CHAPTER 10

Iterative Methods by SPD and Small Subspace Solvers for Nonsymmetric or Indefinite Problems

Jinchao Xu*

Abstract. This paper is devoted to a class of iterative methods for solving nonsymmetric or indefinite problems that are dominated by some SPD (symmetric positive definite) problems. The algorithm is based on a direct solver for the original equation restricted on a small subspace and a given iterative method for the SPD equation. It is shown that any convergent iterative method for the SPD problem will give rise to an algorithm that converges with a comparable rate if the small subspace is properly chosen. Furthermore a number of preconditioners that can be used with GMRES type methods are also obtained.

1. Introduction. In this paper, we shall study a class of iterative methods for solving nonsymmetric or indefinite equations that are governed by some SPD systems. Straight iterative schemes as well as preconditioning techniques will be discussed.

This paper is based on some early work by Xu and Cai [12] and Xu [10]. In [12], a class of preconditioners are presented for GMRES type algorithms and in [10] a class of linear iterative methods are developed. The algorithms in both [12] and [10] are built upon a small subspace solver and a given iterative method for the SPD operator that governs the equation, but the techniques used in these two papers are quite different. In this paper, we shall give a unified treatment for these algorithms, present some improved estimates and also propose some new preconditioners. We would like to mention that certain modifications for the algorithms in [10] have been made by Bramble, Leyk and Pasciak [1].

2. Preliminaries. We assume that $\mathcal{V}$ is a given linear vector space $\mathcal{V}$ equipped with an inner product $(\cdot,\cdot)$. Let $L(\mathcal{V})$ denote the space of all linear operators from $\mathcal{V}$ to itself. We are interested in solving the equation

$$\hat{A}u = f, \tag{2.1}$$

for a given $f \in \mathcal{V}$. Here $\hat{A} \in L(\mathcal{V})$ is a given invertible operator satisfying

$$\hat{A} = A + N,$$

* Department of Mathematics, The Pennsylvania State University, University Park, PA 16802. *email:* xu@math.psu.edu. This work was supported by the National Science Foundation.

and $A \in L(\mathcal{V})$ is SPD in the sense that

$$(Au, v) = (u, Av) \quad \forall u, v \in \mathcal{V} \quad \text{and} \quad (Av, v) > 0 \quad \text{if} \quad v \neq 0;$$

the perturbation operator $N \in L(\mathcal{V})$ is not SPD in general.

As A is SPD, $(\cdot,\cdot)_A = (A\cdot,\cdot)$ defines an inner product on $\mathcal{V}$ and induces a norm on $\mathcal{V}$, denoted by $\|\cdot\|_A$. Given $G \in L(\mathcal{V})$, we define its A-norm by

$$\|G\|_A = \sup_{v \in \mathcal{V}} \frac{\|Gv\|_A}{\|v\|_A}.$$

The construction of an iterative algorithm for (2.1) often amounts to the construction of a $\hat{B} \in L(\mathcal{V})$ which behaves like $\hat{A}^{-1}$. One approach is to use $\hat{B}$ to obtain a linear iterative scheme as follows

$$u^{k+1} = u^k + \hat{B}(f - \hat{A}u^k), \tag{2.2}$$

for $k = 0, 1, 2, \cdots$, and any $u^0 \in \mathcal{V}$. Obvioulsy a sufficient condition for the convergence of scheme (2.2) is

$$\eta = \|I - \hat{B}\hat{A}\|_A < 1,$$

and in this case

$$\|u - u^k\|_A \le \eta^k \|u\|_A.$$

Another approach is to use $\hat{B}$ as a preconditioner for (2.1) in conjunction with GMRES type methods (c.f. [6], [7]). Unlike the conjugate gradient method for SPD problem, the GMRES method may not be convergent without proper preconditioning. A preconditioner for the GMRES method is not only to speed up the convergence but more importantly to guarantee the convergence as well. More precisely, if there are two constants $\alpha_0, \alpha_1 > 0$ such that

$$(\hat{B}\hat{A}v, v)_A \ge \alpha_0 (v, v)_A, \quad \|\hat{B}\hat{A}v\|_A \le \alpha_1 \|v\|_A, \quad \forall v \in \mathcal{V},$$

then, the GMRES method applying to the preconditioned system

$$\hat{B}\hat{A}u = \hat{B}f$$

with the inner product $(\cdot,\cdot)_A$ converges at the rate $1 - \alpha_0^2/\alpha_1^2$ (cf. [6]).

Now we assume that a subspace $\mathcal{V}_0 \subset \mathcal{V}$ is given, we define an operator $\hat{A}_0 : \mathcal{V}_0 \mapsto \mathcal{V}_0$, and three projections $Q_0, P_0, \hat{P}_0 : \mathcal{V} \mapsto \mathcal{V}_0$ by, for all $u_0, v_0 \in \mathcal{V}_0$,

$$(\hat{A}_0 u_0, v_0) = (\hat{A}u_0, v_0),$$

and for all $u \in \mathcal{V}, v_0 \in \mathcal{V}_0$

$$(AP_0 u, v_0) = (Au, v_0), \quad (\hat{A}\hat{P}_0 u, v_0) = (\hat{A}u, v_0), \quad (Q_0 u, v_0) = (u, v_0).$$

It is clear that $\hat{A}_0$, P_0 and Q_0 are well defined. We shall assume that $\hat{A}_0$ is invertible, which implies that $\hat{P}_0$ is also well-defined.

