CHAPTER 4

The Schwarz Algorithm and Multilevel Decomposition Iterative Techniques for Mixed Finite Element Methods

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ABSTRACT. The Schwarz alternating algorithm [18] and the multilevel decomposition iterative method are presented in this paper for mixed finite element methods for second-order elliptic equations. Convergence estimates similar to [3,4,20] are established. Some numerical results illustrate the efficiency of our methods.

1. Introduction. Our object in this paper is to propose and study the use of the Schwarz alternating algorithm and some multilevel decomposition iterative techniques in the mixed finite element method for second-order elliptic equations. For simplicity, we take as our model the homogeneous Neumann boundary value problem. The weak form of the problem seeks $(\boldsymbol{u};p) \in H_0(\text{div};\Omega) \times L^2(\Omega)$ such that

$$(1.1)\qquad \begin{aligned} (c\boldsymbol{u},\boldsymbol{v}) - (\nabla\cdot\boldsymbol{v},p) &= 0, & \boldsymbol{v}&\in\mathcal{V},\\ (\nabla\cdot\boldsymbol{u},w) &= (f,w), & w&\in W,\end{aligned}$$

where Ω is a polygonal domain in $\mathbb{R}^2$, and $\boldsymbol{u} = -a\nabla p$ is the flux variable. The function $a(x) = c^{-1}(x)$ is the coefficient of the problem, which is assumed symmetric and positive definite. For simplicity, we let $\mathcal{V} = H_0(\text{div};\Omega)$ and $W = L^2(\Omega)$. The mixed finite element method for (1.1) seeks $(\boldsymbol{u}_h;p_h)$ from $\mathcal{V}^h \times W^h$, a mixed finite element space associated with a prescribed triangulation $\mathcal{T}_h$, satisfying

$$(1.2)\qquad \begin{aligned} (c\boldsymbol{u}_h,\boldsymbol{v}) - (\nabla\cdot\boldsymbol{v},p_h) &= 0, & \boldsymbol{v}&\in\mathcal{V}^h,\\ (\nabla\cdot\boldsymbol{u}_h,w) &= (f,w), & w&\in W^h.\end{aligned}$$

Many physical problems, for example petroleum reservoir simulation, modeling of ground-water contamination, elasticity problems, and seismic exploration, involve the need for very accurate determination of the flux variable. Very accurate approximations of the flux can be achieved through the use of mixed finite element methods, particularly with discontinuous coefficients $a(x)$, since the flux is introduced as an independent variable in the method. However, mixed finite element methods lead to saddle-point problems whose numerical solution has been quite difficult. Thus, fast and efficient algorithms for solving the discretized problem are needed for the application of mixed methods.

Here we report on some recent work on a solution method for (1.2). Details of the results can be found in [8,9]. We briefly describe mixed finite element methods in §2 and describe the Schwarz alternating algorithm and the convergence estimates in §3. In §4, we introduce a multilevel decomposition iterative method for (1.2) based on the natural multilevel structure of the mixed finite element space. We eliminate the pressure p_h from the saddle-point problem (1.2) by seeking a

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discrete flux $u^* \in \mathcal{V}^h$ whose divergence gives rise to the divergence of u_h. We then apply product algorithms (cf. [3,19,20]) to the reduced problem, which is symmetric and positive definite for the flux variable only. Numerical illustrations are presented in §5.

2. Mixed finite element methods. We outline two families of mixed finite element spaces, one on triangles and one on rectangles:

RT Triangular Elements: Let $x = (x, y)$ be the space variable. The RT (Raviart-Thomas) space [17] of index j on the triangle K for the flux is defined by $\mathcal{V}^h(K) = P_j(K) \oplus x\hat{P}_j(K)$, where $\hat{P}_j(K)$ denotes homogeneous polynomials of degree j on K. The corresponding pressure space is $W^h(K) = P_j(K)$.

BDFM Elements: The BDFM (Brezzi-Douglas-Fortin-Marini) spaces (cf. [6]) are modifications of the rectangular RT spaces. The space of index j for the flux variable is defined by $\mathcal{V}^h(K) = P_j(K)\backslash\{y^j\} \times P_j(K)\backslash\{x^j\}$; the corresponding space for the pressure is defined by $W^h(K) = P_{j-1}(K)$, where $P_i(K)$ denotes the polynomials of total degree no larger than i.

