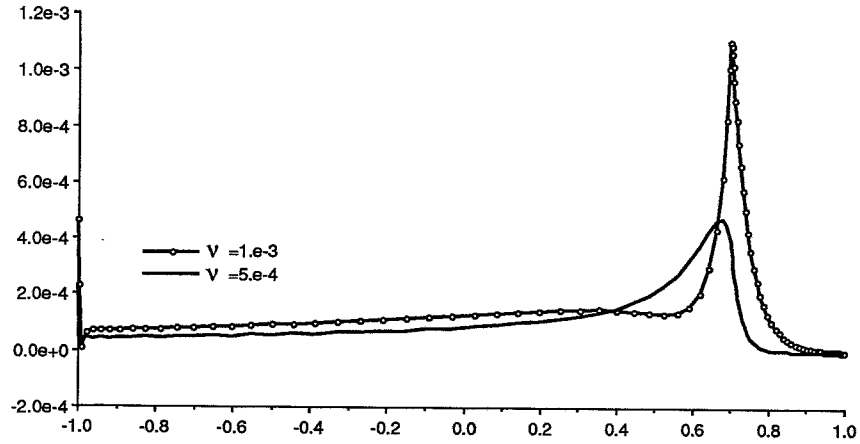

Fig. 13. Comparison between fully viscous and viscous-inviscid approximations to problem (5.17).

Concerning stability, no theoretical analysis is available, though a more stringent time-step restriction is expected for the fully explicit method. We illustrate this fact on the problem at hand, by showing in Tab. 14 for several values of N and ν the ratio $K_{\Delta t}$ between Δt_{si}, the allowable Δt for the semi-implicit method, and Δt_{ex}, the one for the explicit method (the latter being dictated by the CFL condition). Correspondingly, we also report K_{CPU}, the ratio between the CPU-time (on the CRAY 2) of the explicit method (with Δt_{ex}) and that of the semi-implicit one (with Δt_{si}) for a simulation along the time interval $0 \leq t \leq 10$.

	$K_{\Delta t}$			K_{CPU}		
ν / N	0.005	0.001	0.0005	0.005	0.001	0.0005
60	> 28000	814	277	179.1	30.3	10.5
70	> 56000	1314	571	299.5	57.0	26.7
80	> 110000	3200	1086	468.6	121.1	41.3

Tab. 14. The values of $K_{\Delta t}$=$\Delta t_{si}/\Delta t_{ex}$ and K_{CPU}=(CPU-time)$_{ex}$/(CPU-time)$_{si}$.

6. Other applications

Computational aerodynamics is a field that may strongly benefit from a heterogeneous domain decomposition approach. A typical instance is represented by flow simulation around an airfoil or else over a flat plate.

In both cases, the full Navier-Stokes equations need to be solved in a vicinity of the body (including the wake region). However, sufficiently far from the body the flow motion can be modeled by the inviscid Euler equations, or else (whenever the flow is vorticity free), by the full potential equation for the velocity potential. The first approach is pursued in [CQV2], the latter in [GPT].

The coupling of the different set of equations can be accomplished either directly on the original non-linear problem, or else on suitable subproblems. The latter can be obtained, e.g., by the effect of Newton-like linearization procedures on the global non-linear (steady) problem and/or by operator splitting techniques (e.g., [CQV2]), or else by advancing in time by semi-implicit methods ([CQV1]). In all such cases, one resort to convection-diffusion problems like (5.1) (discretized in time) or (5.12), and/or to a generalized form of the Stokes problem (see [BGP], [CQV2]). It is precisely on these subproblems that a coupling procedure can be accomplished (besides the preceding references, see, e.g., [QSV], [CZ]).

We conclude mentioning a different kind of problem, the one of wave propagation in heterogeneous media (e.g., [DL]), whose mathematical description is provided by the Maxwell equations. If the electric conductivity can be assumed to vanish upon one of the media, we are left with the problem of coupling different kind of equations through an interface that represents now the physical separation between the media. This problem, which is faced in [QV2], has a straightforward solution in a two-dimensional case, while a sound treatment is requested in three dimensions (for the case of strictly positive conductivity, see, e.g., [KN]).

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