By the definitions of $\hat{P}_0, \hat{A}_0$ and Q_0,

$$\hat{A}_0 \hat{P}_0 = Q_0 \hat{A}.$$

It follows that, for a given $f \in \mathcal{V}$,

$$\hat{u}_0 = \hat{A}_0^{-1} Q_0 f \quad \text{if and only if} \quad (\hat{A}\hat{u}_0, v_0) = (f, v_0), \quad \forall v_0 \in \mathcal{V}_0.$$

Many estimates in this paper will be established in terms of the following parameter

$$\delta_0 = \sup_{u,v \in \mathcal{V}} \frac{(N(I-\hat{P}_0)u, v)}{\|u\|_A \|v\|_A}. \tag{2.3}$$

The assumption that we shall make late is that δ_0 can be sufficiently small if the subspace $\mathcal{V}_0$ is properly chosen.

In the study of preconditioners, we need to use another parameter defined by

$$\bar{\delta} = \sup_{u,v \in \mathcal{V}} \frac{(Nu, v)}{\|u\|_A \|v\|_A}. \tag{2.4}$$

It is easy to see that

$$\|A^{-1}N\|_A \le \bar{\delta}. \tag{2.5}$$

Observe that $\bar{\delta} = \delta_0$ if $\mathcal{V}_0 = \{0\}$. Without loss of generality, we assume that $\delta_0 \le \bar{\delta}$.

LEMMA 2.1. *For any $u \in \mathcal{V}$*

$$\|(\hat{P}_0 - P_0)u\|_A \le \delta_0 \|u\|_A, \quad \|u - \hat{P}_0 u\|_A \le (1+\delta_0)\|u\|_A. \tag{2.6}$$

Proof. It follows from the definitions of $\hat{P}_0$ and P_0 that

$$(A(\hat{P}_0 - P_0)u, v_0) = (N(I-\hat{P}_0)u, v_0), \quad \forall u \in \mathcal{V}, v_0 \in \mathcal{V}_0,$$

which, with $v_0 = (\hat{P}_0 - P_0)u$, implies the first inequality in (2.6). The second estimate obviously follows from the first one. □

3. Basic algorithms. In this section, a class of linear iterative methods and preconditioners will be presented in a unified framework.

3.1. Linear iterative algorithms. We now present the main algorithm proposed in Xu [10]. The algorithm depends on a given solver, represented by a $B \in L(\mathcal{V})$, for A satisfying

$$\|I - BA\|_A < 1.$$

ALGORITHM 3.1. *Given $u^0 \in \mathcal{V}$. Assume u^k is defined for $k \ge 0$, then*

1. Solve (exactly) the equation on $\mathcal{V}_0$:

$$\hat{A}_0 \hat{u}_0 = Q_0(f - \hat{A}u^k).$$

2. Set $g = f - \hat{A}(u^k + \hat{u}_0)$, for $i = 0, 1, \cdots, p$ and $v^0 = 0$

$$v^{i+1} = v^i + B(g - Av^i),$$

3. $u^{k+1} = u^k + \hat{u}_0 + v^p$.

Like in the classic multigrid method, the first step of the above algorithm plays the role of correction on the small subspace $\mathcal{V}_0$; the second step plays the role of smoothing (by the SPD operator A).

Let us derive the error equation of the above algorithm. Without loss of generality, we assume that $p = 1$. Note that $f = \hat{A}u$, it follows that

$$\hat{u}_0 = \hat{P}_0(u - u^k) \quad \text{and} \quad v^1 = B\hat{A}(I - \hat{P}_0)(u - u^k).$$

Thus

$$u - u^{k+1} = (I - B\hat{A})(I - \hat{P}_0)(u - u^k).$$

Obviously the Algorithm 3.1 is identical to (2.2) if $\hat{B}$ satisfies

$$I - \hat{B}\hat{A} = (I - B\hat{A})(I - \hat{P}_0). \tag{3.1}$$

THEOREM 3.2. *Assume that $\hat{B}$ is given by* (3.1), *then*

$$\|I - \hat{B}\hat{A}\|_A \le \eta,$$

where

$$\eta = \rho^p + 3\delta_0, \quad \rho = \|I - BA\|_A. \tag{3.2}$$

Consequently

$$\|u - u^k\|_A \le (\rho^p + 3\delta_0)^k \|u - u^0\|_A,$$

where u^k are defined by Algorithm 3.1 and u is the solution of (2.1). *Therefore the Algorithm 3.1 is convergent if δ_0 is sufficiently small so that $3\delta_0 < 1 - \rho^p$.*

