CHAPTER 7

Spectral Analysis of the Interface Operators Associated with the Preconditioned Saddle-Point Principle Domain Decomposition Method

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Abstract .

The matrix of the interface problem associated with the domain decomposition method via Lagrange multipliers is compared with the Schur complement matrix. The local contributions of one subdomain to both interface operators are proved to be the inverse each one of the other. Then, some specific spectral features of the matrices of the primal and dual Schur complements are shown that entail the conjugate gradient algorithm to converge faster for the dual Schur complement problem than for the standard primal Schur complement. At last, the preconditioning of the dual Schur complement by the interface stiffness matrix is studied .

1. Introduction .

In a companion paper (Farhat, 1991), the domain decompostion method via Lagrange multipliers, also called saddle point principle domain decomposition method, is thoroughly presented and generalized to the case of piece-wise plynomial Lagrange multipliers. Also, results obtained with the parallel implementation of the method for solving real-life structural analysis problems are presented .

In this paper, some duality properties between the matrix of the condensed operator on the interface associated with the saddle point principle domain decomposition method, that was introduced in Destuynder & Roux (1989) and Farhat & Roux (1991), and the Schur complement matrix (Bjordstad & Widlund, 1986), are shown. In the sequel we will refer to the matrix of the condensed operator on the interface associated with the saddle point principle domain decomposition method as the dual Schur complement matrix .

Some special features of the spectrum of the dual Schur complement matrix are shown in section 3. This matrix has a very few high eigenvalues. The "super-convergence" effects due to these features are shown in section 4. In section 5, we give an explanation of the

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unstability of the dual Schur complement method due to the loss of orthogonality in the conjugate gradient algorithm, and we present a reconjugation procedure in order to get rid of this unstability. In section 6, an unexpensive preconditioner for the dual Schur complement method is introduced. Finally section 7 offers some conclusions upon the practical comparison between the primal and the dual Schur complement methods .

2. Relation between the Schur complement and the dual Schur complement matrices .

Consider a subdomain Ω_s, with boundary intersection with the interface Γ_s. Number the degrees of freedoms located on internal nodes first, and then the ones located on the interface Γ_s .

Using subscipt i for the internal degrees of freedom of subdomain Ω_s , and I for the local interface Γ_s , the local stiffness matrix has the following block pattern :

$$\mathbf{K}_s = \begin{bmatrix} \mathbf{K}_{sii} & \mathbf{K}_{siI} \\ \mathbf{K}_{sIi} & \mathbf{K}_{sII} \end{bmatrix} \quad . \tag{1}$$

The contribution of subdomain Ω_s to the Schur complement matrix is as follows :

$$\mathbf{S}^{(s)} = \mathbf{K}_{sII} - \mathbf{K}_{sIi}\, \mathbf{K}_{sii}^{-1}\, \mathbf{K}_{siI} \quad .$$

Now, if no nodes on the global interface Γ_I belongs to more than two subdomains, the matrix of the restriction on Ω_s of the discrete global trace operator on the interface is the simple boolean matrix :

$$\mathbf{B}_s = \begin{bmatrix} \mathbf{O}_{si} & \mathbf{I}_{sI} \end{bmatrix} \quad ,$$

where $\mathbf{O}_{si}$ is a null matrix, and $\mathbf{I}_{sI}$ the identity matrix with dimension equal to the numbers of degrees of freedom on Γ_s .

The saddle point principle domain decomposition method, consists in introducing a Lagrange multiplier λ in order to remove the continuity constraint upon the displacements fields (Farhat, 1991). In the case of point-wise discrete Lagrange multipliers, the resulting hybrid problem is as follows :

$$\begin{aligned} \mathbf{K}_s\, \mathbf{u}_s &= \mathbf{f}_s - \mathbf{B}_s^t\, \boldsymbol{\lambda} \\ \sum_s \mathbf{B}_s\, \mathbf{u}_s &= 0 \end{aligned} \tag{2}$$

Then, the contribution of the subdomain Ω_s to the dual Schur complement matrix $\mathbf{D}$ obtained by a block Gaussian elimination of the displacements fields $\mathbf{u}_s$ in the hybrid set of equations (2) , takes the form :

$$\mathbf{D}^{(s)} = \mathbf{B}_s^t\, \mathbf{K}_s^{-1}\, \mathbf{B}_s = \begin{bmatrix} \mathbf{O}_{si} & \mathbf{I}_{sI} \end{bmatrix} \begin{bmatrix} \mathbf{K}_{sii} & \mathbf{K}_{siI} \\ \mathbf{K}_{sIi} & \mathbf{K}_{sII} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{O}_{si} \\ \mathbf{I}_{sI} \end{bmatrix} \quad . \tag{3}$$

The following matrices :

$$\begin{bmatrix} \mathbf{C}_{si} \\ \mathbf{C}_{sI} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{sii} & \mathbf{K}_{siI} \\ \mathbf{K}_{sIi} & \mathbf{K}_{sII} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{O}_{si} \\ \mathbf{I}_{sI} \end{bmatrix} \quad ,$$

satisfy the set of equations :

$$\begin{aligned} \mathbf{K}_{sii}\ \mathbf{C}_{si} + \mathbf{K}_{siI}\ \mathbf{C}_{sI} &= \mathbf{O}_{si} \\ \mathbf{K}_{sIi}\ \mathbf{C}_{si} + \mathbf{K}_{sII}\ \mathbf{C}_{sI} &= \mathbf{I}_{sI} \end{aligned} \quad . \tag{4}$$

After eliminating the $\mathbf{C}_{si}$ block in equation (4), we can derive from equation (3) :

$$\mathbf{D}^{(s)} = \mathbf{C}_{sI} = [\ \mathbf{K}_{sII} - \mathbf{K}_{sIi}\ \mathbf{K}_{sii}^{-1}\ \mathbf{K}_{siI}\ ]^{-1} = \left[\mathbf{S}^{(s)}\right]^{-1} \quad . \tag{5}$$

That means that the contributions of subdomain Ω_s to the Schur complement and to the dual Schur complement matrices are the inverse each one of the other. This fact is the algebraic translation of the duality of the two methods .

