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Domain Decomposition Methods using Modified Basis Functions

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ABSTRACT. The paper investigates domain decomposition algorithms based on the use of modified basis functions in projective-difference approximation of problems. We use *usual* basis functions with support diameters of the order of the mesh size inside subdomains ([1-5],[9-11]). On subdomain interface we consider instead basis functions with supports of greater size, which are extended to subdomains in a harmonic manner.
We consider two boundary-value problems. The first of them is the Dirichlet problem in a rectangle Ω, the second one is a problem with natural boundary conditions in a "complicated" domain $\tilde{\Omega} \subset \Omega$, included into Ω. In the Dirichlet problem we introduce modified basis functions and construct a "simple" preconditioner. Then we use the same modified basis functions to solve the problem with natural boundary conditions. We investigate the properties of the matrices arisen in Galerkin's approximations and present the convergence results of domain decomposition algorithms based on iterative processes of minimal corrections and locally optimal three-steps methods. We show the results of some numerical experiments.
In the paper real-valued functions and well-known functional spaces $L_2(\Omega)$, $W_2^1(\Omega)$, $\overset{\circ}{W}{}_2^1(\Omega)$, $W_2^2(\Omega), C^{(1)}(\Omega)$ are used.

1. Statement of Problem

Let us consider the following problem with forced boundary conditions ($u = 0$ on $\partial\Omega$): find $u(x_1, x_2) \in \overset{\circ}{W}{}_2^1(\Omega)$ such that the relationship

$$a(u, v) = (f, v) \qquad \forall v \in \overset{\circ}{W}{}_2^1(\Omega) \tag{1.1}$$

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$$a(u,v)=\int_{\Omega}\Big[\sum_{i,j=1}^{2}p_{ij}(x_1,x_2)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}+q(x_1,x_2)uv\Big]\mathrm{d}x_1\mathrm{d}x_2,$$

$$(f,v)=\int_{\Omega}fv\mathrm{d}\Omega,$$

(1.2) $$a(w,w)\geq C_1||w||^2_{\overset{\circ}{W}{}_2^1(\Omega)},\quad ||w||_{\overset{\circ}{W}{}_2^1(\Omega)}\equiv||\nabla w||_{L_2(\Omega)},$$

$$|a(w,v)|\leq C_2||w||_{\overset{\circ}{W}{}_2^1(\Omega)}\cdot||v||_{\overset{\circ}{W}{}_2^1(\Omega)},\quad 0<C_1,C_2=const<\infty.$$

Here Ω is a rectangle $\Omega=\{(x_1,x_2):\ A<x_1<B,\ C<x_2<D\}$. The functions $p_{ij}=p_{ji},\ q\geq 0$ belong to $C^{(1)}(\Omega)$, $f(x_1,x_2)\in L_2(\Omega)$. (In forthcoming we'll use the notations $x_1\equiv x,\ x_2\equiv y$ too).

For stating techniques of solving problems with natural boundary conditions we will consider the following problem in a bounded domain $\tilde{\Omega}$ with curvilinear boundary $\partial\tilde{\Omega}$: find $u(x_1,x_2)\in W_2^1(\tilde{\Omega})$ such that the relationship

(1.3) $$a(u,v)=f(v)\ \ \forall v\in W_2^1(\tilde{\Omega})$$

is satisfied, where

(1.4) $$a(u,v)=\int_{\tilde{\Omega}}\Big[\sum_{i,j=1}^{2}p_{ij}(x_1,x_2)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}+q(x_1,x_2)uv\Big]\mathrm{d}x_1\mathrm{d}x_2+\int_{\partial\tilde{\Omega}}auv\mathrm{d}\Gamma$$

$$f(v)=\int_{\tilde{\Omega}}fv\mathrm{d}x_1\mathrm{d}x_2+\int_{\partial\tilde{\Omega}}gv\mathrm{d}\Gamma,$$

$$a(u,u)\geq C_1||u||^2_{W_2^1(\tilde{\Omega})},|a(u,v)|\leq C_2||u||_{W_2^1(\tilde{\Omega})}\cdot||v||_{W_2^1(\tilde{\Omega})},$$

$$0<C_1,C_2<\infty,\ \tilde{\Omega}\subset\Omega,\ \ \mathrm{dist}(\partial\tilde{\Omega},\partial\Omega)\geq d_0=const>0,$$

$$g(x_1,x_2)\in L_2(\partial\tilde{\Omega}),\ \ 0<a_0\leq a(x_1,x_2)\leq a_1<\infty,\ \ a_0,\ a_1\ =const>0,$$

$$0\leq q(x_1,x_2)\leq q_1=const<\infty.$$

Using (1.2),(1.4) and known results of boundary value problem theory it is not difficult to prove the existence and the uniqueness of solutions to problems (1.1), (1.3).

2. Approximation using modified basis functions

We'll assume that the domain $\{\Omega\}$ is decomposed into subdomains Ω_i by straight lines parallel to the Y-axis and intersecting X-axis in the points $\tilde{x}_i$, $i = 1, ..., I$. Thus, $\Omega = \cup_{i=1}^{I+1}\Omega_i$, where Ω_i is a rectangle $\Omega_i = \{(x, y) : \tilde{x}_{i-1} \le x \le \tilde{x}_i$, $C \le y \le D\}$, and $\tilde{x}_0 = A$, $\tilde{x}_{I+1} = B$. We denote by γ_i the interface between subdomains Ω_i and Ω_{i+1}. Let us triangulate subdomains $\{\Omega_k\}$ and consider piece-wise linear functions $\{w_i^{(k)}(x, y)\}_{i=1}^{N_k}$ corresponding to interior triangulation nodes in a subdomain Ω_k, $(k = 1, .., I + 1)$.

