CHAPTER 5

Preconditioners for One Class of Elliptic Problems in Not Simply Connected Domains

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Abstract. The fact that coefficients of elliptic equations can be discontinuous along the boundaries of the subdomains but vary slowly in their interiority is the essential assumption in a well-known class of the substructuring methods. In our study we show that the convergence properties of the substructuring algorithms do not change if the subdomains have the regions of highly varying or discontinuous coefficients which are uniformly isolated from the boundaries of the subdomains.

Introduction. In our paper we discuss one of the aspects of the domain decomposition methods for solving a second order elliptic problem with variable coefficients - the dependence of the bounds for the condition number of the iteration operators on the behavior of the coefficients.

The essential assumption used by a well-known family of iterative substructuring methods based on a non-overlapping subdivision of the initial domain [see e.g.3,4,7,8,10,11,12,13,15] is that the coefficients can be discontinuous along the boundaries of the subdomains but vary slowly in each subregion. There are however some problems, e.g. magnetostatic problems in combined formulation [7] or a problem of evaluation of the spectral dimensions of fractals [2], where for an efficient solving we have to deal with the subregions in which this assumption is violated. In such problems for each (or a number) of the substructures we can distinguish the closed "interior" regions with highly varying or discontinuous coefficients which are isolated from the "exterior" boundaries of the substructures by a strip where the coefficients are smooth. In the "interior" regions the coefficients can be equal to zero or infinity and in that case we have the problem in a not simply connected domain with the Dirichlet or homogeneous Neumann conditions on the "interior" boundaries.

In what follows we show that the Poincaré-Steklov operators defined on the "exterior" boundaries for such subproblems are

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spectrally equivalent to the Poincaré-Steklov operator for the Laplacian in the subdomain and the condition number depends only on a geometrical factor - the ratio of the diameters of "interior" region and the whole subdomain. So, in the iterative methods for solving a boundary equation with respect to the trace of an unknown function on the "exterior" boundaries of the subdomains all the preconditioners constructed for the piecewise constant coefficient case [see e.g. 1,3,7,8,9,14,15] can be applied. We also present numerical experiments.

Formulation of the problem. Let Ω be a bounded domain with a Lipschitz boundary $\partial\Omega$, $\Omega\subset\mathbb{R}^m, m=2,3$, decomposed into n regular subdomains Ω_i with Lipschitz boundaries $\partial\Omega_i$, $\bar{\Omega}=\bigcup_{i=1}^{n}\bar{\Omega}_i$. We also suppose that each (or a number) of subdomains can be represented as $\bar{\Omega}_i=\bar{D}_i^1\cup D_i^2$, where $\bar{D}_i^2\subset\Omega_i$, and for a distance d between ∂D_i^2 and $\partial\Omega_i$ holds $d(\partial D_i^2,\partial\Omega_i)\ge\delta\cdot diam\Omega_i$ for all i, with $\delta>0$. We denote as Γ the union of the interior boundaries, $\Gamma=(\bigcup_i\partial\Omega_i)\setminus\partial\Omega$.

Let $V\subset H^1(\Omega)$, $X\subset H^{1/2}(\Gamma)$ or $X\subset H^{1/2}(\partial\Omega)$. We consider the following boundary problem with respect to the trace $u=\gamma w\in X$ of the unknown function $w\in V$: find $u\in X$ such that

$$\sum_{i=1}^{n}(S_i^{-1}u,\gamma z)_{L_2(\partial\Omega_i)}+\alpha(S_E^{-1}u,\gamma z)_{L_2(\partial\Omega)}=(\psi,\gamma z)_{L_2(\Gamma)} \tag{1}$$

holds for all $z\in V$. Here ψ is a given function; $S_E:H^{-1/2}(\partial\Omega)\longrightarrow H^{1/2}(\partial\Omega)$ - is the symmetric, positive-definite Poincaré-Steklov operator for an exterior boundary value problem for the Laplacian [7]; S_i is the Poincaré-Steklov operator for a boundary value problem in the substructure Ω_i for the operator

$$\Lambda_i w = \sum_{j=1}^{m}\frac{\partial}{\partial x_j}\beta_i(x)\frac{\partial}{\partial x_j}w, \qquad x\in\Omega_i.$$