Proof. Without loss of generality, we assume that $p = 1$. Given $u \in \mathcal{V}$, denote $u_0 = \hat{P}_0 u, v = A^{-1}\hat{A}(u - u_0)$ and $w = u - u_0$. We shall first show that

$$\|w - v\|_A \le \delta_0 \|u\|_A, \quad \|v\|_A \le (1 + 2\delta_0)\|u\|_A. \tag{3.3}$$

In fact

$$\begin{aligned} \|w - v\|_A^2 &= (A(w - v), w - v) = ((A - \hat{A})(u - u_0), w - v) \\ &= -(N(u - u_0), w - v) \le \delta_0 \|u\|_A \|w - v\|_A. \end{aligned}$$

The first estimate in (3.3) then follows. To see the second estimate in (3.3), by LEMMA 2.3

$$\begin{aligned} \|v\|_A^2 &= (Av, v) = (\hat{A}(u - u_0), v) = (A(u - u_0), v) + (N(u - u_0), v) \\ &\le (1 + \delta_0)\|u\|_A \|v\|_A + \delta_0 \|u\|_A \|v\|_A \le (1 + 2\delta_0)\|u\|_A \|v\|_A. \end{aligned}$$

Therefore (3.3) is justified.

Thanks to (3.3), the rest of the proof is easy:

$$\begin{aligned} &\|(I - B\hat{A})(I - \hat{P}_0)u\|_A = \|w - B(Av)\|_A \\ &\le \|w - v\|_A + \|v - B(Av)\|_A \le \delta_0 \|u\|_A + \rho \|v\|_A \\ &\le (\delta_0 + \rho(1 + 2\delta_0)) \|u\|_A \le (\rho + 3\delta_0) \|u\|_A \end{aligned}$$

as desired. □

3.2. Preconditioners for GMRES type methods. Based on the theory just developed, a number of preconditioners can be derived in a straightforward fashion.

In particular the preconditioners presented in Xu and Cai [12] can be obtained easily with weaker assumptions.

First, as a direct consequence of Theorem 2.1, we have

THEOREM 3.3.

$$\hat{B} = (I - B\hat{A})\hat{A}_0^{-1}Q_0 + B, \tag{3.4}$$

Then, for all $v \in \mathcal{V}$

$$(\hat{B}\hat{A}v, v)_A \ge (1-\eta)(v,v)_A, \quad \|\hat{B}\hat{A}v\|_A \le (1+\eta)\|v\|_A.$$

The proof of the above theorem is straightforward and hence omitted.

We shall now derive the theory developed in [12].

THEOREM 3.4. *Let*

$$\hat{B} = \omega\hat{A}_0^{-1}Q_0 + B. \tag{3.5}$$

Then, for η *given by* (3.2) *and for all* $v \in \mathcal{V}$

$$(\hat{B}\hat{A}v, v)_A \ge \frac{1}{2}(1-\eta)(v,v)_A, \quad \|\hat{B}\hat{A}v\|_A \le (\omega+2)(1+\bar{\delta})\|v\|_A, \tag{3.6}$$

provided that ω *is sufficiently large and* δ_0 *is sufficiently small, e.g.*

$$\omega \ge \frac{(1+2\bar{\delta})^2}{1-\eta}, \quad \delta_0 \le \frac{1}{4}\frac{1-\eta}{\omega+1+2\bar{\delta}}. \tag{3.7}$$

Proof. Obviously

$$\begin{aligned} \hat{B}\hat{A} &= \omega\hat{P}_0 + B\hat{A} = (\omega - 1 + B\hat{A})\hat{P}_0 + \hat{P}_0 + B\hat{A}(I-\hat{P}_0) \\ &= (\omega - 1 + B\hat{A})P_0 + (\omega - 1 + B\hat{A})(\hat{P}_0 - P_0) + \hat{P}_0 + B\hat{A}(I - \hat{P}_0) \end{aligned}$$

By (2.5) and the fact that $\|I - BA\|_A < 1$, it is easy to show that

$$\|I - B\hat{A}\|_A \le 1 + 2\bar{\delta}.$$

Hence, by (2.6)

$$((\omega - 1 + B\hat{A})(\hat{P}_0 - P_0)v, v)_A \le (\omega + 1 + 2\bar{\delta})\delta_0\|v\|_A^2.$$

An application of Cauchy-Schwarz inequality gives

$$((I - B\hat{A})P_0v, v)_A \le \|I - B\hat{A}\|_A\|P_0v\|_A\|v\|_A \le \frac{1}{1-\eta}(1+2\bar{\delta})^2\|P_0v\|_A^2 + \frac{1-\eta}{4}\|v\|_A^2.$$

Combining the above two estimates with Theorem 3.3 yields

$$(\hat{B}\hat{A}v, v)_A \ge (\omega - \frac{(1+2\bar{\delta})^2}{1-\eta})\|P_0v\|_A^2 + \left(\frac{3(1-\eta)}{4} - (\omega + 1 + 2\bar{\delta})\delta_0\right)\|v\|_A^2.$$

The first estimate in (3.6) then follows if (3.7) holds. The rest of the proof is straigt-forward. □

We are now in a position to derive the main result in [12].

