CHAPTER 21

Two-Level Iterative Refinement Preconditioners

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Abstract. We present a comparative numerical study of the two-level BEPS preconditioner from [6] and two-level, hierarchical basis, local refined preconditioners for solving second-order elliptic equations discretized on two-level patched grids. The latter preconditioners allow inexact subgrid solvers.

1. Introduction and Preliminaries. Let Ω be a given two-dimensional (2–D) polygon and let Γ_N be a given part of the boundary $\partial\Omega$ on which Neumann boundary conditions are imposed. We assume that the Dirichlet portion, $\Gamma_D \equiv \partial\Omega \backslash \Gamma_N$, is a non-trivial part of $\partial\Omega$. We denote the standard L^2-based Sobolev spaces on Ω by $W_2^j(\Omega)$.

We consider the second-order boundary value problem in a variational setting:

Given $f \in L_2(\Omega)$ and $g_N \in W_2^{-\frac{1}{2}}(\Gamma_N)$, find $u \in W_2^1(\Omega)$ such that,

$$a(u,\phi) \equiv \int_\Omega \sum_{i,j} k_{i,j}(x) \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_j} dx = \int_\Omega f\phi dx + \int_{\Gamma_N} g_N \phi d\Gamma$$

for all $\phi \in W_2^1(\Omega), \phi = 0$ on $\Gamma_D \equiv \partial\Omega \backslash \Gamma_N$, and

$$u = 0 \text{ on } \Gamma_D.$$

The bounded measurable coefficients $k_{i,j}(x)$ define a symmetric matrix that is assumed to be uniformly positive definite for $x \in \overline{\Omega}$.

We discretize the above boundary value problem on an initial coarse mesh $\tilde{\omega}$. For definiteness, we consider the case of triangular grids. The finite element discretization space $\tilde{\mathbf{V}} \subset W_2^1(\Omega)$ is assumed to be spanned by piece-wise linear functions that are continuous in Ω and vanish on Γ_D. The nodes (vertices of triangles that do not lie on Γ_D)

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define the initial (coarse) grid $\tilde{\omega}$. We denote by $\tilde{h}$ the discretization parameter on $\tilde{\omega}$. Thediscrete finite element problem reads:

Find $u_{\tilde{h}} \in \tilde{\mathbf{V}}$ such that for all $\phi \in \tilde{\mathbf{V}}$ we have

$$a(u_{\tilde{h}}, \phi) = \int_{\Omega} f\phi dx + \int_{\Gamma_N} g_N \phi d\Gamma.$$

Let $\left\{\tilde{\phi}_i\right\}_{i=1}^{\tilde{n}}$ be a standard nodal basis in $\tilde{\mathbf{V}}$. Then we can compute the corresponding stiffness matrix $\tilde{A}$,

$$\tilde{A} = \left\{a\left(\tilde{\phi}_j, \tilde{\phi}_i\right)\right\}_{i,j=1}^{\tilde{n}}$$

and formulate the finite element problem as a linear algebraic system,

$$\tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}},$$

where $\tilde{\mathbf{x}}$ is the coefficient vector of $u_{\tilde{h}}$ expanded in terms of the basis $\left\{\tilde{\phi}_i\right\}_{i=1}^{\tilde{n}}$ and

$$\tilde{\mathbf{b}} = \left(\int_{\Omega} f\tilde{\phi}_i dx + \int_{\Gamma_N} g_N \tilde{\phi}_i d\Gamma\right)_{i=1}^{\tilde{n}}.$$

In general, we may not be satisfied by the solution $u_{\tilde{h}}$ on the coarse grid $\tilde{\omega}$. Thus, after *a posteriori* analysis, one may decide to refine the mesh. This process can be successively repeated. In practice, the refinement can take place only in certain subregions, where the currently computed approximate solution has a large gradient.