It is known that these two families are stable, which means that the Babuška-Brezzi stability condition is satisfied. For a more detailed discussion of the mixed method, see [1,5,6,7,12,13,17].

The iterative methods proposed in this paper are applied to a positive definite problem defined on a subspace $\mathcal{H}^h$ of $\mathcal{V}^h$. The subspace $\mathcal{H}^h$ consists of those discrete fluxes that are divergence free; i.e., $\mathcal{H}^h = \{v \in \mathcal{V}^h;\ \nabla \cdot v = 0\}$. Thus, any flux $v \in \mathcal{H}^h$ can be expressed as the **curl** of a stream function $\phi \in H^1(\Omega)$. Furthermore, the stream function ϕ is uniquely determined in $H_0^1(\Omega)$, since the flux has zero boundary values in the normal direction to $\partial\Omega$. Denote by $\mathcal{S}^h$ the set of stream functions with vanishing boundary value. The space $\mathcal{S}^h$ shall be termed the stream-function space. Note that any stream function ψ is a continuous piecewise polynomial. Thus, $\mathcal{S}^h$ is a finite element space of C^0-piecewise polynomials associated with $\mathcal{T}_h$.

The stream-function space for the families mentioned above can be characterized as follows.

Theorem 2.1. *Let $\mathcal{S}^h$ denote the stream-function space. Then,*

(1) *for the RT triangular element of index $j \geq 0$, $\mathcal{S}^h = \{\phi \in C^0(\Omega); \phi|_K \in P_{j+1}(K), K \in \mathcal{T}_h\}$;*

(2) *for the BDFM element of index j, $\mathcal{S}^h = \{\phi \in C^0(\Omega); \phi|_K \in P_{j+1}(K)\backslash\{x^{j+1}, y^{j+1}\}, K \in \mathcal{T}_h\}$.*

Proof. We illustrate the proof for the RT triangular elements only; the analysis for the BDFM is similar. Let $\mathcal{S}^h$ be defined as in the theorem. It is obvious that **curl** ϕ is a discrete flux in the RT space of index j. Further, it is divergence free. Thus, $\mathcal{S}^h$ is a subspace of the stream-function space for the RT element of index j. Conversely, for any $v \in \mathcal{H}^h$, let $\phi \in H_0^1(\Omega)$ be the stream function of v. Since v is divergence free, we know that $v|_K \in P_j(K)^2$ on any $K \in \mathcal{T}_h$. Thus, ϕ is a continuous piecewise polynomial of order $j+1$, which implies $\phi \in \mathcal{S}^h$.

3. Schwarz alternating algorithm. Assume that we have an overlapping domain decomposition for Ω which aligns with $\mathcal{T}_h$ on the boundary; i.e., there exist subdomains $\Omega_i \subset \Omega$, for $i = 1, \cdots, J$, such that $\Omega = \cup_{i=1}^J \Omega_i$. Further, for any element $K \in \mathcal{T}_h$ and index i, K either is entirely in Ω_i or has an empty intersection with Ω_i. Thus, the restriction of $\mathcal{T}_h$ on Ω_i provides a regularly defined triangulation $\mathcal{T}_i$ for Ω_i. Let $\mathcal{V}_i^h \times W_i^h$ be the corresponding finite element space associated with $\mathcal{T}_i$. Analogously, set $\mathcal{H}_i^h = \{v \in \mathcal{V}_i^h;\ \nabla \cdot v = 0\}$.

The first step in the Schwarz alternating method involves seeking a discrete flux $u^* \in \mathcal{V}^h$ such that

$$\nabla \cdot u^* = f^h, \tag{3.1}$$

where $f^h \in W^h$ is the standard L^2 projection of f on W^h. To obtain such a flux u^*, let $\mathcal{T}_0 = \{K_i\}_{i=1}^L$ be a 'coarse' triangulation of Ω whose elements align with those of $\mathcal{T}_h$ on the boundary. Hence, $\mathcal{T}_h$ can be regarded as a refinement of $\mathcal{T}_0$. As before, let $\tilde{\mathcal{V}}_i^h \times \tilde{W}_i^h$ be the finite element space associated with the triangulation $K_{i,h}$, which is the restriction of $\mathcal{T}_h$ on K_i. Let f_0^h be the L^2 projection of f^h in the space $\tilde{W}_0^h$, and $f_i^h \in \tilde{W}_i^h$ be the restriction of $f^h - f_0^h$ on K_i. It follows that $f^h = f_0^h + \sum_{i=1}^L f_i^h$.