THEOREM 3.5. *Assume that $\bar{B}$ is a SPD preconditioner for A and*

$$\hat{B} = \omega \hat{A}_0^{-1} Q_0 + \bar{B}. \tag{3.8}$$

Then, for all $v \in \mathcal{V}$

$$(\hat{B}\hat{A}v, v)_A \geq \frac{\lambda_0 + \lambda_1}{4}\left(\frac{2\lambda_0}{\lambda_1 + \lambda_0} - 3\delta_0\right)A(v,v), \quad \|\hat{B}\hat{A}v\|_A \leq (\omega + 2)(1 + \bar{\delta})\frac{\lambda_0 + \lambda_1}{2}\|v\|_A,$$

provided that ω is sufficiently large and δ_0 is sufficiently small. Here

$$\lambda_0 = \lambda_{\min}(BA), \lambda_1 = \lambda_{\max}(BA).$$

Proof. Let $B = \frac{2}{\lambda_0 + \lambda_1}\bar{B}$. Then

$$\rho = \|I - BA\|_A \leq \frac{\lambda_1 - \lambda_0}{\lambda_1 + \lambda_0} < 1.$$

The desired result can be derived from Theorem 3.4. □

4. Subspace correction method. The algorithm we have studied above are based on a given iterative algorithm for the SPD problem. In this section, we shall discuss a special class of iterative methods for SPD problem and discuss the corresponding Algorithm 3.1 and its modification.

4.1. Subspace correction method for the SPD problem. Following the theory in Xu [9], many linear iterative methods for SPD problems can be formulated by space decomposition and subspace corrections. We shall now give a brief review of this theory.

The main ingredient of the theory is a decomposition of $\mathcal{V}$ that consists of a number of subspaces $\mathcal{V}_i \subset \mathcal{V}$, $(0 \leq i \leq J)$ such that

$$\mathcal{V} = \sum_{i=0}^{J} \mathcal{V}_i. \tag{4.1}$$

This means that, for each $v \in \mathcal{V}$, there exist $v_i \in \mathcal{V}_i$ $(0 \leq i \leq J)$, such that

$$v = \sum_{i=0}^{J} v_i.$$

This representation of v may not be unique in general.

For each i, define $Q_i : \mathcal{V} \mapsto \mathcal{V}_i$ and $A_i : \mathcal{V}_i \mapsto \mathcal{V}_i$ by

$$(Q_i u, v_i) = (u, v_i), \quad u \in \mathcal{V}, v_i \in \mathcal{V}_i, \quad (A_i u_i, v_i) = (A u_i, v_i), \quad u_i, v_i \in \mathcal{V}_i.$$

Note that A_i is SPD. The algorithm for the SPD system for A will be designed based on given algorithms for solving the subspace equations $A_i u_i = f_i$ for $f_i \in \mathcal{V}_i$. Again these algorithms are characterized by some SPD operators, denoted by R_i.

ALGORITHM 4.1. *Preconditioner for A:*

$$\bar{B} = \sum_{i=0}^{J} R_i Q_i. \tag{4.2}$$

ALGORITHM 4.2. *Given $v^0 \in \mathcal{V}$. Assume that $v^k \in \mathcal{V}$ is obtained. Then v^{k+1} is defined by*

$$v^{k+(i+1)/(J+1)} = v^{k+i/(J+1)} + R_i Q_i (f - A v^{k+i/(J+1)})$$

for $i = 0, 1, \cdots, J$.

Denote $T_i = R_i Q_i A$. A direct manipulation gives that

$$v - v^{k+1} = (I - BA)(v - v^k),$$

where

$$I - BA = (I - T_J)(I - T_{J-1}) \cdots (I - T_1)(I - T_0). \tag{4.3}$$

The theory for the above algorithms depends on two parameters, K_0 and K_1, defined as follows: for any $v \in \mathcal{V}$, there exists a decomposition $v = \sum_{i=1}^J v_i$ for $v_i \in \mathcal{V}_i$ such that

$$\sum_{i=0}^{J} (R_i^{-1} v_i, v_i) \le K_0 (Av, v);$$

and for any $S \subset \{0, 1, 2, \cdots J\} \times \{0, 1, 2, \cdots J\}$ and $u_i, v_i \in \mathcal{V}$ $(0 \le i \le J)$

$$\sum_{(i,j) \in S} (T_i u_i, T_j v_j)_A \le K_1 \left(\sum_{i=0}^{J} (T_i u_i, u_i)_A \right)^{\frac{1}{2}} \left(\sum_{j=0}^{J} (T_j v_j, v_j)_A \right)^{\frac{1}{2}}.$$

THEOREM 4.3. *For the preconditioner given by* (4.2),

$$\lambda_{\min}(BA) \ge K_0^{-1}, \quad \lambda_{\max}(BA) \le K_1, \quad \kappa(BA) \le K_0 K_1.$$

The Algorithm 4.2 converges if $\omega_1 = \max_i \lambda_{\max}(R_i A_i) < 2$, and furthermore

$$\|E_J\|_A^2 \le 1 - \frac{2 - \omega_1}{K_0 (1 + K_1)^2}. \tag{4.4}$$

The proof of the above theorem can be found in [9]. It has been shown in [9] that the above theorem provides optimal estimates for a large class of iterative methods including the classic multigrid methods, BPX multigrid preconditioner, additive and multiplicative domain decomposition methods and hierarchical basis methods.