So, if there are no crosspoints on the interface Γ_I , the global Schur complement and dual Schur complement matrices are as follows :

$$\mathbf{S} = \sum_s \mathbf{S}^{(s)} \ , \ \mathbf{D} = \sum_s \mathbf{D}^{(s)} = \sum_s \left[\ \mathbf{S}^{(s)}\ \right]^{-1} \quad .$$

Thus, the so called local Neumann preconditioner of the Schur complement method (Bjorstadt & Wildlund, 1986, Le Tallec & al., 1990), is nothing else than the dual matrix **D** . So, the results that has been proved for the local Neumann preconditioner of the Schur complement method, apply for the dual Schur complement method with a local Dirichlet preconditioner .

Moreover, the condition numbers of the **D** and **S** matrices should be very similar, since it has been proved that the condition number of the **DS** matrix, does not depend upon the mesh size h, in the case where there is no crosspoint on the interface .

If there are some crosspoints on the interface, the local Neumann preconditioner of the Schur complement method differs from the dual Schur complement matrix at the crosspoints only, because there are several Lagrange multipliers for each degree of freedom attached with these nodes (Farhat & Roux 1991) .

3. Spectral patterns of the primal and dual Schur complement matrices .

To illustrate this section, we have performed some tests with a three dimensional cantilever beam, with a rectangular cross-section, splitted into two identical subdomains, each one featuring a cantilever beam with a square cross-section and a 5 ratio of the length upon the width .

We have implemented both primal and dual Schur complement methods for this problem, with two different meshes, for a shear stress loading. With the first mesh, each sub-domain has 400 nodes, and 1200 degrees of freedom, and the interface has 64 nodes and 192 degrees of freedom. With the second one, the numbers of points per subdomain is 2720, that means 8160 degrees of freedom, and the number of points on the interface is 224, that means 672 degrees of freedom .

The dual Schur complement method has also been implemented with an even finer third mesh with 20032 nodes and 60096 degrees of freedom in each subdomain, and 832 nodes and 2496 degrees of freedom on the interface. The others methods have not been implemented because, in the particular case presented here, as the subdomains are geometrically identical, the local matrices of the Neumann problem are identical too, whereas the local matrices of the Dirichlet problem are not, due to the different location of the interface within each subdomain. The memory requirements for the storage of the $\mathbf{LDL}^t$ decomposition of all the local matrices should have exceeded 128 Mwords .

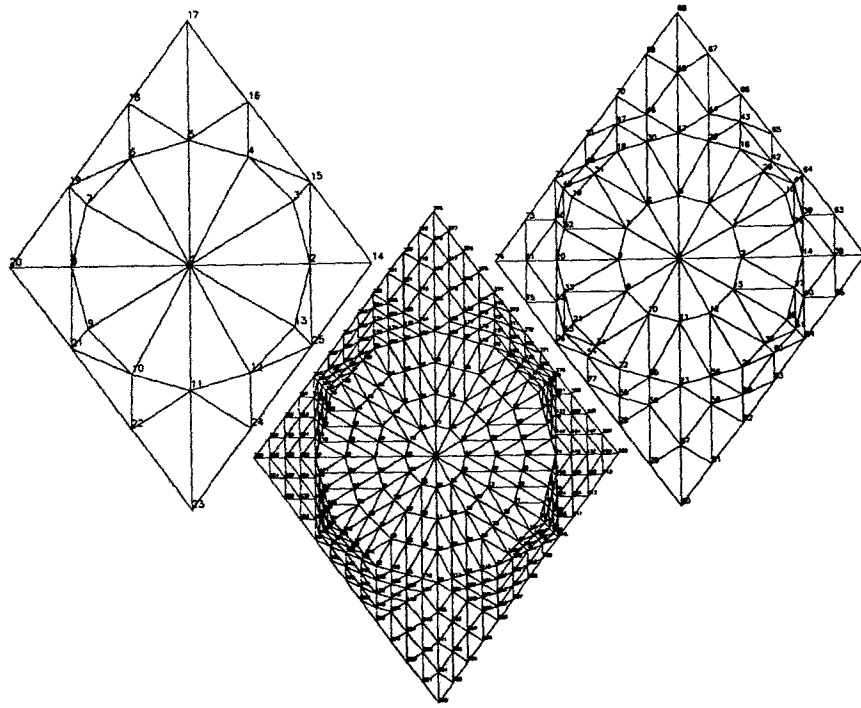

Figure 1 : different meshes of a cross-section of the cantilever beam .

The numbers of iterations to obtain 10^{-2} , 10^{-4} and 10^{-6} global residuals $||\mathbf{Ku} - \mathbf{b}||/||\mathbf{b}||$ for preconditioned and non preconditioned primal and dual Schur complement methods figure in the following tables .

	Mesh 1 : 192 interface d.o.f.		
global residual	10^{-2}	10^{-4}	10^{-6}
primal	22	28	34
preconditioned primal	4	7	10
dual	7	16	23
preconditioned dual	4	7	10

	Mesh 2 : 672 interface d.o.f.		
global residual	10^{-2}	10^{-4}	10^{-6}
primal	21	27	34
preconditioned primal	4	7	11
dual	7	17	24
preconditioned dual	4	7	11

	Mesh 3 : 2496 interface d.o.f.		
global residual	10^{-2}	10^{-4}	10^{-6}
dual	9	18	27

The first conclusion we can derive from these results is that the dependance of the number of iterations upon the mesh size is low, and that, without preconditioning, the dual method performs better than the primal .

