CHAPTER 26

Numerical Solution of a Turning Point Problem*

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Abstract. The turning point problem

$$\begin{cases} -\varepsilon\Delta u + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 & (x,y) \in [(-1,1)\times(-1,1)] \\ u(-1,y) = V_a, \quad u(1,y) = V_b, \\ u(x,-1) = V_c, \quad u(x,1) = V_d, \end{cases}$$

is known to have some extremely small eigenvalues. No successful numerical solution to this problem has been reported. In this paper, a numerical procedure is proposed. All four boundary layers are well defined and the numerical singularity is successfully removed.

Key words. boundary layer, domain decomposition, overlap, Schwarz Alternating Method(SAM), turning point problem

AMS(MOS) subject classifications. 65F10, 65N20

1. Introduction. The singularly perturbed elliptic partial differential equation

$$-\varepsilon\Delta u + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + k(x,y)u = 0 \tag{1}$$

has been extensively studied by many researchers. When $\varepsilon \to 0$, the solution to this problem becomes difficult. A typical property of the solution is the existence of boundary layers. This problem has many important applications including the solution of the Navier-Stokes equations and stochastic differential equations [13].

In (1), the lower order operator represents the deterministic flow field while the second order part represents a slow diffusion of particles. Therefore, the results will depend on the nature of the underlying flow. In [13], Matkowsky classifies the singularly perturbed elliptic boundary value problem into three cases according to how the particles are diffusing:

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Type (1). with a flow,
Type (2). across a flow,
Type (3). against a flow.

In particular, the problem of type (3) is called the turning point problem, since the flow changes its direction inside of the region. Asymptotic analysis of the three types of diffusion can be traced back to the early 1950's [10, 11]. The third type of diffusion is the most difficult one. The first few results were obtained by O'Malley; Ventsel and Freidlin [14, 20]. Later, a stronger result was reported by Ludwig [12]. More results have been published since then [7, 15].

The investigation of the numerical solution of the turning point problem started from stiff-ODE problems. Dorr (1971) first reported that the turning point problem in the one-dimensional case is extremely ill-conditioned [4]. For example, the condition number of the matrix equation for the following problem

$$\begin{cases} -\varepsilon y'' + xy = 0, & x \in (-1,1) \\ y(-1) = a, & y(1) = b \end{cases}$$

is $\kappa(A) = 1.1021 \cdot 10^{14}$, where $\varepsilon = 0.003$ and $n = 100$. Many numerical techniques for the stiff-ODE and turning point problems in the one-dimensional case are found in [1]. However, successful numerical techniques for turning point problems in higher dimensional spaces have not been reported. This is because of the extreme numerical singularity of the resulting matrix equation. B. Zhu has shown [21] that the smallest eigenvalue of the following problem

$$\begin{cases} -\varepsilon\Delta u + x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \lambda u & (x,y) \in [(-1,1) \times (-1,1)] \\ u(x,y)|_\Gamma = 0 \end{cases}$$

has the estimate

$$|\lambda| \leq O(\varepsilon^{-1/2} e^{-\alpha/\varepsilon})$$

where $\alpha > 0$.

Much work has been done for boundary layer problems in higher dimensional space. For example, Hedstrom and Osterheld first studied the effect of ε on the boundary layer [6]. Segal discussed the different aspects of numerical methods which relate to the computation for singular perturbation problems [18]. Rodrigue and Reiter [16] investigated the application of the Schwarz Alternating Method (*SAM*) to (1). Brown, Chin, Hedstrom, Manteuffel and Scroggs [2, 3, 9, 17] studied other domain decomposition techniques. Elman and Golub reported their sequence of studies on iterative methods for the convection-diffusion problem [5].

In this paper, a numerical procedure using the domain decomposition approach, or more precisely *SAM*, for the solution of turning point problems is presented. This procedure can remove the severe numerical singularity in the original form; therefore, a successful numerical solution for this problem becomes feasible. The same *SAM* can also be used to accurately define the sharp boundary layers.

2. Turning point problem. In this paper, the following turning point problem is considered:

$$(2)\qquad \begin{cases} -\varepsilon\Delta u + x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0 & (x,y) \in [(-1,1)\times(-1,1)] \\ u(-1,y) = V_a, \quad u(1,y) = V_b, & \\ u(x,-1) = V_c, \quad u(x,1) = V_d, & \end{cases}$$

This problem was suggested by C. Holland [8]. It is typical of a wide class of turning point problems. The solution to this problem has four sharp boundary layers of width ε at each side of the square and the solution at the origin is

$$u(0,0) = \frac{1}{4}(V_a + V_b + V_c + V_d).$$

The technique we use for solving this problem can also be applied to other cases [19].