Schwarz Algorithm (Part 1):

(1) For each $i, 0 \leq i \leq L$, find $(u_i^*; p_i^*) \in \tilde{\mathcal{V}}_i^h \times \tilde{W}_i^h$ such that

$$\begin{aligned} (\tilde{c}u_i^*, v) - (\nabla \cdot v, p_i^*) &= 0, & v &\in \tilde{\mathcal{V}}_i^h, \\ (\nabla \cdot u_i^*, w) &= (f_i^h, w), & w &\in \tilde{W}_i^h, \end{aligned} \tag{3.2}$$

where $\tilde{c}$ is an arbitrary positive function on Ω.

(2) Set $u^* = \sum_{i=0}^{L} u_i^*$.

Remark 3.1. As indicated by (3.2), the coefficient $\tilde{c}$ may differ from c. This is so because we only care to have some discrete flux satisfying the second equation of (3.2). Therefore, one may, for instance, take $\tilde{c} = 1$ or $\tilde{c} = c$ for the sake of convenience in the real computation.

Theorem 3.1. *Let the discrete flux u^* be obtained as above. Then, $\nabla \cdot u^* = f^h$.*

The proof of Theorem 3.1 is straightforwad from the definition of u^* combined with (3.2) and the decomposition for f. Now the saddle-point problem (1.2) can be reduced to a positive definite problem for the flux by setting $\hat{u}^h = u_h - u^*$ and then seeking $\hat{u}^h \in \mathcal{H}^h$ satisfying

$$(3.3) \qquad (c\hat{u}^h, v) = -(cu^*, v), \qquad v \in \mathcal{H}^h.$$

Let P_i be the projection operator from $\mathcal{H}^h$ to $\mathcal{H}_i^h$ defined by

$$(3.4) \qquad (c\, P_i\xi, v) = (c\, \xi, v), \qquad \xi \in \mathcal{H}^h,\ v \in \mathcal{H}_i^h.$$

Assume, here and throughout this paper, that ω is a real number in $(0, 2)$.

Schwarz Algorithm-1 (Part 2): Given $\hat{u}_n^h \in \mathcal{H}^h$, an approximation to (3.4), we seek the next approximate solution $\hat{u}_{n+1}^h \in \mathcal{H}^h$ as follows:

(1) Let $Z_0 = \hat{u}_n^h$, and define $Z_i \in \mathcal{H}^h$, for $i = 1, \cdots, J$, by $Z_i = Z_{i-1} + \omega P_i(\hat{u}^h - Z_{i-1})$.
(2) Set $\hat{u}_{n+1}^h = Z_J$.

For the general substructure $\{\Omega_i\}_{i=1}^J$ used in the Schwarz algorithm-1 (Part 2), we establish a convergence estimate as in [3,20]. The result is stated in Theorem 3.3, below. Also, we consider the Schwarz method for a 'two-level' domain decomposition. Namely, if one introduces a coarse level in the Schwarz algorithm, the convergence rate will be improved dramatically. This 'two-level' Schwarz method can be described as follows. Starting from a 'coarse' triangulation $\mathcal{T}_0$ of mesh size h_0, which could be the one that was used to construct u^* in Part 1 (for instance), we construct subdomains Ω_i by expanding the element $K_i \in \mathcal{T}_0$ by a prescribed distance $d = O(h_0)$; the part outside Ω will be omitted. It follows that $\{\Omega_i\}_{i=1}^J$ forms an overlapping domain decomposition of Ω. The Schwarz algorithm-1 (Part 2) can be applied to this substructure and, as in the case for second-order elliptic equations, yields a convergence rate bounded by $1 - O(h_0^2)$ (see Theorem 3.3). From the multigrid method, we make use of the 'coarse' triangulation $\mathcal{T}_0$. Let $\mathcal{V}_0^h \times W_0^h$ be the finite element space associated with $\mathcal{T}_0$ and $\mathcal{H}_0^h$ be a subspace of $\mathcal{V}_0^h$ consisting of divergence free flux elements.