Furthermore, the preconditioned algorithms perform quite well, but, as preconditioning roughly doubles the computational cost and memory requirement, because both local Neumann and Dirichlet problems have to be solved at each iteration, the unpreconditioned dual method compares quite well with the preconditioned ones .

We obtained the same kind of results with bigger and stiffer problems, with even a sharper difference between the unpreconditioned primal and dual Schur complement methods .

After computing the various interface matrices and their eigenvalues we obtained the values of the condition numbers :

Mesh 1 : 192 interface d.o.f.	
primal	64.0
preconditioned primal	2.87
dual	64.6
preconditioned dual	2.87

Mesh 2 : 672 interface d.o.f.	
primal	56.0
preconditioned primal	2.82
dual	60.1
preconditioned dual	2.82

Mesh 3 : 2496 interface d.o.f.	
dual	99.5

The reason why both preconditioned methods give the same results is clear. The matrix of the preconditioned Schur complement method is **DS**, when the matrix of the preconditioned dual Schur complement method is **SD**. As the matrices **S** and **D** are symmetric, $\mathbf{DS} = (\mathbf{SD})^t$.

But the values of the condition numbers cannot explain why the dual method performs better than the primal one. In order to understand why it does, the spectral density of each matrix has to be studied .

The following bar diagrams show the density of the spectra of the preconditioned and unpreconditioned primal and dual Schur complement matrices, for meshes 1 and 2 .

The ratio of the eigenvalue upon the smaller one figures on the x axis, and the number of eigenvalues per interval, on the y axis. Both axes are drawn with logarithmic scaling. Only one figure is presented for both preconditioned matrices, because their spectra are identical, as we proved it above .

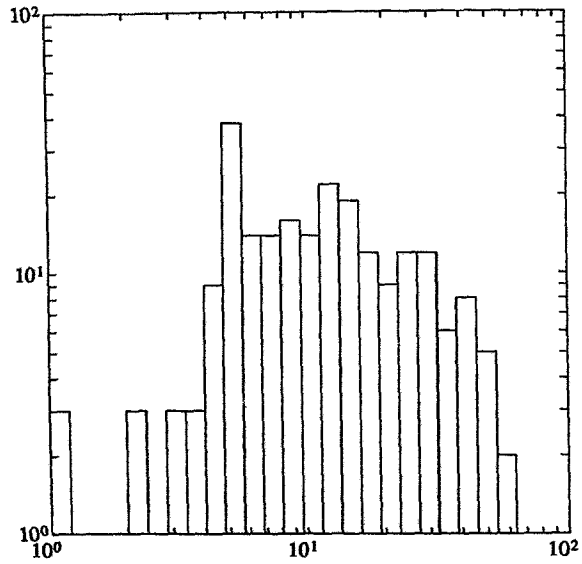

Figure 2 : primal Schur complement , mesh 1 .

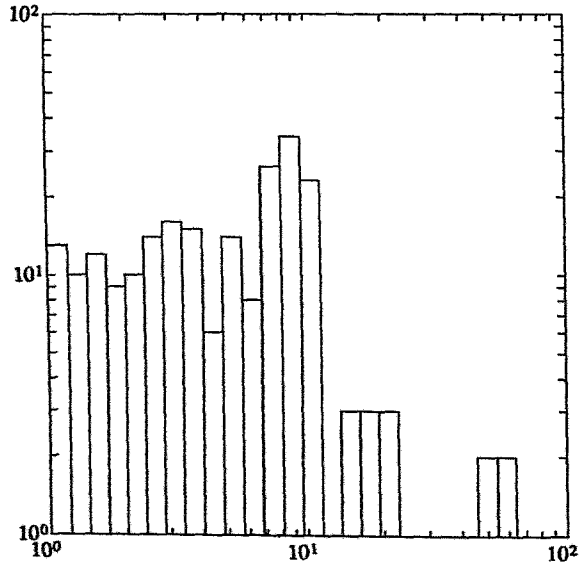

Figure 3 : dual Schur complement , mesh 1 .

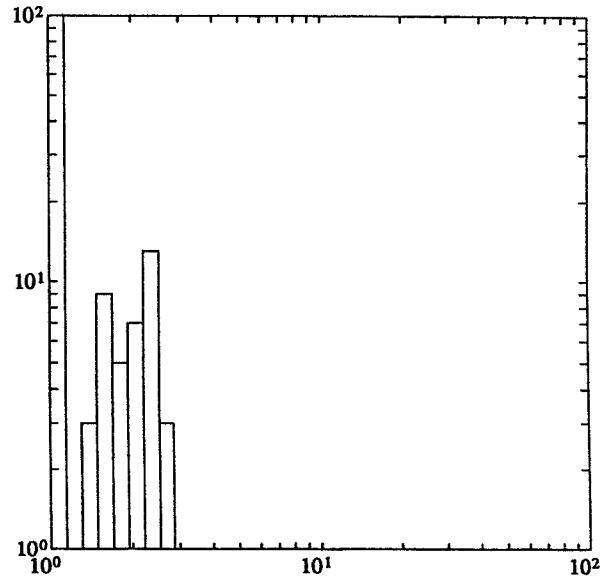

Figure 4 : preconditioned Schur complement , mesh 1 .