Applying Zhu's result to this problem, the linear system of equations discretized from (2) will have an extremely ill-conditioned matrix if ε is small. No meaningful numerical solution can be obtained if special techniques are not used [8]. However, if we carefully examine the relationship between ε and the condition number of the matrix, the following three observations are very important to the construction of our numerical procedure for the solution of (2).

1. The smallness of ε is relative to the size of the solution domain. If we reduce the size of the solution region in (2) the same small ε will result in a different condition number. It is clear that the solution for a smaller region is less difficult. Thus, if a domain decomposition approach is used, the solution of each subproblem will be easier.
2. When the domain decomposition approach is applied to the turning point problem, most of the subproblems are *not* turning point problems! Therefore, the solution of these subproblems is not an issue. There is only one subdomain which contains a turning point[1]. Fortunately, it can be made to have a very small size.
3. The asymptotic analysis of this problem [14, 21] showed that the expansion of the solution of (2) in a series of ε has one base term. In particular, this term is independent of ε . Therefore, if ε_1 and ε_2 are close, the corresponding solutions of the same boundary value problem (2) are also close. Most of the changes happen around the boundary layers. Numerical computations have verified their analysis.

Based on the first two observations domain decomposition approaches (and the Schwarz alternating method (*SAM*) in particular) seem to be very helpful in overcoming the numerical singularity which appeared in the original problem. Specifically, if we decompose the solution region into many small overlapping subregions, there is only one subdomain which contains a turning point. The rest of the subproblems are "easier" boundary layer problems, for which many known numerical techniques can be used to solve them. Since the size of that subdomain which contains the turning

[1] We only discuss the single turning point case in this paper. There is no conceptual difficulty in generalizing our procedure to the case of more than one turning point.

Procedure Turning_point_problem ()

Choose ε_0 such that

$$\begin{cases} -\varepsilon_0 \Delta u^{(0)} + x\dfrac{\partial u^{(0)}}{\partial x} + y\dfrac{\partial u^{(0)}}{\partial y} = 0 & (x,y) \in [(-1,1)\times(-1,1)] \\ u^{(0)}(-1,y) = V_a, \quad u^{(0)}(1,y) = V_b, \\ u^{(0)}(x,-1) = V_c, \quad u^{(0)}(x,1) = V_d, \end{cases}$$

can be solved without stability problem.

Let $i = 0$

While $\varepsilon_i > \varepsilon$ **do**

Pick $\varepsilon_{i+1} < \varepsilon_i$ and a new set of overlapping subdomains.
Use $u^{(i)}$ as initial guess start the *SAM* iteration for

$$\begin{cases} -\varepsilon_{i+1}\Delta u^{(i+1)} + x\dfrac{\partial u^{(i+1)}}{\partial x} + y\dfrac{\partial u^{(i+1)}}{\partial y} = 0 \\ \qquad (x,y) \in [(-1,1)\times(-1,1)] \\ u^{(i+1)}(-1,y) = V_a, \quad u^{(i+1)}(1,y) = V_b, \\ u^{(i+1)}(x,-1) = V_c, \quad u^{(i+1)}(x,1) = V_d, \end{cases}$$

until it converges.
$i = i + 1$

end

end

FIG. 1. *Numerical procedure for turning point problem.*

point is smaller, the numerical difficulty of solving this subproblem will not be as severe as the original problem. However, this naive approach still faces the stability problem if a poor initial guess is given for a very small ε . Fortunately, the third observation can lead us to a successful solution process. We start from a "large" ε for which the turning point problem can be solved without a stability problem. Then we reduce the ε and decompose the solution region into several overlapping subregions. Using the solution from the large ε as the initial guess for the new decomposition, *SAM* can be applied to obtain the solution for the new smaller ε . Two issues in this process are important:

- The new ε should not change too rapidly so that the solution corresponding to the new problem has no large changes in parts of the region.
- The size of the subdomain which contains the turning point has to be small enough such that the solution of this problem has no stability problems.

3. Convergence analysis. For problem (2), the matrices resulting from many finite difference or finite element discretizations are unsymmetric. The convergence of the *SAM* for these matrices needs to be justified.

It is known that the matrix from the discretization of (2) is diagonally dominant, if an upwind scheme is used. When the grid size h is small enough, the matrix

Given initial guess $x^{(0)}$,

While $\|Ax^{(l)} - b\| \geq Tol$ do
$x^{(l+1)} = x^{(l)}$
For $i = 1$ to k
Solve[a]

$$A_{\Omega_i} x_{\Omega_i}^{(l+1)} = b_{\Omega_i}^{(l+1)},$$

Update $x^{(l+1)}$ by $x_{\Omega_i}^{(l+1)}$
end
$l = l + 1$
end

[a] Note that $b_{\Omega_i}^{(l+1)}$ contains the information from the artificial boundary as well.