Schwarz Algorithm-2 (Part 2): Given $\hat{u}_n^h \in \mathcal{H}^h$, an approximate solution from (3.3), we seek the next approximate solution $\hat{u}_{n+1}^h \in \mathcal{H}^h$ as follows:

(1) Let $Z_{-1} = \hat{u}_n^h$, and define $Z_i \in \mathcal{H}^h$, for $i = 0, 1, \cdots, J$, by $Z_i = Z_{i-1} + \omega P_i(\hat{u}^h - Z_{i-1})$.
(2) Set $\hat{u}_{n+1}^h = Z_J$.

The Schwarz Algorithm (Part 2) involves some projection operator P_i onto the subspace $\mathcal{H}_i^h$. Since it is generally very hard to find a nodal basis for $\mathcal{H}_i^h$, a direct computation of $P_i v$ is almost impossible in practice. We propose two approaches which lead to an easy determination of $P_i v$. The first is based on the stream-function space. Let $\mathcal{S}_i^h$ be the corresponding stream-function space over Ω_i. Denote by $\tilde{a}(\cdot,\cdot)$ the bilinear form defined by $\tilde{a}(\phi, \psi) = (c\ \mathbf{curl}\ \phi,\ \mathbf{curl}\ \psi)$ for all $\phi,\ \psi \in \mathcal{S}_i^h$.

Theorem 3.2. *For any $\xi \in \mathcal{H}^h$, let $\eta_i \in \mathcal{S}_i^h$ be defined by*

$$(3.5) \qquad \tilde{a}(\eta_i, \psi) = (c\ \xi,\ \mathbf{curl}\ \psi), \qquad \psi \in \mathcal{S}_i^h.$$

Then $P_i\xi = \mathbf{curl}\ \eta_i$.

Remark 3.2. Theorem 3.2 shows that the action $P_i\xi$ can be calculated through the computation of a second-order elliptic problem in the standard Galerkin finite element space. This idea can obviously be applied to the computation of $\hat{u}^h$. Actually, the same reasoning shows that $\hat{u}^h = \textbf{curl}\ \eta$ for some $\eta \in \mathcal{S}^h$ defined by

$$\tilde{a}(\eta, \psi) = -(cu^*,\ \textbf{curl}\ \psi), \qquad \psi \in \mathcal{S}^h, \tag{3.6}$$

where, as before, $\tilde{a}(\eta, \psi) = (c\ \textbf{curl}\ \eta,\ \textbf{curl}\ \psi)$. Equation (3.6) is clearly the standard Galerkin method for a second-order elliptic equation. Thus, the reduced mixed finite element method is equivalent to a standard Galerkin method and all the existing results in domain decomposition and preconditioning techniques are applicable. However, (3.5) is different from the standard Galerkin method applied directly to (1.1) for the pressure only. The elliptic problem (3.5) is equivalent to the mixed finite element method for (1.1), and hence provides a more accurate approximate flux, especially for problems with discontinuous coefficient $a(x)$.

Remark 3.3. The technique developed in this section can be extended to problems with mixed Dirichlet-Neumann boundary values for the second-order elliptic equation. See [8,9] for details.

The second approach to the computation of $P_i\xi$, as suggested in [16], can be obtained by solving a saddle-point problem on Ω_i. Let $(\xi_i^h; \theta_i^h) \in \mathcal{V}_i^h \times W_i^h$ be defined such that

$$\begin{aligned} (c\ \xi_i^h, v) - (\nabla \cdot v, \theta_i^h) &= (c\xi, v), \qquad & v \in \mathcal{V}_i^h, \\ (\nabla \cdot \xi_i^h, w) &= 0, & w \in W_i^h. \end{aligned} \tag{3.7}$$

Then, it is obvious that $\xi_i^h = P_i\xi$.