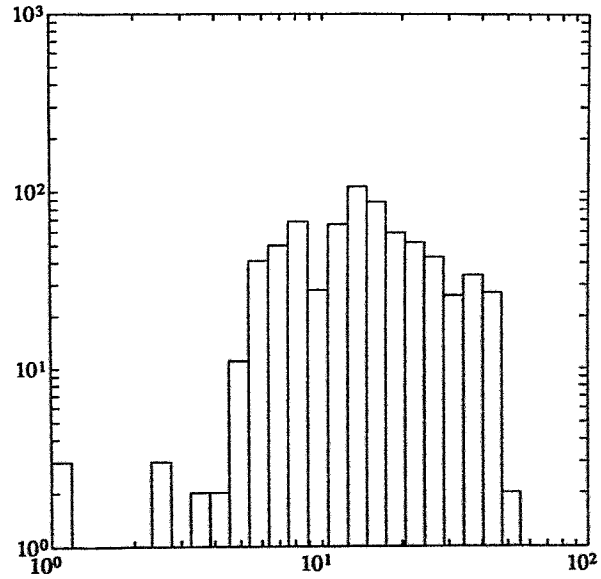

Figure 5 : primal Schur complement , mesh 2 .

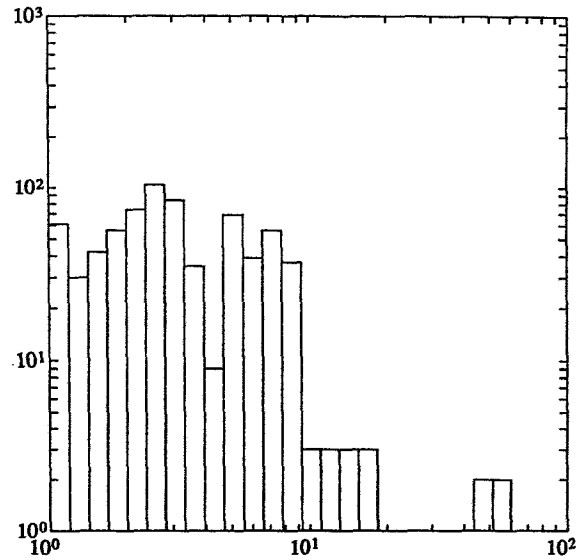

Figure 6 : dual Schur complement , mesh 2 .

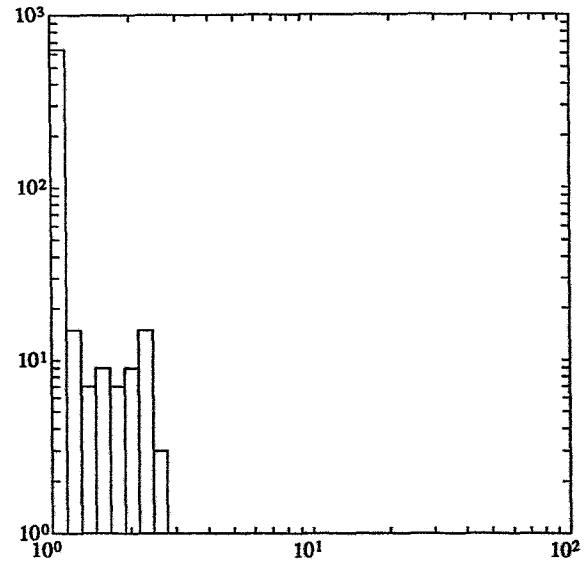

Figure 7 : preconditioned Schur complement , mesh 2 .

The dual Schur complement matrix has a few large eigenvalues. On the opposite, the primal Schur complement has a few small eigenvalues, that have actually the order of magnitude of the inverses of the largest eigenvalues of the dual matrix .

The faster convergence observed with the dual Schur complement method can be explained by the "super-linear" convergence of the conjugate gradient method, that occurs when the matrix has only a few extremal eigenvalues. We shall give more details in the next section .

These different patterns of the spectra of the matrices can be explained with both mechanical and mathematical viewpoints .

The Schur complement matrix is the matrix of a condensed primal problem on the interface. It is a stiffness matrix. Its lowest eigenvalues correspond to the low frequencies of the interface, and so to the lower frequencies of the global structure. The largest eigenvalues correspond to the highest frequencies, that means to the local modes .

When decreasing the mesh size, the number of medium and high frequencies that can be numerically captured increase, while the lowest eigenvalues, that are already a good approximation of the physical low frequencies, do not change .

On the opposite, the dual Schur complement matrix is a compliance matrix, wich few highest eigenvalues are of the order of magnitude of the inverses of the lowest eigenvalues of the global structure .

As the first iterations of the conjugate gradient capture mostly the coordinates of the solution according to the eigenvectors associated with the highest eigenvalues, the dual Schur complement method gives very quickly a good approximation of the global displacements .

On a mathematical point of view, the spectral patterns of the matrices can be explained by the compactness of the continuous operators .

The local Schur complement matrix $\mathbf{S}^{(s)}$ is the discretisation of the mapping of the trace on the interface of the diplacements field solution of the linear elasticity equation on the subdomain onto the external forces to be applied on the interface, in order to get the same displacements .

Under certain assumptions, this mapping, that is called a Steklov-Poincaré operator (Agoshkov, 1988), is a one to one continuous mapping of $(\mathbf{H}_{00}^{1/2}(\Gamma_s))^3$ onto $(\mathbf{H}^{-1/2}(\Gamma_s))^3$.

The local dual Schur complement matrix $\mathbf{D}^{(s)}$ is the dicretisation of the reciproqual mapping from $(\mathbf{H}^{-1/2}(\Gamma_s))^3$ on $(\mathbf{H}_{00}^{1/2}(\Gamma_s))^3$. Due to the compactness of the mapping of $(\mathbf{H}_{00}^{1/2}(\Gamma_s))^3$ into $(\mathbf{H}^{-1/2}(\Gamma_s))^3$, the continuous operator associated with $\mathbf{D}^{(s)}$ is compact from $(\mathbf{H}^{-1/2}(\Gamma_s))^3$ on $(\mathbf{H}^{-1/2}(\Gamma_s))^3$. The same is true for the global interface operator that is the sum of the local ones .