For the coefficients $\beta_i(x)$ holds

$$\beta_i(x) = \begin{cases} \mu_i(x)\ge 0, & x\in D_i^2 \\ 0<const_i<\infty, & x\in D_i^1 \end{cases}. \tag{2}$$

If we choose in (1) $V=\{w\in H^1(\Omega):(\gamma w,g_0)_{L_2(\partial\Omega)}=0\}$, $X=\{u\in H^{1/2}(\partial\Omega): (u,g_0)_{L_2(\partial\Omega)}=0\}$, g_0 is the Robin potential, $\alpha=1$, $n=1$, $\Gamma=\partial D_1^2$ we obtain a problem in combined formulation [7]. Choosing in (1) $V=H_0^1(\Omega)$, $X=H^{1/2}(\Gamma)$, $\alpha=0$, $n>1$ we obtain the homogeneous Dirichlet problem in Ω with highly varying coefficients in the substructures.

The properties of the operators S_i, S_i^{-1} in the case when in (2) $\bar{D}_i^1\equiv\bar{\Omega}_i$ (the case of slowly varying coefficients in the substructures) and methods for solving the problem (1) have been studied in [e.g 1,7,8,9,12,14]. Below, the principal aim of our study is to analyze the properties of these operators in the dependence of highly varying coefficients when the condition (2) holds true.

Analysis of operators. For simplicity we discuss the two-dimensional case. Analysis of the three-dimensional one is the same but more cumbersome.

Let us consider the following variational problem in the circle $\bar{\Omega}_c=\{\ 0\le\rho\le R,\ 0<\varphi\le 2\pi\ \}$ with the boundary $\partial\Omega$: for a given function $t\in X^*$ find $w\in V$ such that

(3) $$a(w,z) = \langle t,\gamma z\rangle \quad \text{holds for all } z\in V\ .$$

Here $V\subset H^1(\Omega_c)$, $X\subset H^{1/2}(\partial\Omega)$; $\gamma\ :\ V\ \longrightarrow\ X$ is the operator of traces; $\langle\cdot,\cdot\rangle=(\cdot,\cdot)_{L_2(\partial\Omega)}$ is the duality paring between X and X^*. The bilinear form $a(w,z)$ is defined by

(4) $$a(w,z) = \int_\Omega \beta(x)\nabla w\nabla z dx,$$

with the coefficients

(5) $$\beta(x) = \begin{cases} 0 \le \mu=const \le \infty, & x\in D^2=\{0\le\rho<r,\ 0<\varphi\le 2\pi\} \\ 1 & ,\ x\in D^1=\{r<\rho\le R,\ 0<\varphi\le 2\pi\} \end{cases}.$$

If in (3) we set

(6) $$V_\Delta=H^1(\Omega_c),\quad X=H^{1/2}(\partial\Omega),\quad t\in X^*\ :\ \langle t,1\rangle=0,$$

then the problem (3)-(6) is equivalent to the Neumann boundary value problem in Ω_c.

We denote the Poincaré-Steklov operator for this problem as S_μ when $\mu\ne 1$ and as S_Δ when $\mu=1$. The properties of the operator S_Δ as well as $S_\Delta^{-1}:X\ \longrightarrow\ X^*$ are well-known [1,12,14]: the later is symmetric, positive-definite, bounded in corresponding norms and gives an equivalent norm in X. The following lemma determines the properties of the operator S_μ^{-1}:

LEMMA 1. Let S_Δ and S_μ be the Poincaré-Steklov operators for the problem (3)-(6) when $\mu=1$ and when $\mu\ne 1$ correspondingly. Then for each $u\in X$ the following inequalities hold true

(7) $$\begin{aligned} &\frac{1}{q(\mu)}\langle S_\Delta^{-1}u,u\rangle \le \langle S_\mu^{-1}u,u\rangle \le \langle S_\Delta^{-1}u,u\rangle \quad \text{if } 0\le\mu<1 \\ &\langle S_\Delta^{-1}u,u\rangle \le \langle S_\mu^{-1}u,u\rangle \le q(\mu)\langle S_\Delta^{-1}u,u\rangle \quad \text{if } 1<\mu\le\infty \end{aligned},$$

where $q(\mu)=\dfrac{1+\mu-(r/R)^2(1-\mu)}{1+\mu+(r/R)^2(1-\mu)}$ is bounded independently of μ.