Theorem 3.3. *There exists a constant $\tilde{C}$ such that the convergence of the Schwarz algorithm-1 (Part 2) is bounded by*

$$\gamma_0 = 1 - \omega(2-\omega)d^2/(\tilde{C}J). \tag{3.8}$$

We see from (3.8) that the convergence rate for the Schwarz algorithm-1 (Part 2) has an upper bound dependent upon two parameters d and J; d characterizes the size of the overlapped subdomain and J is the number of subdomains. We emphasize that, in (3.8), J could be replaced by N_0 defined by $N_0 = \max_{x\in\Omega} N_x$, where N_x denotes the number of subdomains containing $x \in \Omega$. The number N_0 is apparently bounded from above by J. However, in some important applications, the number N_0 could be independent of the number of subdomains J. We consider, for example, the substructure $\{\Omega_i\}_{i=1}^J$ obtained by expanding each element of the coarse level $\mathcal{T}_0$ by the prescribed distance $d = O(h_0)$, which was used to define the Schwarz algorithm-2 (Part 2). It is clear that the number N_0 is a constant independent of J. However, as a small number, the parameter d, which is proportional to h_0, contributes a negative effect to the convergence. As in the case for second-order elliptic problem, the use of the coarse level can balance this negative effect and yield uniform convergence for the method. The result is presented as follows.

Theorem 3.4. *Let $\{\Omega_i\}_{i=1}^J$ be the substructure of Ω used to define the Schwarz algorithm-2. Then there exists a constant C such that the convergence of the Schwarz algorithm-2 (Part 2) is bounded by $\gamma_1 = 1 - \frac{\omega(2-\omega)}{C}$.*

4. Multilevel decomposition algorithms. In this section we consider multilevel decomposition techniques applied to the mixed finite element method (1.2). To begin, let $\mathcal{T}_0$ be an intentionally coarse initial triangulation of Ω. For $i = 1, \cdots, J$, let $\mathcal{T}_i$ be the triangulation obtained by breaking every triangle (or rectangle) of $\mathcal{T}_{i-1}$ into four subtriangles (or subrectangles) by connecting the mid-points of each edge (or of opposite edges). Denote also by $\mathcal{T}_h = \mathcal{T}_J$ the finest triangulation of Ω. The triangles of the triangulation $\mathcal{T}_i$ are called level i elements. The vertices of level i elements are called level i nodes.

Let $\mathcal{V}_i^h \times W_i^h$ be the finite element space associated with $\mathcal{T}_i$. For any $f^h \in W^h$, let

$$f_i^h = Q_i^h f^h - Q_{i-1}^h f^h, \tag{4.1}$$

for $i = 0, \cdots, J$, where Q_i^h is the standard L^2 projection operator onto the space W_i^h and $Q_{-1}^h = 0$.

Lemma 4.1. *Let f_i^h be as above. Then $f^h = \sum_{i=0}^J f_i^h$. Furthermore, the functions f_i^h, for $i = 1, \cdots, J$, have vanishing mean values on each element of level $i-1$.*

As in the Schwarz alternating algorithm, we first find a discrete flux whose divergence is f^h. Our method is based on Lemma 4.1. Let $\mathcal{M}_0^h = \mathcal{V}_0^h$ and $\mathcal{M}_i^h$ be the subspace of $\mathcal{V}_i^h$ consisting of those fluxes whose boundary values in the normal component of the boundary of level $i-1$ elements are zero; i.e.,

$$\mathcal{M}_i^h = \left\{ v \in \mathcal{V}_i^h,\ v \cdot \nu_{\partial T} = 0, \quad \text{on } \partial T \text{ and } T \in \mathcal{T}_{i-1} \right\}, \tag{4.2}$$

for $i = 1, \cdots, J$. The corresponding pressure spaces are defined by taking $\tilde{W}_0^h = W_0^h$ and

$$\tilde{W}_i^h = \{ w \in W_i^h,\ \int_T w dx = 0, \qquad \text{for all } T \in \mathcal{T}_{i-1} \} \tag{4.3}$$

for $i = 1, \cdots, J$. It is clear from Lemma 4.1 that $f_i^h \in \tilde{W}_i^h$.

Multilevel Decomposition Algorithm (MDA) (Part 1): Let f^h be the L^2 projection of the right-hand side function f of (1.1) in W^h, which is decomposed as in Lemma 4.1.