Hence, the eigenvalues of the continuous dual operator form a decreasing series converging to zero. This fact explains why the eigenvalues of the discrete **D** matrix tend to accumulate to zero when the meshsize decreases, as it can be seen on figures 3 and 6, and on the figure 8 below .

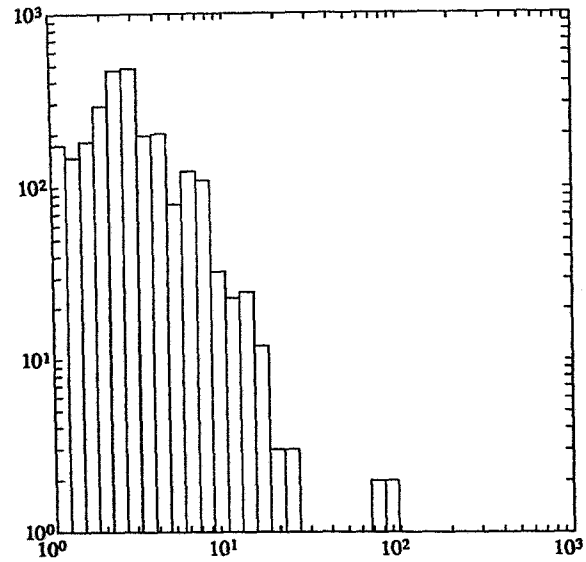

Figure 8 : dual Schur complement , mesh 3 .

4. Convergence of the conjugate gradient algorithm for matrices with a few extremal eigenvalues .

The conjugate gradient algorithm consists in computing, at iteration number j, the projection, with respect to the dot product associated with the matrix **K**, of the solution of the equation $\mathbf{Kx} = \mathbf{b}$ onto the space $\mathbf{x}_0 + \text{span}\left\{ \mathbf{r}_0 , \mathbf{Kr}_0 , \mathbf{K}^2\mathbf{r}_0 , \cdots , \mathbf{K}^{j-1}\mathbf{r}_0 \right\}$, where $\mathbf{x}_0$ is the starting vector, and r_0 the residual $\mathbf{b} - \mathbf{Kx}_0$.

The j eigenvalues, $\lambda_1^j < \lambda_2^j < \cdots < \lambda_j^j$, of the orthogonal projection of the operator **K** upon the Krylov space span $\mathbf{r}_0 , \mathbf{Kr}_0 , \mathbf{K}^2\mathbf{r}_0 , \cdots , \mathbf{K}^{j-1}\mathbf{r}_0$ are called the Ritz values .

It can be shown, using a minimax argument, that for any fixed k, λ_k^j decreases and λ_{j-k}^j increases. Hence, if we note $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, the eigenvalues of the **K** matrix, when λ_k^j is in the interval $[\lambda_k , \lambda_{k+1}]$ it is also true for any superior value of j . In the same way, once λ_{j-k}^j is in the interval $[\lambda_{n-k-1} , \lambda_{n-k}]$, it is also the case for any larger value of j .

In van der Sluis & van der Vorst (1986), it was proved that once the extremal Ritz values have entered what is refered as their final intervals, then the following conjugate gradient iterations behave as if the corresponding eigenvalues of the matrix were not present. That means as if the spectrum of the matrix and, so, the condition number were reduced .

In order to illustrate this points, we have performed some tests with a 200×200 diagonal matrix, with eingenvalues between $10^{-0.5}$ to $10^{5.5}$. So, the condition number of the matrix is 10^6 , that is representative of lots of real-life structural analysis problem .

The eigenvalues are clustered into two subsets. The largest eigenvalues vary from $10^{4.5}$ to $10^{5.5}$, and the smallest ones from $10^{-0.5}$ to $10^{0.5}$.

In the figures above, we present, for various dispatchings of the eigenvalues, the evolution of the residual $\|\mathbf{Kx} - \mathbf{b}\|/\|\mathbf{b}\|$, with $\mathbf{b} = diag\,(\mathbf{K})$.

The number of iterations figures on the x axis, and the logarithm of the residual on the right y axis .

At each iteration, we computed all the Ritz values, and they figure with a x mark on a vertical line. The horizontal lines show the exact position of the eigenvalues belonging to the coarser cluster. The left y axis figures the logarithm of the eingenvalues .

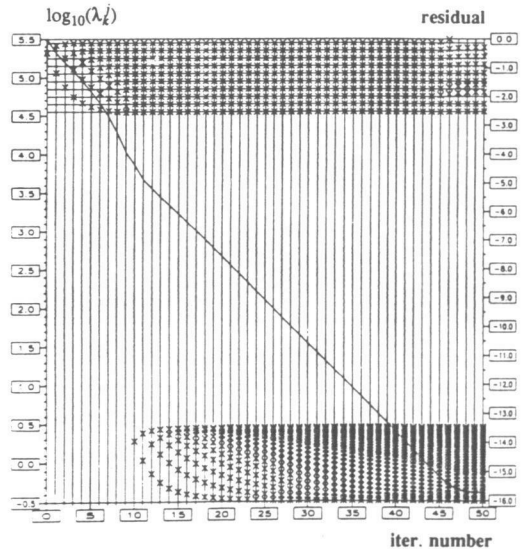

Figure 9 : 10 large , 190 small eigenvalues .

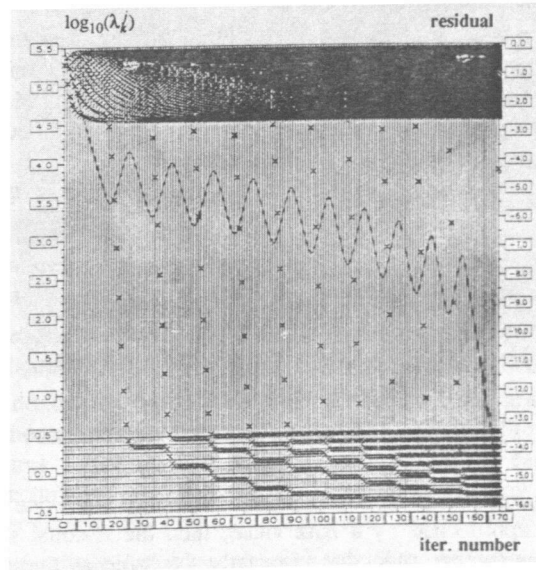

Figure 10 : 190 large , 10 small eigenvalues .