We study the following model situation. Let $\Omega_1 \subset \Omega$ be a subregion where we introduce a finer mesh ω_1. We assume that Ω_1 is covered by coarse-grid elements. The refinement is done by subdividing any coarse-grid element in Ω_1 into a fixed number of congruent ones. At the interfaces that arise between the refined and unrefined elements, we have to introduce so-called "slave" nodes. The values of the functions at these nodes in the corresponding fine finite element space $\mathbf{V} \supset \tilde{\mathbf{V}}$ are obtained by interpolation (linear interpolation between the vertices of the adjacent coarse-grid element that is not refined). This means that the values at these nodes are not degrees of freedom. We denote by ω the finer grid (called also composite grid) obtained in this manner. We have $\omega_1 = \Omega_1 \cap \omega$; that is, ω_1 consists of all the nodes (old and added in the refinement) in Ω_1. The fine-grid space can also be defined as follows,

$$\mathbf{V} = \tilde{\mathbf{V}} + \mathbf{V}_0^{(1)},$$

where $\mathbf{V}_0^{(1)}$ is the standard finite element space of piece-wise linear functions in Ω_1 that vanish on the interfaces between Ω_1 and the unrefined part of Ω. Let $\{\phi_i\}_{i=1}^{n}$ be the standard nodal basis in $\mathbf{V}$. Then one can consider the following hierarchical basis functions in $\mathbf{V}$:

$$\left\{\hat{\phi}_i\right\}_{i=1}^{n} = \{\phi_i\}_{i=\tilde{n}+1}^{n} \cup \left\{\tilde{\phi}_i\right\}_{i=1}^{\tilde{n}}.$$

We compute the stiffness matrix with respect to the hierarchical basis, obtaining

$$A = \left\{a\left(\hat{\phi}_j, \hat{\phi}_i\right)\right\}_{i,j=1}^{n}.$$

We note that, in the computation, the hierarchical basis stiffness matrices are used only implicitly (*cf.* Yserentant [16] or Bank, Dupont, and Yserentant [5]).

Our basic assumption is the following: *Problems on the standard, say uniform, grids can be solved efficiently; for example, there exists a standard software code for such solution processes.* In our case, such grids will be $\tilde{\omega}$, ω_1, and any grid obtained by uniform refinement of any of their regular parts. For example, we may think of rectangular subgrids.

Now we are in position to formulate a number of preconditioners. To do this, we partition the composite-grid matrix into the following natural hierarchical two-by-two block form:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{matrix} \}\omega\backslash\tilde{\omega} \\ \}\tilde{\omega} \end{matrix}.$$

The blocks are readily computed as follows:

$$A_{11} = \{a(\phi_j, \phi_i)\}_{i,j=\tilde{n}+1}^{n},$$

$$A_{12} = \left\{a\left(\tilde{\phi}_j, \phi_i\right)\right\}_{1\le j\le\tilde{n},\ \tilde{n}+1\le i\le n},$$

$$A_{22} = \left\{a\left(\tilde{\phi}_j, \tilde{\phi}_i\right)\right\}_{i,j=1}^{\tilde{n}} \equiv \tilde{A}.$$

That is, A_{22} is nothing but $\tilde{A}$, the coarse-grid matrix. We also need the stiffness matrix $A_1 = \left\{a_{\Omega_1}\left(\hat{\phi}_j, \hat{\phi}_i\right)\right\}_{x_j, x_i\in\Omega_1}$, that is, the stiffness matrix in the subregion Ω_1 using the finite element space $\mathbf{V}_0^{(1)}$ of functions that vanish on the interfaces between Ω_1 and the unrefined part of Ω. We similarly have,

$$A_1 = \begin{pmatrix} A_{11} & A_{12}^{(1)} \\ A_{21}^{(1)} & A_{22}^{(1)} \end{pmatrix} \begin{matrix} \}\omega_1\backslash\tilde{\omega}_1 \\ \}\tilde{\omega}_1 \end{matrix}.$$

Here $\tilde{\omega}_1$ and ω_1 are the coarse and fine grids in Ω_1, respectively. Note that A_1 has the same first block on its diagonal as A. We also need the following Schur complement W_1,

$$W_1 = A_{11}^{(1)} - A_{12}^{(1)} A_{22}^{(1)^{-1}} A_{21}^{(1)}.$$

We emphasize that one can solve problems with W_1 based on solvers for A_1 on the regular grid ω_1. Alternatively, we can use a coarse-grid solver in the subregion Ω_1 and then interpolate the solution on the fine grid in Ω_1. In practice, this will depend on the data structure used for storing the matrices in order to take advantage of possible vectorization properties of the corresponding solver.