(1) For $i = 0, \cdots, J$, solve $u_i^h \times \theta_i^h \in \mathcal{M}_i^h \times \tilde{W}_i^h$ by

$$\begin{aligned} (\tilde{c} u_i^h, v) - (\nabla \cdot v, \theta_i^h) &= 0, & v &\in \mathcal{M}_i^h, \\ (\nabla \cdot u_i^h, w) &= (f_i^h, w), & w &\in \tilde{W}_i^h, \end{aligned} \tag{4.4}$$

where $\tilde{c}$ is an arbitrary positive function defined on Ω.

(2) Set $u^* = \sum_{i=0}^J u_i^h$.

Lemma 4.2. *Let u^* be obtained by the algorithm above. Then $\nabla \cdot u^* = f^h$.*

Remark 4.1. The solution of (4.4) can be obtained by solving some local problems. To see this, we note that the space $\mathcal{M}_i^h$ is the direct sum of subspaces defined on disjoint subdomains $\mathcal{T}_\kappa$, where $\mathcal{T}_\kappa$ are elements of level $i-1$. Thus, the problem (4.4) is quivalent to subproblems restricted to the subdomain $\mathcal{T}_\kappa$ (with vanishing boundary value in the normal direction). The only exception is the solution of $u_0^h \times \theta_0^h$ which is defined on the coarse level and, therefore, can not be split into some local problems. However, the coarse-level problem does not cost much to compute.

Remark 4.2. The MDA (Part 1) provides a way to find the desired u^* based on the natural multilevel decomposition (4.1) for f^h. In practice, there are other methods available. The numerical experiments in §5 use a different approach to construct u^* for rectangular domains.

Similarly to (3.3), we can utilize u^* to reduce the saddle-point problem (1.2) to a positive definite problem (3.3), where $u_h = \hat{u}^h + u^*$. Note that the problem (3.3) can be solved as the standard Galerkin method by employing the stream-function spaces (see Remark 3.1). However, we would like to study this problem using its present form. Following the idea in the Schwarz algorithm, we propose two iterative algorithms solving (3.6) based on a certain multilevel structures of the finite element space $\mathcal{V}^h$. The method is similar to the multilevel decomposition iterative method proposed in [20, 21] for the Galerkin method. In general, the method can be regarded as an extension of the standard multigrid method. The idea here is to replace the smoothing (for instance, the Gauss-Seidel and Jacobi smoothings) in the multigrid method by the Schwarz alternating method or additive Schwarz for each level.

First, let $\mathcal{N}_i$ be the set of level i nodes for $i = 0, 1, \cdots, J$. Associated with each node $x_{ik} \in \mathcal{N}_i$, let $\Omega_{i,k}$ be the subdomain of Ω consisting of level i elements having x_{ik} as a common vertex. $\{\Omega_{i,k}\}_{k=1}^{m_i}$ forms an overlapping decomposition of Ω, where m_i is the number of level i nodes. Let $\mathcal{V}_{i,k} \times W_{i,k}$ be the corresponding finite element space of level i defined on $\Omega_{i,k}$ with the natural partition induced from $\mathcal{T}_i$. Accordingly, let $\mathcal{H}_{i,k}$ be the divergence-free subspace of $\mathcal{V}_{i,k}$. The second part of the MDA can then be stated as follows.

Multilevel Decomposition Algorithm-1 (MDA-1) (Part 2): Given $\hat{u}_n^h \in \mathcal{H}^h$, an approximate solution of (3.3), we seek the next approximate solution $\hat{u}_{n+1}^h \in \mathcal{H}^h$ as follows:

(1) Define $Z_0 \in \mathcal{H}^h$ by $Z_0 = \hat{u}_n^h + \omega P_0(\hat{u}^h - \hat{u}_n^h)$.

(2) For $i = 1, \cdots, J$, let $Y_0 = Z_{i-1}$ and $Y_k = Y_{k-1} + \omega P_{i,k}(\hat{u}^h - Y_{k-1}), k = 1, \cdots, m_i$, where $P_{i,k}$ is the projection operator onto $\mathcal{H}_{i,k}$ with respect to the $(\cdot,\cdot)_c \equiv (c\ \cdot,\cdot)$ inner product. Then, we let $Z_i = Y_{m_i}$.

(3) Set $\hat{u}_{n+1}^h = Z_J$.