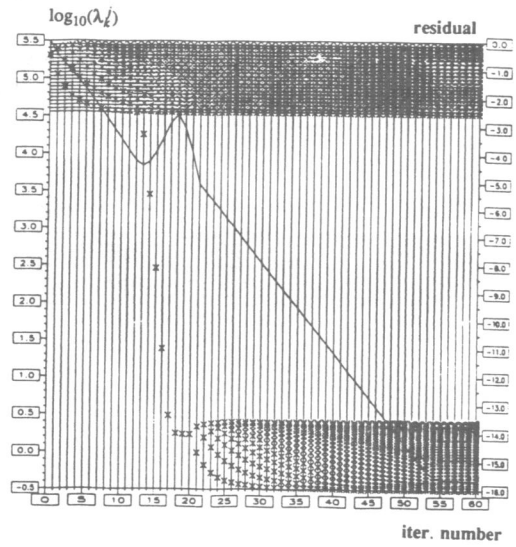

Figure 11 : 20 large , 180 small eigenvalues .

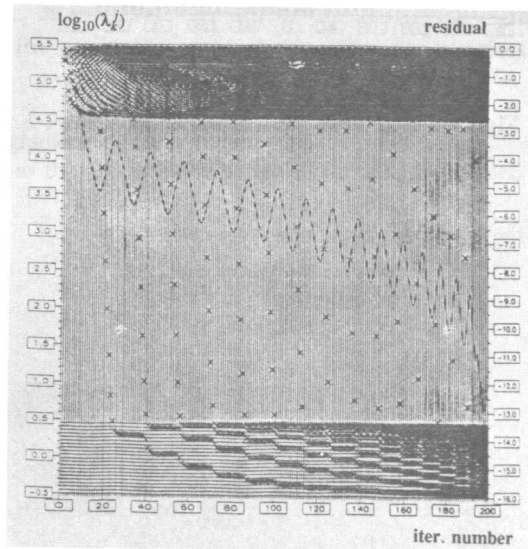

Figure 12 : 180 large , 20 small eigenvalues .

On all these figures, it can be actually seen that, once each one of the eigenvalues of the coarser cluster is approached by a Ritz value, then the residual tends to decrease in a linear way, with a convergence ratio that is exactly the same as the one we obtained for a matrix with only one cluster of eigenvalues and a 10^1 condition number .

Accordingly with a suggestion made in van der Sluis & van der Vorst (1986), the residual tends to be the most oscillating when one of the Ritz value is far from any of the actual

eigenvalues of the matrix .

But the most interesting point lies in the comparison between the situation where there are only a few large eigenvalues, and the opposite situation. Because, firstly, it is vey clear that the upper part of the spectrum is approximated much faster than the lower part, that is hardly a surprise considering the way the Krylov space is build. So, the starting point of the fast linear convergence is reached much earlier in the case where the highest part of the spectrum is coarse than in the opposite case. Secondly, the residual tends to decrease faster at the first iterations when there are few large eigenvalues. Because having a good approximation of the coordinates of the solution according to the eigenvectors associated with a part of the spectrum is more effective in reducing the residual when this part is the upper end than when it is the lower end of the spectrum .

These conclusions are totally conforming with the results we obtained in comparing the beheavior of the primal and dual Schur complement methods and the spectral patterns of the associated interface matrices .

5. Stability and orthogonality .

The rapid convergence of the largest Ritz values to the eingenvalues of the matrix with a coarse upper end of the spectrum was already studied in Parlett (1980). But it was also shown that such a spectral pattern causes rapid loss of orthogonality of the direction vectors .

The effects of rounding errors on the orthogonality are all the more amplified at the following iterations as they concern eigenvectors associated with eigenvalues that are large in proportion with most of the others. Increasing the precision of the computation is not effective, because the propagation of the errors depends in a polynomial way upon the eigenvalues ratio.

A simple way to get rid of the distrasous consequence of the loss of orthogonality upon the convergence of the conjugate gradient algorithm consists in filtering the roundin errors wit the help of a reconjugation process (Roux, 1990) .

The most efficient one consists in keeping all the direction vectors $\mathbf{p}_k$ and their products by the matrix $\mathbf{Kp}_k$. Then, at ech iteration of the conjugate gradient algorithm, the new direction vector is computing in such a way that it is actually conjugate to all the previous ones with respect to the dot product associated with the matrix, according to the following formula .

$$\mathbf{p}_j = \mathbf{r}_j + \sum_{k=0}^{k=j-1} \gamma_k \, \mathbf{p}_k \text{ , with } \gamma_k = - \frac{(\mathbf{r}_j \text{ , } \mathbf{Kp}_k)}{(\mathbf{Kp}_k \text{ , } \mathbf{p}_k)} \text{ .}$$

This reconjugation process entails, at iteration number j, the computation of j dot products ($\mathbf{r}_j$, $\mathbf{Kp}_k$), and of a $n \times j$ matrix vector product to determin the new direction vector.

Using such a process would be a non-sense, in the case of a sparse matrix **K**, because it would cause a tremendous increase of both CPU time and memory requirement .

But in the scope of domain decomposition method, the conjugate gradient algorithm applies only to interface unknowns, whereas computing the product by the primal or dual Schur complement matrix requires a forward and backward substitution for each subdomain. So, the relative cost of the reconjugation process on the interface is very low. This has been proved by all the experiments we realized with real-life stuctural analysis problem .