We define:

PRECONDITIONER 1.

$$C = \begin{pmatrix} W_1 & A_{12} \\ 0 & \tilde{A} \end{pmatrix} \begin{pmatrix} I & 0 \\ \tilde{A}^{-1}A_{21} & I \end{pmatrix} \begin{matrix} \}\omega\backslash\tilde{\omega} \\ \}\tilde{\omega} \end{matrix}$$

PRECONDITIONER 2.

$$D = \begin{pmatrix} W_1 & 0 \\ 0 & \tilde{A} \end{pmatrix} \begin{matrix} \}\omega\backslash\tilde{\omega} \\ \}\tilde{\omega} \end{matrix}.$$

It is clear that both preconditioners incorporate solvers on regular grids $\tilde{w}$ and ω_1. The second preconditioner, which is block-diagonal, gives rise to the so-called AFAC method studied in McCormick and Quinlan [14]. In a sense, all these preconditioners are variations

of the two-level hierarchical preconditioners introduced in the paper of Bank and Dupont [4] and studied further in Axelsson and Gustafsson [1] (see also Maitre and Musy [10]). Here we emphasize their modification and application to the local refinement preconditioning techniques.

For the BEPS method, proposed in Bramble, Ewing, Pasciak, and Schatz [6], which is also based on solvers utilized only on the regular grids, the solution starts on the finer grid ω_1, then a coarse-grid correction is performed, and the preconditioning step is completed by one more fine-grid solver. In the related FAC-method of McCormick [12] and McCormick and Thomas [14] (see also McCormick [13] and Mandel and McCormick [11]), the last fine-grid solution step is replaced by interpolation. This approach allows inexact coarse-grid solvers (see Ewing, Lazarov, and Vassilevski [8]), but the fine grid solvers must be exact (or must be handled up to h-dependent accuracy), which means that if one solves these problems by an optimal iterative method, $\mathcal{O}(\log 1/h)$ iterations are required on the fine grid ω_1.

The main goal of the present paper is to demonstrate the performance of the two-grid preconditioners with approximations to the block W_1, which is of main importance when we generalize the method for multilevel refinement, versus the BEPS method, [6] (see also [8]). The numerical results are presented in the next section. A relevant theory for the hierarchical basis two- and multi-level iterative refinement preconditioners with inexact subgrid solvers is given in Ewing and Vassilevski [9].

2. Comparative Numerical Experiments. In this section, we compare the performance of the (two-level) BEPS preconditioner from [6] (see also [8]), with the Preconditioners 1 and 2 with inexact solvers on the refined subdomain. The test problems are

$$-\nabla \cdot a(x,y)\nabla u = f(x,y), \qquad (x,y) \in \Omega = (0,1]^2$$

$$\begin{aligned} \partial u/\partial n &- \text{given on } \{x=1\}, \text{ and } \{y=1\}, \\ u &- \text{given on } \{x=0\}, \text{ and } \{y=0\}. \end{aligned}$$

We use piece-wise linear functions on right-angled triangles for the finite element spaces $\tilde{\mathbf{V}}$ and $\mathbf{V}_0^{(1)}$, where the subdomain in which we introduce finer mesh is

$$\Omega_1 = \{(x,y) : 7/8 < x < 1,\ 7/8 < y < 1\}.$$

The test problems correspond to the following diffusion coefficients:

PROBLEM 1: (a smooth coefficient):

$$a(x,y) = 1 + x^2 + y^2.$$

The right-hand side f corresponds to the following exact solution in this case:

$$u(x,y) = e^{-x}(x-1)^2 + (y-1)^2.$$

PROBLEM 2. (a discontinuous coefficient):

$$a(x,y) = \begin{cases} 1, & x > 15/16 \text{ or } y > 15/16, \\ 1000, & x < 15/16 \text{ and } y < 15/16. \end{cases}$$

The exact solution u in this case is

$$u(x,y) = (1-x)^2(1-y)^2(x-15/16)(y-15/16)/a(x,y).$$

Note that u is almost zero outside the domain $(15/16, 1] \times (15/16, 1]$.