4.2. Subspace correction method for (2.1). Suppose that $\mathcal{V}_0$ used in the definition of Algorithm 3.1 coincides with that in the decomposition (4.1). Then, if the Algorithm 3.1 is applied with Algorithm 4.2, the subspace problems on $\mathcal{V}_0$ are solved twice in each iteration, once for A_0 and once for $\hat{A}_0$. We shall remove the solver for A_0 from Algorithm 4.2 and modify the Algorithm 3.1 as follows:

ALGORITHM 4.4. *Given $u^0 \in \mathcal{V}$. Assume that u^k is defined for $k \ge 1$, then we define $u^{k+1} = \hat{u}^k + v^J$ where*

$$\hat{u}^k = u^k + \hat{A}_0^{-1} Q_0 (f - \hat{A} u^k)$$

and, for $i = 1, \cdots, J$,

$$v^i = v^{i-1} + R_i Q_i (g - A v^{i-1})$$

with $g = f - \hat{A}\hat{u}^k$ *and* $v^0 = 0$.

The error equation of the above algorithm is

$$u - u^{k+1} = (I - \tilde{B}\hat{A})(I - \hat{P}_0)(u - u^k)$$

where

$$I - \tilde{B}A = (I - T_J)(I - T_{J-1})\cdots(I - T_1). \tag{4.5}$$

THEOREM 4.5. *Assume that* $\omega_1 < 2$. *Then the Algorithm 4.4 converges if* δ_0, *given by* (2.3), *is sufficiently small. Furthermore the error operator* $\tilde{E} = (I - \tilde{B}\hat{A})(I - \hat{P}_0)$ *satisfies*

$$\|\tilde{E}\|_A \leq \eta$$

where

$$\eta = 1 + 5\delta_0 - \frac{2 - \omega_1}{K_0(1 + K_1)^2}. \tag{4.6}$$

Proof. Obviously, for B defined by (4.3) with $R_0 = A_0^{-1}$

$$I - BA = (I - \tilde{B}A)(I - P_0). \tag{4.7}$$

A direct manipulation yields

$$\begin{aligned}(I - \tilde{B}\hat{A})(I - \hat{P}_0) &= (I - B\hat{A})(I - \hat{P}_0)\\ &+ (I - \tilde{B}A)(P_0 - \hat{P}_0) + (B - \tilde{B})N(I - \hat{P}_0).\end{aligned}$$

Thus

$$\begin{aligned}\|(I - \tilde{B}\hat{A})(I - \hat{P}_0)u\|_A &\leq \|(I - B\hat{A})(I - \hat{P}_0)u\|_A\\ &+ \|(I - \tilde{B}A)(P_0 - \hat{P}_0)u\|_A + \|(B - \tilde{B})N(I - \hat{P}_0)u\|_A\\ &\equiv I_1 + I_2 + I_3.\end{aligned}$$

The estimate of I_1 is given by THEOREM 3.2

$$I_1 \leq (\rho + 3\delta_0)\|u\|_A$$

where $\rho = 1 - \frac{2-\omega_1}{K_0(1+K_1)^2}$ by Theorem 4.3. By the assumption on R_i, $\|I - T_i\|_A \leq 1$ which implies that $\|I - \tilde{B}A\|_A \leq 1$. Hence, by (2.6)

$$I_2 \leq \|(P_0 - \hat{P}_0)u\|_A \leq \delta_0\|u\|_A.$$

It remains to estimate I_3. We first note that, by (4.7), $(B - \tilde{B})A = (I - \tilde{B}A)P_0$. Thus

$$\|(B - \tilde{B})A\|_A = \|I - \tilde{B}A\|_A\|P_0\|_A \leq 1.$$

Let "t" and "$*$" denote the transpositions with respect to the inner products $(\cdot, \cdot)$ and $(A\cdot, \cdot)$ respectively, then

$$\|(B - \tilde{B})^t A\|_A = \|[(B - \tilde{B})A]^*\|_A = \|(B - \tilde{B})A\|_A \leq 1.$$

Consequently

$$\begin{aligned}\|(B - \tilde{B})N(I - \hat{P}_0)u\|_A^2 &= ((B - \tilde{B})N(I - \hat{P}_0)u, A(B - \tilde{B})N(I - \hat{P}_0)u)\\ &\leq \delta_0\|u\|_A\|(B - \tilde{B})^t A(B - \tilde{B})N(I - \hat{P}_0)u\|_A\\ &\leq \delta_0\|u\|_A\|(B - \tilde{B})^t A\|_A\|(B - \tilde{B})N(I - \hat{P}_0)u\|_A\\ &\leq \delta_0\|u\|_A\|(B - \tilde{B})N(I - \hat{P}_0)u\|_A.\end{aligned}$$

Hence

$$I_3 = \|(B - \tilde{B})N(I - \hat{P}_0)u\|_A \le \delta_0\|u\|_A.$$

The desired estimate then follows. □

4.3. Preconditioners. With the subspace correction methods for the SPD problem, we shall now disucss the corresponding preconditioners studied in Section 3.2.