FIG. 2. *Description of the SAM algorithm.*

resulting from a central scheme is also diagonally dominant [5]. With this diagonal dominance condition, a convergence result can be shown.

Let

$$Ax = b$$

be the matrix equation from the discretization of (2), and Ω_i, $i = 1, \cdots, k$ be the overlapping subdomains such that all grid nodes on the artificial boundary are located at least in the interior of one subdomain. Denote x_{Ω_i} as the unknown vector in subdomain Ω_i and and A_{Ω_i} the corresponding principal submatrix for these unknowns. The description of the Schwarz alternating method applied for this particular decomposition is given in Fig. 2.

Then the following lemma can lead to the convergence of the *SAM* algorithm in this case.

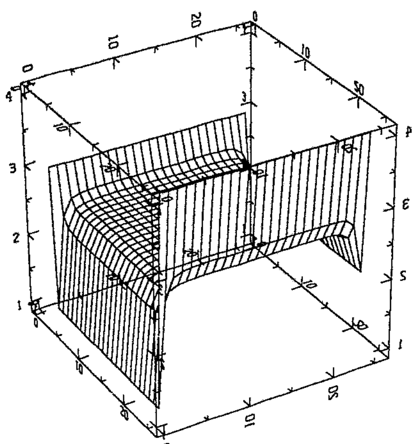
LEMMA 3.1. *If the matrix A_{Ω_i} is diagonally dominant, then*

$$\|x_{\Omega_i}\|_\infty < \gamma \|x_{\Gamma_i}\|_\infty$$

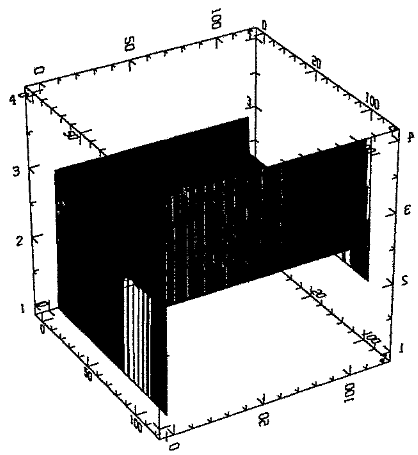
where x_{Γ_i} is a vector which contains all boundary node values (including both true and artificial boundary nodes) of Ω_i and $\gamma < 1$

Proof. We prove this lemma by contradiction. □

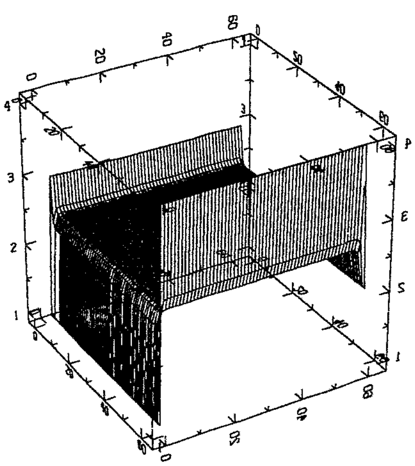
4. Boundary layers. To gain efficiency and to maintain the diagonal dominance, the upwind scheme is used for defining the overall solution. However, it is known that this scheme is diffusive on the boundary layer. The central scheme is used around the boundary layer to improve accuracy. Many much smaller overlapping subregions (say size of $30\varepsilon \times 30\varepsilon$) along the boundary are allocated. Of course, the grid size of these subdomains is refined to the size of ε . Apply *SAM* to these new boundary subdomains and a convergence solution is ensured. Our numerical results indicate the improvement of sharpness in the boundary layers.



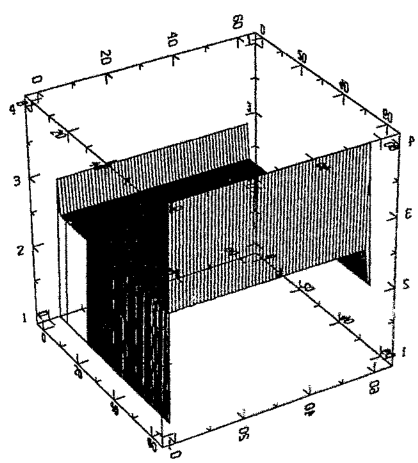
ϵ = 1/50 - unsharpened



ϵ = 1/1600 - unsharpened



ϵ = 1/200 - unsharpened



ϵ = 1/200 - sharpened

FIG. 3. *Surface plots for* $\varepsilon = 1/50,\ 1/1600,\ 1/200$.