All the results presented in this paper, and particularly the ones of the previous section, were obtained with applying the total reconjugation process .

Nevertheless, the cost of the process can be even more reduced by limiting the number of the direction vectors actually kept. In such a case, the optimal strategy consists in keeping the

first direction vectors, instead of the last ones as in the GMRES algorithm for instance, because the subspace generated by these vectors is to be close to the subspace generated by the eigenvectors associated with the highest eigenvalues .

If the maximum number of direction vectors stored is equal to $nd+1$ the formula to compute the j-th direction vector, with $j > nd$ becomes :

$$\mathbf{p}_j = \mathbf{r}_j + \gamma_{j-1}\,\mathbf{p}_{j-1} + \sum_{k=0}^{k=nd-1} \gamma_k\,\mathbf{p}_k \text{ , with } \gamma_k = -\frac{(\mathbf{r}_j\ ,\ \mathbf{K}\mathbf{p}_k)}{(\mathbf{K}\mathbf{p}_k\ ,\ \mathbf{p}_k)} .$$

The tests we performed shew that keeping a small number of direction vectors (a few tens for several thousands degrees of freedom on the interface) was enough to get almost the same stability of the conjugate gradient than with the complete reconjugation .

6. An efficient preconditioner for the dual Schur complement .

For all the tests we have made with real-life problems, preconditioning the dual Schur complement method with the local Dirichlet solvers, that is, in the case where there is no crosspoint on the interface, with the primal Schur complement matrix, never entailed a diminution of the number of iterations by a factor larger than two .

So, this preconditioner is not actually efficient, even though it drastically reduces the condition number of the problem, because the CPU time and memory requirement are almost doubled, when using it .

Another way to precondition the dual Schur complement matrix, wich was introduced in (Farhat & Roux, 1991), consists in using the restriction on the interface of the global stiffness matrix $\mathbf{K}_{II}$.

As the dual Schur complement matrix is a compliance matrix, it must be preconditioned by a stiffness matrix. The primal Schur complement is the optimal one, theoretically, but it is too expensive in practice .

Computing the product by the interface stiffness matrix, consists in computing in each subdomain Ω_s, the product by the bloc matrix $\mathbf{K}_{sII}$, and then assemble all the local contributions as for the Schur complement .

The main difference with the local Dirichlet preconditioner lies in the fact that the block matrices $\mathbf{K}_{sII}$ are sparse, and so, this process is unexpensive on both CPU time and memory requirement viewpoints .

All the tests performed with this preconditioner leads to the same conclusion that it does not modify the convergence at the first iterations, but it increase the convergence ratio when the starting point of the fast linear convergence is reached .

In order to understand this phenomenon, we computed the eigenvalues of the preconditioned dual Schur complement matrix, for the three different problems introduced in section 2 . The condition numbers were as in the following table .

Condition number of the preconditioned dual Schur complement	
Mesh 1 : 192 interface d.o.f.	38.3
Mesh 2 : 672 interface d.o.f.	69.0
Mesh 3 : 2496 interface d.o.f.	129.5

The comparison with the tables in section 2 shows that the condition number is not generally reduced. But the spectral patterns of the preconditioned dual Schur complements presented in

the figures below, show a concentration of the lower parts of the spectra, whereas the highest eigenvalues do not change a lot .

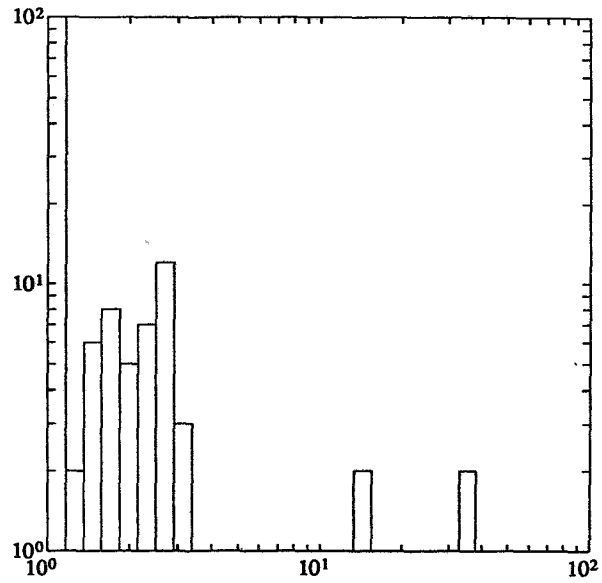

Figure 13 : spectrum of the preconditioned dual matrix, mesh 1 .

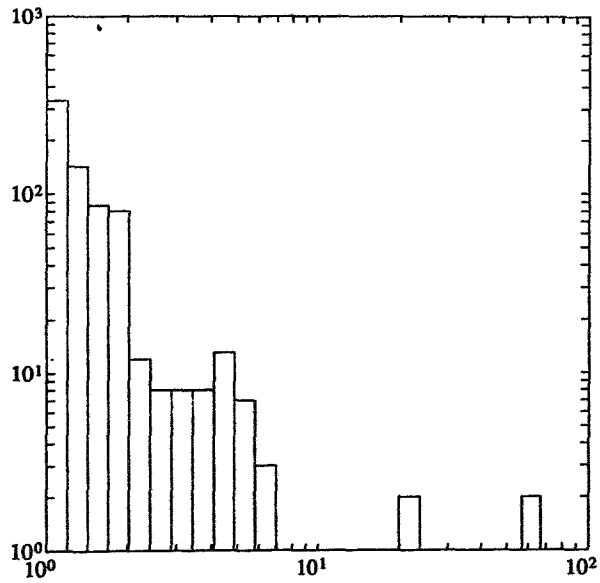

Figure 14 : spectrum of the preconditioned dual matrix, mesh 2 .