Remark 4.3. The projection operator $P_{i,k}$ is defined locally on each (macro-element) $\Omega_{i,k}$. In practice, we have to solve a local problem on $\Omega_{i,k}$ to determine this operator; the computation of such local problems is cheap. As in the Schwarz algorithm, the operator $P_{i,k}$ can be obtained by solving either a saddle-point problem (3.7) or an elliptic problem (3.5); they are small problems associated with macro-elements.

Since the Schwarz alternating method is an analogue of the SOR iterative method for matrix computation (cf. [3,19,20]), we see that the MDA-1 (Part 2) is actually an analogue of the multigrid algorithm based on the SOR smoothing for each level. Due to the connection between the SOR method and the Schwarz alternating method (cf. [3, 20]), we propose to modify MDA-1 (Part 2) by using the additive Schwarz (cf. [10]) on each level i. Let $R_i = \sum_{k=1}^{m_i} P_{k,i}$ be a symmetric and nonnegative operator on $\mathcal{H}^h$ with respect to the $(\cdot,\cdot)_c \equiv (c\ \cdot,\cdot)$ inner product. Let λ_i be the largest eigenvalue of R_i which is bounded by a constant C uniformly in i. Set $T_i = \lambda_i^{-1} R_i$.

Multilevel Decomposition Algorithm-2 (MDA-2) (Part 2): Given $\hat{u}_n^h \in \mathcal{H}^h$, an approximate solution of (3.3), we seek the next approximate solution $\hat{u}_{n+1}^h \in \mathcal{H}^h$ as follows:

(1) Define $Z_0 \in \mathcal{H}^h$ by $Z_0 = \hat{u}_n^h + \omega P_0(\hat{u}^h - \hat{u}_n^h)$.
(2) For $i = 1, \cdots, J$, define Z_i by $Z_i = Z_{i-1} + \omega T_i(\hat{u}^h - Z_{i-1})$.
(3) Set $\hat{u}_{n+1}^h = Z_J$.

Theorem 4.2. *There exists a constant C such that the convergence of the MDA-1 and MDA-2 (Part 2) are bounded by*

$$\gamma_2 = 1 - \frac{\omega(2-\omega)}{CJ}. \tag{4.5}$$

The estimate (4.5) is established without any regularity assumption beyond $H(\text{div}, \Omega)$ necessary to define the weak form. However, the estimates are level dependent, though the dependence is very weak. In [9] we have also proved a uniform convergence for the MDAs if more regularity is assumed. The result follows:

Theorem 4.3. *Assume that the H^2 regularity is satisfied for the problem defined by the bilinear form $\tilde{a}(\cdot,\cdot)$. Then, there exists a constant C such that the convergence of the MDA-2 (Part 2) is bounded by $\gamma_4 = 1 - \frac{\omega(2-\omega)}{C}$.*

5. Numerical experiments. In this section, we present some numerical experiments to illustrate our theory. For simplicity, we consider (1.2) with $c = 1$. The domain $\Omega = (0,1) \times (0,1)$ is the unit square. The homogeneous Neumann boundary condition is imposed on $\partial\Omega$. The right-hand side function is $f(x,y) = 2\pi^2 \cos x \cos y,\ (x,y) \in \Omega$, and the exact solution is $p(x,y) = \cos x \cos y$. It is clear from the definition of the flux that $u = (u_1, u_2)$, where $u_1 = \pi \sin x \cos y$ and $u_2 = \pi \cos x \sin y$.

The RT space of the lowest order is used in the computation. We began with a uniform 2×2 grid. The code was used to refine each coarse element (square) into four congruent small squares, obtaining a fine mesh with $(2^J + 1)^2$ nodal points. The number J is said to be the number of levels of the refinement. In the MDA (Part 1), the code did not quite follow the idea presented in the paper because of the use of a different decomposition (the procedure suggested in the paper is more applicable to adaptively refined meshes). More precisely, the discrete flux u^* satisfying (3.1) was obtained according to the following decomposition for Ω. Let $\mathcal{T}_h$ be the fine mesh of Ω. Define $\mathcal{T}_0$ to be the collection of 2^J 'thin' strips (see Figure 1). Each element (strip) has a partition inherited from $\mathcal{T}_h$. Thus, we can use the MDA (Part 1) to construct the desired discrete flux u^*.