The initial uniform discretization has mesh sizes $h_c = 1/16, 1/32, 1/64$. In the subdomain Ω_1 we introduce a finer mesh with a size $h = h_c/n_0$, $n_0 = 2, 4, 8$.

The preconditioners are:

BEPS PRECONDITIONER: Consider the following domain decomposition block form of A,

$$A = \begin{bmatrix} A_1 & P \\ Q & A_2 \end{bmatrix} \begin{matrix} \}\omega_1 \\ \}\omega_2 \end{matrix}.$$

Then the BEPS preconditioner is defined as follows:

$$B = \begin{pmatrix} A_1 & 0 \\ Q & \tilde{S} \end{pmatrix} \begin{pmatrix} I & A_1^{-1}P \\ 0 & I \end{pmatrix} \begin{matrix} \}\omega_1 \\ \}\omega_2 = \tilde{\omega}\backslash\omega_1 \end{matrix},$$

where $\tilde{S}$ is the Schur complement of the coarse-grid matrix $\tilde{A}$ partitioned into the following block form:

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \begin{matrix} \}\tilde{\omega}_1 = \tilde{\omega}\backslash\omega_2 \\ \}\tilde{\omega}_2 = \omega_2 \end{matrix};$$

i.e.,

$$\tilde{S} = \tilde{A}_{22} - \tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12}.$$

For more details we refer to Ewing, Lazarov, and Vassilevski [8]. We only note that to solve a system with the Schur complement $\tilde{S}$, we can use solvers for the coarse-grid matrix, since

$$\tilde{A}^{-1} = \begin{pmatrix} * & * \\ * & \tilde{S}^{-1} \end{pmatrix}.$$

Then $\tilde{S}^{-1}v_2 = \left(\tilde{A}^{-1}\begin{pmatrix} 0 \\ v_2 \end{pmatrix}\right)_2$. The other preconditioners are Preconditioners 1 and 2 from Section 1, modified by inexact fine-grid solvers. We now define an approximation Z_1 to the Schur complement W_1 of the fine-grid matrix A_1.

The finite element stiffness matrix A_1 for the subdomain Ω_1 admits a block-tridiagonal structure if we use a line ordering of the nodes, say, in the vertical direction; i.e., we have

$$A_1 = \begin{pmatrix} V_{11} & V_{12} & & \\ V_{21} & V_{22} & V_{23} & 0 \\ & \ddots & \ddots & \\ 0 & & V_{n,n-1} & V_{nn} \end{pmatrix}, \quad n = 1/h.$$

Then we use the incomplete block-factorization for block-tridiagonal matrices as proposed in Axelsson and Polman [2] (see also [7]); i.e., we compute

$$B_1 = (Y^{-1} - L)(I - YU),$$

where $-L$ and $-U$ are the strictly lower and strictly upper block diagonal parts of A_1, respectively. The matrix Y is block diagonal with blocks $\{Y_i\}_{i=1}^n$ that are computed by the following algorithm. Below, for any matrix X, we denote by $X^{(p)}$ a $(2p+1)$-banded approximation to X. Set $Y_i = \left(V_{11}^{-1}\right)^{(p)}$; for $i = 2, 3, \ldots, n$ compute

$$Y_i = \left[(V_{ii} - V_{i,i-1}Y_{i-1}V_{i-1,i})^{-1}\right]^{(p)}.$$

Since the matrices $V_{ii} - V_{i,i-1}Y_{i-1}V_{i-1,i}$ are kept $(2p+1)$-banded, one can compute Y_i based only on the $LD^{-1}U$ factorization of $V_{ii} - V_{i,i-1}Y_{i-1}V_{i-1,i}$, without any inversion of matrices. For more details, we refer to Vassilevski [15]; see also Axelsson and Polman [2].