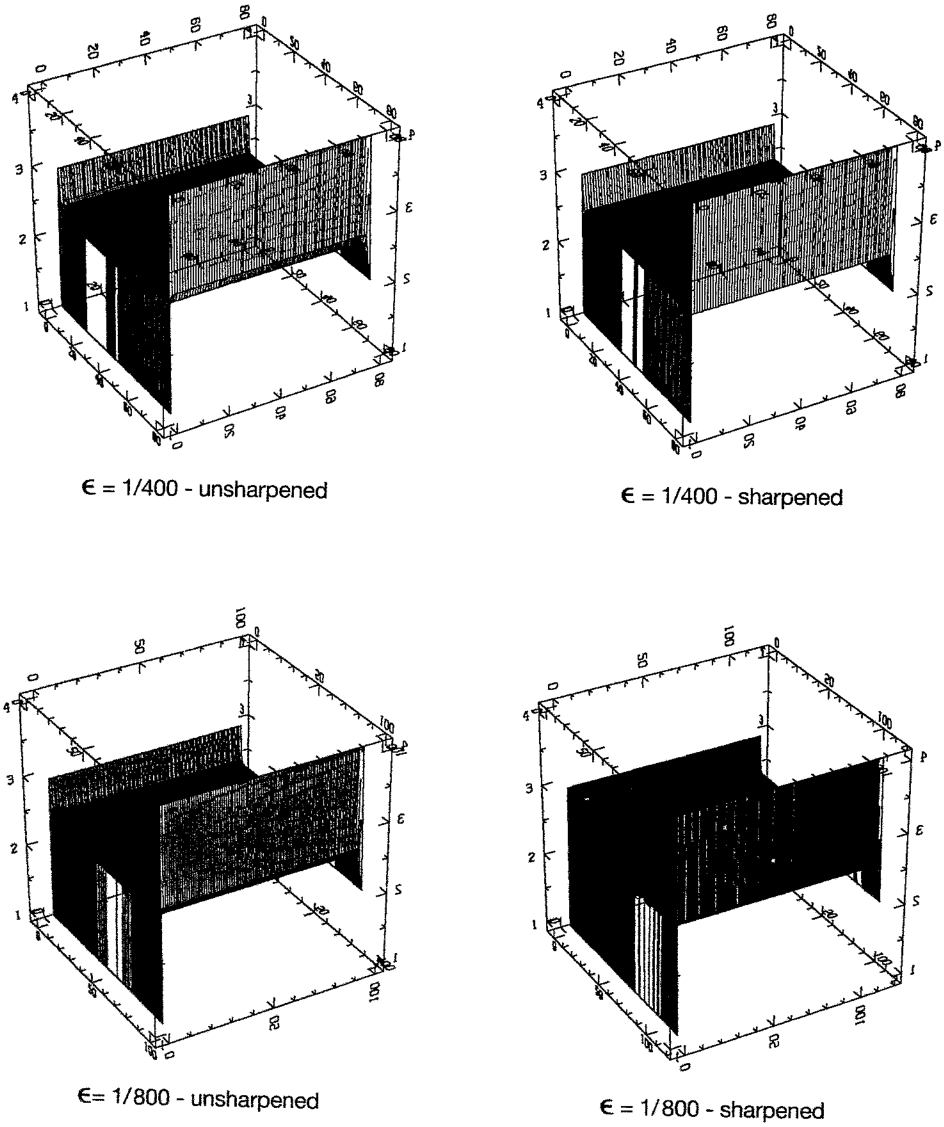


FIG. 4. *Surface plots for* $\varepsilon = 1/400, 1/800$.

5. Numerical testing. Numerical testing is carried out for this model turning problem (2). We start with $\varepsilon_0 = 0.02$. The solution of this problem has no stability problem. Then we set $\varepsilon_k = 0.5 * \varepsilon_{k-1}$; decompose the square into $(k+1) \times (k+1)$ equal sized subdomains and apply *SAM* to the turning point problem with the smaller ε_k . It is interesting that the solution of the turning point subproblem does not change any further after $k > 3$. In a real application, many of the subproblems which are away from the boundary layers need not be recomputed after reducing ε a few times. This phenomenon can be explained by the asymptotic expansion for this problem [14, 21]. We present the surface plots of the solution for $\varepsilon = 0.0025$, 0.00125, and 0.000625. In particular, we present both solutions before and after using central scheme to sharpen the boundary layers. Our computation stopped when $\varepsilon_k = 1/1600$. If the computation would continue for even smaller ε , the decomposition strategy we used here need to be updated to ensure the stability problem of the turning point subproblem.

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