The proof of this lemma is based on the following property of the Poincaré-Steklov operator [see e.g. 1,12]:

(8) $$\langle S_k^{-1}u,u\rangle = a(w,w),\ w\in V_\Delta,\ u=\gamma w\in X,\ k=\Delta,\mu$$

and the explicit representation for the solution w of the Dirichlet problem:

$$(9)\qquad w=\begin{cases}\sum_{n=0}^{\infty}\dfrac{2(r/R)^{n}(\rho/r)^{n}}{(r/R)^{2n}(1-\mu)+(1+\mu)}(\alpha_n\cos n\varphi+\beta_n\sin n\varphi), & x\in D^2\\ & \mu\neq 1\\ \sum_{n=0}^{\infty}\dfrac{(1-\mu)r^{2n}/(R^n\rho^n)+(1+\mu)(\rho/R)^n}{(r/R)^{2n}(1-\mu)+(1+\mu)}(\alpha_n\cos n\varphi+\beta_n\sin n\varphi), & x\in D^1\end{cases}$$

$$w=\sum_{n=0}^{\infty}(\rho/R)^n(\alpha_n\cos n\varphi+\beta_n\sin n\varphi),\qquad \mu=1$$

where α_n and β_n are Fourier coefficients of u. Now straightforward evaluations for the bilinear form in (8) give the inequalities (7).

It can also be easily seen that for $q(\mu)$ holds

$$(10)\qquad Q^{-1}\le q(\mu)\le Q\quad \text{for all } \mu, \text{ where } Q=\frac{1+(r/R)^2}{1-(r/R)^2}.$$

So, lemma is proved ■.

REMARK 1. Note that the variational problems (3) for the two limiting cases with $\mu=\infty$ and $\mu=0$ are equivalent to those in the ring $\bar{\Omega}_r=\{0<r\le\rho\le R,\ 0<\varphi\le 2\pi\}$ with the Dirichlet and homogeneous Neumann conditions on the interior boundary $\Gamma_0=\{\rho=r,\ 0<\varphi\le 2\pi\}$ correspondingly and with the following choice of the spaces V

$$V_D=\{z\in H^1(\Omega_r):\ \gamma z=const \text{ on } \Gamma_0\},\ V_N=H^1(\Omega_r).$$

From inequalities (7) and (10) it follows that for the operators S_D^{-1} and S_N^{-1}, $S_k^{-1}:X\to X^*$ $k=D,N$, for these problems holds

$$Q^{-1}\langle S_\Delta^{-1}u,u\rangle\le\langle S_N^{-1}u,u\rangle\le\langle S_\Delta^{-1}u,u\rangle$$
$$\langle S_\Delta^{-1}u,u\rangle\le\langle S_D^{-1}u,u\rangle\le Q\langle S_\Delta^{-1}u,u\rangle$$
$$Q^{-1}\langle S_N^{-1}u,u\rangle\le\langle S_\mu^{-1}u,u\rangle\le Q\langle S_D^{-1}u,u\rangle,\ u\in X,$$

where the operator S_D^{-1} is uniquely defined on the traces of functions from V_D with $const=\alpha_0=\int_{\partial\Omega}u\,d\sigma$ ■.

Suppose now that P is a bi-lipschitz mapping of Ω_c on a Lipschitz domain $\Omega_L=D_L^1\cup D_L^2$ such that

$$c_1R\le diam\Omega_L\le c_2R\ ;\quad c_3r\le diamD_L^2\le c_4r\ ;$$
$$c_5(R-r)\le d(x,y)\le c_6(R-r) \text{ for all } x\in\partial D_L^2,\ y\in\partial\Omega_L;\ 0<c_i<\infty$$

holds, here d is the distance between ∂D_L^2 and $\partial\Omega_L$. Consider the problem (3)-(5) where (5) holds for the regions D_L^2 and D_L^1 correspondingly. Then the following lemma holds true:

LEMMA 2. Let $\bar{S}_\Delta,\bar{S}_\mu,\bar{S}_N,\bar{S}_D$ be the Poincaré-Steklov operators for the problem (3)-(5) in a Lipschitz domain Ω_L defined above, with $\mu=1$, $\mu\neq1$, $\mu=0,\mu=\infty$ correspondingly. Then the following inequalities hold true:

$$cQ^{-1}\langle\bar{S}_\Delta^{-1}u,u\rangle\le\langle\bar{S}_N^{-1}u,u\rangle\le C\langle\bar{S}_\Delta^{-1}u,u\rangle,$$
$$c\langle\bar{S}_\Delta^{-1}u,u\rangle\le\langle\bar{S}_D^{-1}u,u\rangle\le CQ\langle\bar{S}_\Delta^{-1}u,u\rangle,$$
$$c_1\langle\bar{S}_N^{-1}u,u\rangle\le\langle\bar{S}_\mu^{-1}u,u\rangle\le C_1\langle\bar{S}_D^{-1}u,u\rangle$$

for all $u\in H^{1/2}(\partial\Omega_L)$, where C and c depend only on the Lipschitz constants of P and P^{-1}, Q is defined in (10), C_1 and c_1 are independent of μ.

The proof of this lemma follows from *REMARK 1* and the invariance of norms and seminorms of Sobolev spaces under sufficiently smooth transformations of variables [5].

REMARK 2. Note that if we consider the operators $S_k^{-1}, \bar{S}_k^{-1}$ on a finite dimensional subspace $X^h \subset H^{1/2}(\partial\Omega)$ of functions $u^h = \gamma w^h$, $w^h \in V_k^h \subset V_k$, $k = \Delta, N, D$, then the results of *LEMMAs 1* and *2* remain valid ∎.

From the results of this paragraph it follows that for iterative solving of the problem (1)-(2) all the preconditioners constructed for the piecewise constant coefficient case [see e.g. 1,3,4,6,7,8,9,13,15] can be applied without crucial deterioration of the convergence properties of the iterative algorithms.

Numerical experiments. Numerical experiments demonstrating independence of the convergence properties of the iterative algorithms of the behavior of μ in the interiority of Ω when the problem (1)-(2) corresponds to the three-dimensional magnetostatic one in combined formulation have been presented in [7, p.134, Table.5].

Below we present numerical experiments for the two-dimensional problem (1)-(2) in the case when we have the Dirichlet problem in initial domain.

The experiments have been done using a preconditioned conjugate gradient (PCG) method to solve the boundary equation (1) with respect to the trace of unknown function on the boundaries of the subdomains with the preconditioners for the piecewise constant coefficient case proposed in [7,8]. Note that from *LEMMA 2* it follows that for the number of iterations N of the PCG method holds $N \simeq 1/\sqrt{\delta}$, where $\delta = d(\partial D_i^2, \partial\Omega_i)/diam\partial\Omega_i \simeq 1-r/R$.

EXAMPLE 1. In the first example Ω is the unit square decomposed by vertical and horizontal lines into 9 identical square subdomains Ω_{kl}, $k,l=1\div3$. In local systems of coordinates associated with the centers of subdomains, the subdomain Ω_{kl} itself and the region D_{kl}^2 are represented as

$$\bar{\Omega}_{kl} = \{|x| \le R,\ |y| \le R\},\quad \bar{D}_{kl}^2 = \{|x| \le r,\ |y| \le r\},\ k,l=1\div3. \tag{11}$$

For the coefficients in (2) we put $\mu(x)=0$, $x \in D_{kl}^2$, and so we have the boundary value problem in $(\Omega \setminus \bigcup_{k,l=1}^{3} D_{kl}^2)$ with the homogeneous Neumann boundary conditions on the "interior" boundaries $\bigcup_{k,l=1}^{3} \partial D_{kl}^2$.

Table 1 shows the dependence of the number N of iterations necessary to reduce the initial residual of the solution by a factor 10^{-5} on the geometrical factor $\delta=(1-r/R)$. The case with $\delta=1$ corresponds to the piecewise constant coefficient case.

δ	1	2/3	1/3	1/6	1/12
N	10	10	11	13	16

Table 1

EXAMPLE 2. The second example demonstrates the behavior of the number of iterations N when the number of subdomains D_{kl}^2 increases but the parameter δ remains fixed for all subdomains. First, Ω is decomposed into 4, then into 16 subdomains Ω_{kl}, which are defined in (11), but for the two cases $\delta=2/3$. In both the cases the number of iterations N is equal to *10*.

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