The above block-factored matrix B_1 gives rise to a vectorizable solution algorithm since, when applying B_1^{-1} to certain vectors, we use only matrix-vector products with the sparse (banded) blocks Y_i, $V_{i,i-1}$, $V_{i,i+1}$, which can be vectorized well. For more details, we refer to Axelsson and Polman [2] (see also [7] and [15]). In our test we have used the so-called CHOL[p] approximation for X^{-1}; see [7] and [15].

The product $Z_1^{-1}\mathbf{v}_1$, for any vector $\mathbf{v}_1$ defined on $\omega\backslash\tilde{\omega}_1$, is then defined as follows:

$$Z_1^{-1}\mathbf{v}_1 = \tilde{B}_1^{-1}\begin{pmatrix} 0 \\ \mathbf{v}_1 \end{pmatrix}\begin{matrix}\tilde{\omega}_1 \\ \omega\backslash\tilde{\omega}_1\end{matrix}, \quad \text{restricted to } \omega_1\backslash\tilde{\omega}_1.$$

$\tilde{B}_1^{-1}\begin{pmatrix} 0 \\ \mathbf{v}_1 \end{pmatrix}$ stands for the approximate solution of the system $A_1\mathbf{x}_1 = \begin{pmatrix} 0 \\ \mathbf{v}_1 \end{pmatrix}\begin{matrix}\}\tilde{\omega}_1 \\ \}\omega\backslash\tilde{\omega}_1\end{matrix}$ by the preconditioned conjugate gradient method using B_1 as a preconditioner. Strictly speaking, $\tilde{B}_1^{-1}$ is a nonlinear mapping; but as demonstrated in [3], this makes no significant difficulty for the performance of the preconditioners C and D (see below) if $\tilde{B}_1^{-1}\begin{pmatrix} 0 \\ \mathbf{v}_1 \end{pmatrix}$ is an accurate enough solution of the system $A_1\mathbf{x}_1 = \begin{pmatrix} 0 \\ \mathbf{v}_1 \end{pmatrix}$. In our test, we have used a high relative accuracy $\epsilon_1 = 10^{-4}$, i.e., the B_1^{-1}-norm of the residual $\begin{pmatrix} 0 \\ \mathbf{v}_1 \end{pmatrix} - A_1\tilde{B}_1^{-1}\begin{pmatrix} 0 \\ \mathbf{v}_1 \end{pmatrix}$ to be smaller than ϵ_1.

In our test, we have chosen the halfbandwidth $p = 4$. Preconditioners 1 and 2 are then modified as follows using approximate solutions on the ω_1 grid.

PRECONDITIONER 1. (a block two-level Gauss-Seidel preconditioner with inexact fine-grid solvers):

$$C = \begin{bmatrix} Z_1 & A_{12} \\ 0 & \tilde{A} \end{bmatrix}\begin{bmatrix} I & 0 \\ \tilde{A}^{-1}A_{21} & I \end{bmatrix}\begin{matrix}\}\omega\backslash\tilde{\omega} \\ \}\tilde{\omega}\end{matrix}.$$

PRECONDITIONER 2. (a block two-level Jacobi preconditioner with inexact fine-grid solver):

$$D = \begin{bmatrix} Z_1 & 0 \\ 0 & \tilde{A} \end{bmatrix}\begin{matrix}\}\omega\backslash\tilde{\omega} \\ \}\tilde{\omega}\end{matrix}.$$

We solve the linear algebraic problems

$$A\mathbf{x} = \mathbf{b},$$

corresponding to the standard (nodal basis) finite element discretization of problems 1 and 2 above. We use the PCG method with the preconditioners specified above. In the tables below we show the number of iterations, iter, and average reduction factors, ρ, defined by

$$\rho = \left(\frac{\sqrt{\Delta}}{\sqrt{\Delta_0}}\right)^{\frac{1}{\text{iter}}},$$

where $\Delta = r^T r$ and $\Delta_0 = r_0^T r$. Here r and r_0 stand for the current and initial residuals, respectively. We also show the CPU time in seconds of the solution part of the method. The initial iterate is chosen to be

$$\mathbf{x}^0 = B^{-1}\mathbf{b},\ C^{-1}\mathbf{b},\ \text{or}\ D^{-1}\mathbf{b}$$

in the corresponding cases. For the hierarchical basis preconditioners, C and D, we transform the corresponding data to the hierarchical basis coefficient vectors (only on the preconditioning step). The stopping criterion is $\sqrt{r^T r} < \epsilon = 10^{-6}$. The tests were run on the Alliant FX/8.