THEOREM 4.6. *For $\tilde{B}$ given by* (4.5), *we have*

$$\hat{B} = (I - \tilde{B}\hat{A})\hat{A}_0^{-1}Q_0 + \tilde{B}, \tag{4.8}$$

Then, for η given by (4.6) *and for all $v \in \mathcal{V}$*

$$(\hat{B}\hat{A}v, v)_A \ge (1-\eta)(v,v)_A, \quad \|\hat{B}\hat{A}v\|_A \le (1+\eta)\|v\|_A.$$

Because of Theorem 4.5, the proof of this theorem or the next one is identical to that of Theorem 3.3 or Theorem 3.4.

THEOREM 4.7. *For $\tilde{B}$ given by* (4.5), *define*

$$\hat{B} = \omega\hat{A}_0^{-1}Q_0 + \tilde{B}. \tag{4.9}$$

Then, for all $v \in \mathcal{V}$

$$(\hat{B}\hat{A}v, v)_A \ge \frac{1}{2}(1-\eta)A(v,v), \quad \|\hat{B}\hat{A}v\|_A \le (\omega+2)(1+\bar{\delta})\|v\|_A,$$

provided that ω is sufficiently large and δ_0 is sufficiently small.

Note preconditioner (4.9) may also be applied in the SPD case.

THEOREM 4.8.

$$\hat{B} = \omega\hat{A}_0^{-1}Q_0 + \sum_{i=1}^{J} R_i Q_i. \tag{4.10}$$

Then, for all $v \in \mathcal{V}$

$$(\hat{B}\hat{A}v, v)_A \ge \frac{\lambda_0+\lambda_1}{4}\left(\frac{2\lambda_0}{\lambda_1+\lambda_0} - 4\delta_0\right)(v,v)_A, \quad \|\hat{B}\hat{A}v\|_A \le (\omega+2)(1+\bar{\delta})\frac{\lambda_0+\lambda_1}{2}\|v\|_A,$$

provided that ω is sufficiently large and δ_0 is sufficiently small.

Proof. Using the obvious identity

$$\hat{B}\hat{A} = \hat{P}_0 - P_0 + (\omega-1)\hat{P}_0 + \tilde{B}\hat{A},$$

the desired result then follows by (2.6) and Theorem 3.5. □

5. Algorithms in terms of vector and matrices. In this subsection, we shall represent some algorithms studied earlier in terms of vectors and matrices which are more convenient to code. As an example, we shall only discuss the Algorithm 4.4. For the derivation detail and the equivalence between these algorithms, we refer to Xu [9] for a general technique.

Assume that $\{\phi_i\}_{i=1}^{n}$ is a given basis of $\mathcal{V}$ and $\{\phi_i^l\}_{i=1}^{n_l}$ is a given basis of $\mathcal{V}_l$ $(0 \le l \le J)$. The stiffness matrices of operator A and $\hat{A}$ on these bases are given by

$$\mathcal{A} = ((A\phi_i, \phi_j)) \in \mathbb{R}^{n\times n}, \quad \hat{\mathcal{A}} = ((\hat{A}\phi_i, \phi_j)) \in \mathbb{R}^{n\times n}, \quad \hat{\mathcal{A}}_l = ((\hat{A}\phi_i^l, \phi_j^l)) \in \mathbb{R}^{n_l\times n_l}.$$

The equation (2.1) is then equivalent to

$$\hat{\mathcal{A}}x = b. \tag{5.1}$$

The relation between (2.1) and (5.1) is given by $u = \sum_{i=1}^{n} x_i\phi_i$ and $b_i = (f, \phi_i)$.

Since $\mathcal{V}_l \subset \mathcal{V}$, there are $t^l_{ij} \in \mathbb{R}$ such that

$$\phi^l_j = \sum_{i=1}^{n} t^l_{ij}\phi^l_i, \quad 1 \le j \le n_l.$$

In this way, for each l, we get a matrix $T_l = (t^l_{ij}) \in \mathbb{R}^{n\times n_0}$. Assume that $\mathcal{R}_l$ is a solver for $\mathcal{A}_l$ (for example, as the exact solver, $\mathcal{R}_l = \mathcal{A}_l^{-1}$), the Algorithm 4.4 can be written in the following equivalent form.

ALGORITHM 5.1. *Given $x^0 \in \mathbb{R}^n$. Assume x^k is defined for $k \ge 0$, then we define $x^{k+1} = \hat{x}^k + z^J$ where*

$$\hat{x}^k = x^k + T_0\hat{\mathcal{A}}_0^{-1}T_0^t(b - \hat{\mathcal{A}}x^k)$$

and for $i = 1, 2, \cdots, J$,

$$z^i = z^{i-1} + T_i\mathcal{R}_iT_i^t(\gamma - \mathcal{A}z^{i-1}),$$

where $\gamma = b - \hat{\mathcal{A}}\hat{x}^k$.