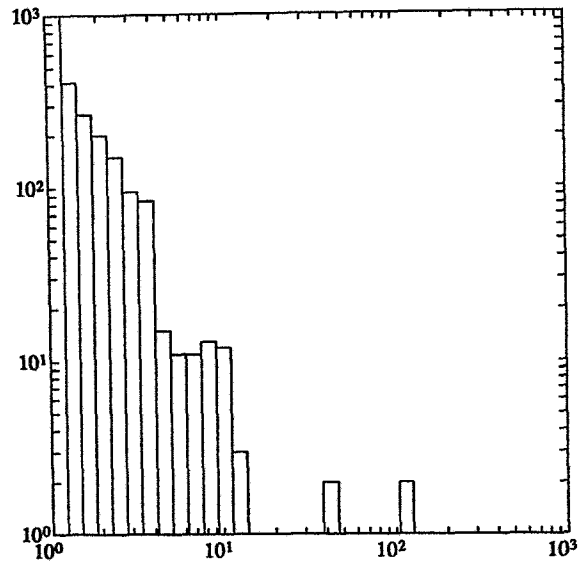

Figure 15 : spectrum of the preconditioned dual matrix, mesh 3 .

These patterns explain why the linear convergence ratios obtained once the largest Ritz values have entered their final intervals are better than the ones obtained without preconditioning. The possible increase of the condition number does not cause any trouble because it is only due to the increase of a very small number of large eingenvalues .

So, preconditioning with the restriction on the interface of the global stiffness matrix and with the primal Schur complement matrix appears to have similar consequences for the lower part of the spectrum of the dual matrix .

In the first case, the contribution of subdomain Ω_s to the preconditioner is the block $\mathbf{K}_{sII}$, and in the second case, it is the local Schur complement matrix : $\mathbf{K}_{sII} - \mathbf{K}_{sIi}\ \mathbf{K}_{sii}^{-1}\ \mathbf{K}_{siI}$.

The smallest eigenvalues of the dual Schur complement correspond to the inverse of the largest eigenvalues of the primal Schur complement. As the block $\mathbf{K}_{sIi}\ \mathbf{K}_{sii}^{-1}\ \mathbf{K}_{siI}$ is symmetric and positive definite, for a vector v such that :

$$\left(\left[\sum_s \mathbf{K}_{sII} - \mathbf{K}_{sIi}\ \mathbf{K}_{sii}^{-1}\ \mathbf{K}_{siI}\right]\mathbf{v}\ ,\ \mathbf{v}\right) \gg \left(\left[\sum_s \mathbf{M}_s\right]\mathbf{v}\ ,\ \mathbf{v}\right)\ ,$$

$$\text{then}\quad \left(\left[\sum_s \mathbf{K}_{sII}\right]\mathbf{v}\ ,\ \mathbf{v}\right) > \left(\left[\sum_s \mathbf{K}_{sIi}\ \mathbf{K}_{sii}^{-1}\ \mathbf{K}_{siI}\right]\mathbf{v}\ ,\ \mathbf{v}\right)\ ,$$

$$\text{and}\quad \left(\left[\sum_s \mathbf{K}_{sII}\right]\mathbf{v}\ ,\ \mathbf{v}\right) \gg \left(\left[\sum_s \mathbf{M}_s\right]\mathbf{v}\ ,\ \mathbf{v}\right)\ .$$

The $\mathbf{M}_s$ matrices are the local mass matrices .

The minimax argument suggests that the highest eigenvalues of the matrix $\left[\sum_s \mathbf{K}_{sII}\right]$ should be close of the highest eigenvalues of matrix $\left[\sum_s \mathbf{K}_{sII} - \mathbf{K}_{sIi}\ \mathbf{K}_{sii}^{-1}\ \mathbf{K}_{siI}\right]$, and so, the subspaces generated by the associated eigenvectors should be close too .

This gives an heuristic explanation of the fact that, when used as preconditioners of the dual Schur complement matrix, both matrices have similar effects upon the lower end of the spectrum of the dual matrix, that appears to be tightly related to the upper end of the spectrum of the primal one .

7. Conclusion .

Although both primal and dual Schur complement matrices have similar condition numbers, the conjugate gradient algorithm converges much faster for the dual problem than for the primal one, provided that the reconjugation procedure presented in section 5 is implemented .

The primal Schur complement method can be very efficiently preconditioned by solving local Neumann problems, but in this case, as it was shown in Le Tallec & al. (1990), the same unstability occurs than with the dual Schur complement method, and so, the use of the reconjugation process is mandatory .

One iteration of the primal Schur complement method with the local Neumann preconditioner is roughly twice more expensive than one iteration of the dual Schur complement with the unexpensive preconditionner presented in section 6, as the main part of the computations lies in the solution of the local problems, and that the solutions of the local Dirichlet and Neumann problems require similar amounts of computation .

Furthermore, the main part of the data lies in the Choleski or Crout factorizations of the matrices of the local problems, and so, the use of the local Neumann preconditioner for the primal Schur complement method nearly doubles the total memory requirement .

So, the primal Schur complement method with the local Neumann preconditioner is twice more expensive, per iteration, than the dual Schur complement with the preconditioner presented in section 5, on both computation cost and memory requirement viewpoints .

Futhermore, each time we compared both methods for solving real-life structural analysis problems, the number of iterations to get a given value for the global residual with the primal method has never been less than half the number of iterations necessary with the dual one .

The conclusion that can be derived from these tests is that the dual Schur complement method is more cost-effective than the primal Schur complement method, as it requires at worst the same global CPU time and twice less memory. And it is better suited for being implemented on distributed memory parallel machines, as it was shown in Farhat & Roux (1991) .

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