3. Conclusions. From the tests presented we see a very good vectorization of all preconditioners, with a best performance of the BEPS preconditioner B, both on convergence rate and CPU-time. The two-level Gauss-Seidel preconditioner appeared to give a number of iterations bounded independently of the mesh ratio h_c/h. However, the two-level Jacobi preconditioner showed a certain deterioration when increasing the mesh ratio h_c/h. Another observation is that all preconditioners are robust with respect to discontinuous coefficients. Perhaps a more careful implementation of the hierarchical two-level preconditioners would make their performance more competitive with respect to the BEPS preconditioner.

Table 1. Iterative Convergence Results for Problem 1

$h_c = 1/16$

h_c/h	B.E.P.S.			Gauss-Seidel			Jacobi		
	iter	ρ	CPU (sec)	iter	ρ	CPU (sec)	iter	ρ	CPU (sec)
2	3	0.009	0.57	5	0.010	1.65	12	0.29	1.86
4	4	0.02	0.91	5	0.016	1.93	15	0.37	2.91
8	4	0.02	1.36	5	0.016	2.58	16	0.36	4.70

$h_c = 1/32$

h_c/h	B.E.P.S.			Gauss-Seidel			Jacobi		
	iter	ρ	CPU (sec)	iter	ρ	CPU (sec)	iter	ρ	CPU (sec)
2	3	0.009	1.79	5	0.04	5.27	15	0.38	7.46
4	4	0.02	2.58	5	0.05	6.01	19	0.47	11.30
8	4	0.02	4.41	5	0.05	8.46	21	0.51	19.90

$h_c = 1/64$

h_c/h	B.E.P.S.			Gauss-Seidel			Jacobi		
	iter	ρ	CPU (sec)	iter	ρ	CPU (sec)	iter	ρ	CPU (sec)
2	3	0.006	6.48	5	0.04	21.46	14	0.37	28.56
4	3	0.01	7.86	5	0.05	24.05	20	0.49	47.27
8	3	0.01	14.49	5	0.05	34.61	23	0.54	90.39

Table 2. Iterative Convergence Results for Problem 2

$h_c = 1/16$

	B.E.P.S.			Gauss-Seidel			Jacobi		
h_c/h	iter	ρ	CPU (sec)	iter	ρ	CPU (sec)	iter	ρ	CPU (sec)
2	2	0.009	0.42	3	0.001	1.12	10	0.22	1.53
4	3	0.007	0.68	3	0.003	1.25	12	0.28	2.25
8	4	0.014	1.26	3	0.004	1.68	13	0.33	3.70

$h_c = 1/32$

	B.E.P.S.			Gauss-Seidel			Jacobi		
h_c/h	iter	ρ	CPU (sec)	iter	ρ	CPU (sec)	iter	ρ	CPU (sec)
2	2	0.0006	1.30	4	0.020	4.33	14	0.37	6.87
4	3	0.008	2.02	5	0.010	5.80	18	0.46	10.36
8	3	0.008	3.31	5	0.014	7.94	19	0.48	16.78

$h_c = 1/64$

	B.E.P.S.			Gauss-Seidel			Jacobi		
h_c/h	iter	ρ	CPU (sec)	iter	ρ	CPU (sec)	iter	ρ	CPU (sec)
2	3	0.004	6.37	5	0.03	20.85	15	0.38	28.34
4	3	0.004	7.66	5	0.05	23.04	20	0.49	43.36
8	3	0.007	14.09	5	0.05	32.35	24	0.55	81.09

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