6. Second order elliptic equations. In this section, we shall discuss some applications of our algorithms to second order elliptic boundary value problems:

$$\begin{cases} \hat{\mathcal{L}}U &= F \quad \text{in} \quad \Omega, \\ U &= 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{6.1}$$

where Ω is a bounded domain in R^d, and $\hat{\mathcal{L}} = \mathcal{L} + \mathcal{N}$ with

$$\mathcal{L}U = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}(a_{ij}(x)\frac{\partial U}{\partial x_i}) \quad \text{and} \quad \mathcal{N}U = \sum_{i=1}^{d} b_i(x)\frac{\partial U}{\partial x_i} + c(x)U.$$

We assume all the coefficients are sufficiently smooth and the matrix $(a_{ij}(x))$ is symmetric and uniformly positive definite for any $x \in \Omega$. We also assume that (6.1) is uniquely solvable for any $F \in L^2(\Omega)$.

Let $H^1(\Omega)$ be the standard Sobolev space consisting of square integrable functions with square integrable (weak) derivatives of first order and $H^1_0(\Omega)$ a subspace of $H^1(\Omega)$ consisting of functions that vanish on $\partial\Omega$ (in an appropriate sense). Then $U \in H^1_0(\Omega)$ is the solution of (6.1) if and only if

$$\hat{A}(U, \chi) = (F, \chi), \quad \forall\chi \in H^1_0(\Omega), \tag{6.2}$$

where

$$\hat{A}(U, \chi) = \int_\Omega \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial U}{\partial x_i}\frac{\partial \chi}{\partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial U}{\partial x_i}\chi + c(x)U\chi \quad \text{and} \quad (F, \chi) = \int_\Omega F\chi.$$

We first define the finite element approximation scheme. Assume that Ω is a polyhedral domain and that Ω has been triangulated with $\Omega = \cup_i\tau_i$, where τ_i's are simplexes of size h with $h \in (0,1]$ and quasi-uniform. By this we mean that there exist constants C_0 and C_1 not depending on h such that each simplex τ_i is contained in (contains) a ball of radius $C_1 h$ (respectively $C_0 h$). The finite element space $\mathcal{V}_h$ is defined to be the functions which are continuous on Ω, piecewise linear with respect to the triangulation $\{\tau_i\}$, and vanish on $\partial\Omega$.

In the sequel, we shall drop the subscript h and denote $\mathcal{V} = \mathcal{V}_h$.

The finite element approximation to the solution of (6.2) is the function $u \in \mathcal{V}$ satisfying

$$\hat{A}(u,v) = (F,v), \quad \forall v \in \mathcal{V}. \tag{6.3}$$

This equation is uniquely solvable if h is sufficiently small (cf. Schatz [8]).

Define $\hat{A} \in L(\mathcal{V})$ by

$$(\hat{A}u,v) = \hat{A}(u,v), \quad u,v \in \mathcal{V}. \tag{6.4}$$

The equation (6.3) is then equivalent to

$$\hat{A}u = f \tag{6.5}$$

with some $f \in \mathcal{V}$.

If we define

$$A(U,\chi) = \int_\Omega \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial U}{\partial x_i}\frac{\partial \chi}{\partial x_j},$$

an SPD operator $A \in L(\mathcal{V})$ can be defined similarly. Introducing the Galerkin projection $\hat{P}_h : H_0^1(\Omega) \mapsto \mathcal{V}$ defined by

$$\hat{A}(\hat{P}_h v, \phi) = \hat{A}(v,\phi), \quad \forall v \in H_0^1(\Omega), \phi \in \mathcal{V}.$$

Then $u = \hat{P}_h U$, where U and u are solutions of (6.3) and (6.3) respectively.

Following Schatz [8], there is a constant $\alpha \in (0,1]$ (depending on the regularity of the adjoint equation of (6.1)) such that

$$\|v - \hat{P}_h v\|_{L^2(\Omega)} \le Ch^\alpha \|v\|_{H^1(\Omega)}, \quad \forall v \in H_0^1(\Omega). \tag{6.6}$$