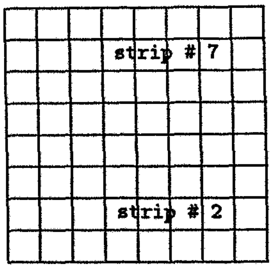

Figure 1. Illustration of an 8×8 *rectangular partition and decomposition.*

After we had u^*, we applied the MDA (Part 2) to approximate the new flux $\hat{u}^h$. We summarize the numerical result obtained by using MDA-1 (Part 2) in Tables 5.1–5.2, from which we see that the relaxation parameter ω can speed up the convergence of the algorithm.

The number of accurate digits is defined by

$$\text{Digits} = -\log\left(\frac{\|u_n^h - u\|_0}{\|u\|_0}\right),$$

where $\|\cdot\|_0$ is the standard L^2 norm and $u_n^h = \hat{u}_n^h + u^*$ is the approximate solution of the finite element approximation u^h.

The results in Tables 5.3–5.4 are obtained by using MDA-2 with different choices of λ_i. We found that the best choice for this number is 1.

The average rate of convergence of the MDAs is presented in Table 5.5. We emphasize that, according to our theory, the rate of convergence of the MDAs is independent of the number of levels in our computational example. This was verified by the numbers in the Table as well.

Table 5.1
Convergence of the MDA-1 with $\omega = 1$

Iteration	1	2	3	4	5	6	7	8	9
Digits ($J = 5$)	0.71	1.44	2.20	2.96	3.36	3.39	3.40	-	-
Digits ($J = 6$)	0.71	1.44	2.19	2.98	3.70	3.97	4.00	-	-
Digits ($J = 7$)	0.71	1.44	2.19	2.98	3.75	4.33	4.56	-	-
Digits ($J = 8$)	0.71	1.44	2.19	2.97	3.76	4.40	4.89	5.15	5.20

Table 5.2
Convergence of the MDA-1 with $\omega = 1.2$

Iteration	1	2	3	4	5	6	7	8	9
Digits ($J = 5$)	0.95	1.97	2.71	3.22	3.37	3.39	3.40	-	-
Digits ($J = 6$)	0.95	1.96	2.76	3.41	3.84	3.98	4.00	-	-
Digits ($J = 7$)	0.95	1.95	2.78	3.45	4.01	4.47	4.59	4.60	-
Digits ($J = 8$)	0.95	1.95	2.78	3.46	4.03	4.64	5.12	5.20	-

Table 5.3
Convergence of the MDA-2 with $\lambda_i^{-1} = 1$

Iteration	1	2	3	4	5	6	7	8	9	10
Digits ($J = 5$)	0.61	1.23	1.84	2.47	3.05	3.35	3.39	-	-	-
Digits ($J = 6$)	0.61	1.22	1.84	2.47	3.09	3.67	3.96	3.99	4.00	-
Digits ($J = 7$)	0.61	1.22	1.84	2.46	3.09	3.72	4.30	4.57	4.60	-
Digits ($J = 8$)	0.61	1.22	1.84	2.46	3.09	3.73	4.36	4.94	5.18	5.20

Table 5.4
Convergence of the MDA-2 with $\lambda_i^{-1} = .9$

Iteration	6	7	8	9	10	11	12	13	14	15
Digits ($J = 5$)	2.88	3.22	3.36	3.39	3.40	-	-	-	-	-
Digits ($J = 6$)	2.90	3.33	3.70	3.92	3.98	4.00	-	-	-	-
Digits ($J = 7$)	2.90	3.34	3.76	4.14	4.43	4.56	4.59	4.60	-	-
Digits ($J = 8$)	2.90	3.34	3.77	4.17	4.55	4.87	5.08	5.17	5.19	5.20

Table 5.5
The average rate of convergence of MDA

	δ ($J=5$)	δ ($J=6$)	δ ($J=7$)	δ ($J=8$)
MDA-1 ($\omega=1$)	0.33	0.27	0.27	0.26
MDA-1 ($\omega=1.2$)	0.33	0.27	0.27	0.22
MDA-2 ($\lambda_i^{-1}=.9$)	0.46	0.43	0.44	0.45
MDA-2 ($\lambda_i^{-1}=1$)	0.33	0.32	0.31	0.30

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