We shall take the space $\mathcal{V}_0 = \mathcal{V}_{h_0}$, the finite element space defined on a triangulation with mesh size $h_0 \in (h,1)$, which satisfies

$$\mathcal{V}_{h_0} \subset \mathcal{V}_h.$$

Correspondingly $\hat{P}_0 = \hat{P}_{h_0}$. We assume that h_0 is sufficiently small so that $\hat{A}_0$ is invertible and hence $\hat{P}_0$ is well-defined. Applying (6.6) with h_0 in place of h, we then get by integration by parts

$$(N(I-\hat{P}_0)u,v) \le C\|(I-\hat{P}_0)u\|_{L^2(\Omega)}\|v\|_A \le c_0 h_0^\alpha \|u\|_A\|v\|_A,$$

and by Poincaré's inequality

$$(Nu,v) \le c_0\|u\|_A\|v\|_A,$$

Hence for δ_0 defined by (2.3) and $\bar{\delta}$ defined by (2.4),

$$\delta_0 \le c_0 h_0^\alpha, \quad \bar{\delta} \le c_0.$$

6.1. Domain decomposition methods. Assume that we are given a set of overlapping subdomains $\{\Omega_i\}_{i=1}^J$ of Ω whose boundaries align with the mesh triangulation defining $\mathcal{V}$. One way of defining the subdomains and the associated partition is by starting with disjoint open sets $\{\Omega_i^0\}_{i=1}^J$ with $\bar{\Omega} = \cup_{i=1}^J \bar{\Omega}_i^0$ and $\{\Omega_i^0\}_{i=1}^J$ quasi-uniform of size h_0. The subdomain Ω_i is defined to be a mesh subdomain containing Ω_i^0 with the distance from $\partial\Omega_i \cap \Omega$ to Ω_i^0 greater than or equal to ch_0 for some prescribed constant c.

Based on the subdomains given above, we can then define subspace $\mathcal{V}_i$ $(1 \le i \le J)$ by

$$\mathcal{V}_i = \{v \in \mathcal{V} : v(x) = 0, \text{ for } x \in \Omega \setminus \Omega_i\}.$$

In addition, we introduce a coarse finite element subspace $\mathcal{V}_0 \subset \mathcal{V}$ defined from a quasi-uniform triangulation of Ω of size h_0. For the subspaces $\mathcal{V}_i$ $(0 \le i \le J)$ defined above, it can be shown that (4.1) holds. We assume the subspace solvers R_i satisfy the following property:

$$\sigma(R_i A_i) \subset [\omega_0, \omega_1], \quad 0 \le i \le J.$$

for some positive constant ω_0. Furthermore we assume that ω_0 and ω_1 are bounded below and above (by 2) respectively uniformly with respect to h_0, h and J.

The corresponding Algorithms 4.1 and 4.2 are known as additive and multiplicative Schwarz domain decomposition methods. It can be shown that (cf. [9])

$$K_0 \le C\omega_0^{-1}, K_1 \le C\omega_1.$$

Consequently, for equation (6.1), we obtain two corresponding domain decomposition algorithms: Algorithms 3.1 and 4.4; and six preconditioners: (3.4), (3.5), (3.8), (4.8), (4.9) and (4.10). The corresponding estimates hold when h_0 is sufficiently small (but independent of h).

In the domain decomposition context, we note that the preconditioner (4.10) was studied by Cai and Widlund [5]; some numerical examples for (3.8) can be found in Xu and Cai [12]; analysis and numerical experiment for (4.10) was done recently by Cai [4].

Similar results obviously hold for other domain decomposition methods such as substructuring methods.

6.2. Multigrid algorithms. To define a multigrid algorithm, we assume the triangulation $\mathcal{T}$ is constructed by a successive refinement process. More precisely, $\mathcal{T} = \mathcal{T}_J$ for some $J > 1$ and $\mathcal{T}_k$, for $k \le J$, are a nested sequence of quasi-uniform triangulations which consist of simplexes $\mathcal{T}_k = \{\tau_k^i\}$ of size h_k for $1 \le k \le J$ such that $\Omega = \cup_i \tau_k^i$, where the quasi-uniformity constants are independent of k. These triangulations should be nested in the sense that any simplex τ_{k-1}^l can be written as a union of simplexes of $\{\tau_k^i\}$. We further assume that there is a constant $\eta > 1$, independent of k, such that h_k is proportional to η^{-k}. As an example, in two dimensional case, a finer grid is obtained by connecting the midpoints of the edges of the triangles of the coarser grid with $\mathcal{T}_0$ being the given coarsest initial quasi-uniform triangulation.

Corresponding to each triangulation $\mathcal{T}_k$, a finite element space $\mathcal{V}_k$ can be defined by $\mathcal{V}_k = \{v \in H_0^1(\Omega) : v|_\tau \in \mathcal{P}_1(\tau), \quad \forall \tau \in \mathcal{T}_k\}$, where $\mathcal{P}_1$ is the space of linear polynomials. Obviously

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_J \equiv \mathcal{V}.$$

With the subspaces v_i given above, (4.1) holds trivially.

If the subspace solvers R_i are given by Gauss-Seidel or damped Jacobi iterations, we have (cf. [9])

$$K_0 \le C, \quad K_1 \le C, \quad \omega_1 = 1.$$

Here C is a positive constant independent of h, h_0 and J. In this case, (4.2) is the BPX preconditioner (cf. [3], [11]) and Algorithm 4.2 is equivalent to the classic ("slash cycle") multigrid method (cf. [2], [9]).

Consequently, we have all the corresponding Algorithms 3.1 and 4.4; and preconditioners: (3.4), (3.5), (3.8), (4.8), (4.9) and (4.10); and the related estimates.

The discussion for the hierarchical basis method is similar.

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