Let h_i be the characteristic size of triangulars in Ω_i.

On $\gamma = \cup_{i=1}^{I}\gamma_i$ let us introduce basis functions computed on the basis of fundamental functions of Poincaré-Steklov operators: precisely on $\gamma_i = \{x = \tilde{x}_i,\ C < y < D\}$ we consider functions of the following form

$$\omega_j^{(\gamma_i)} = sin\Big[\frac{j\pi(D-y)}{D-C}\Big] \times \begin{cases} 0 \quad \text{outside } \Omega_i \cup \gamma_i \cup \Omega_{i+1}; \\ \frac{sh[\frac{j\pi}{D-C}(\tilde{x}_{i+1}-x)]}{sh[\frac{j\pi}{D-C}(\tilde{x}_{i+1}-\tilde{x}_i)]} \quad in\ \Omega_{i+1}; \\ \frac{sh[\frac{j\pi}{D-C}(x-\tilde{x}_{i-1})]}{sh[\frac{j\pi}{D-C}(\tilde{x}_i-\tilde{x}_{i-1})]} \quad in\ \Omega_i, \\ j = 1, ..., N^{(\gamma_i)}, \quad i = 1, .., I. \end{cases} \tag{2.1}$$

We will denote all the functions $\{\omega_j^{(\gamma_i)}\}$ by $\{\omega_j^{(\gamma)}\}_{j=1}^{N^{(\gamma)}}$ where $N^{(\gamma)} = \sum_{i=1}^{I} N^{(\gamma_i)}$. It is easy to notice that the functions $\{\omega_j^{(k)}\}$, $\{\omega_j^{(\gamma)}\}$ are linearly independent.

Let us consider the space $\overset{\circ}{W}{}_2^{1,h}(\Omega)\subset\overset{\circ}{W}{}_2^1(\Omega)$ which consists of functions of the following form

$$u_h = \sum_{k=1}^{I+1}\sum_{i=1}^{N^{(k)}} a_i^{(k)}\omega_i^{(k)} + \sum_{i=1}^{N^{(\gamma)}} a_i^{(\gamma)}\omega_i^{(\gamma)}. \tag{2.2}$$

Approximation properties of functions u_h are given by the following lemma.

LEMMA 2.1 [13]. *For any function* $v \in \overset{\circ}{W}{}_2^1(\Omega) \cap W_2^2(\Omega)$ *there exists a function* v_h *of the form (2.2) (for* $N^{(\gamma)} \simeq 1/h$*) such that:* $||v - v_h||_{W_2^1(\Omega)} \le C\ h\ ||v||_{W_2^2(\Omega)}$, $h_i < h\ \forall i$.

We will look for an approximate solution to the problem (1.1) in the form (2.2). The unknowns $\{a_i^{(k)}\}$, $\{a_i^{(\gamma)}\}$ are derived from the system of linear algebraic equations:

$$a(u_h, \omega_i^{(k)}) = (f, \omega_i^{(k)}), \quad i = 1, ..., N_k, \quad k = 1, ..., I+1, \tag{2.3}$$

$$a(u_h, \omega_j^{(\gamma)}) = (f, \omega_j^{(\gamma)}), \quad j = 1, ..., N^{(\gamma)}.$$

This system can be written in a vector-matrix form:

$$\widehat{A}\vec{a} = \vec{f}, \tag{2.4}$$

where

$$\widehat{A} = \begin{bmatrix} \widehat{A}_1 & \dots & 0 & \widehat{U}_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \widehat{A}_{I+1} & \widehat{U}_{I+1} \\ \widehat{L}_1 & \dots & \widehat{L}_{I+1} & \widehat{A}_\gamma \end{bmatrix}$$

$$(\widehat{A}_k)_{i,j} = a(\omega_j^{(k)}, \omega_i^{(k)}), \ (\widehat{A}_\gamma)_{i,j} = a(\omega_j^{(\gamma)}, \omega_i^{(\gamma)}), \ (\widehat{U}_k)_{i,j} = a(\omega_j^{(\gamma)}, \omega_i^{k)}),$$

$$\widehat{L}_k = (\widehat{U}_k)^T, \ \vec{a} = (\vec{a}_1, ..., \vec{a}_{I+1}, \vec{a}_\gamma)^T, \ \vec{f} = (\vec{f}_1, ..., \vec{f}_{I+1}, \vec{f}_\gamma)^T.$$

If we find vectors $\vec{a}_k$ from the system: $\vec{a}_k = -\widehat{A}_k^{-1}\widehat{U}_k\vec{a}_\gamma + \widehat{A}_k^{-1}\vec{f}_k$, $k = 1, ..., I+1$, and substitute them into the last block equation of the system (2.4) then we obtain " the equation on γ"

$$\mathcal{A}\vec{a}_\gamma = \vec{F}, \tag{2.5}$$

where $\mathcal{A} = \widehat{A}_\gamma - \sum \widehat{L}_k \widehat{A}_k^{-1} \widehat{U}_k, \quad \vec{F} = \vec{f}_\gamma - \sum \widehat{L}_k \widehat{A}_k^{-1} \vec{f}_k$.

The matrix $\mathcal{A}$ in (2.5) is of order $N^{(\gamma)}$. The choice of one or another iterative method for the system (2.5) is the basis for corresponding algorithm of domain decomposition method. ¿From the projection methodss theory we can conclude that the approximate solutions $\{u_{h_i}\}$, $i = 1, 2, ...$ to the problem (1.1) defined according to (2.3) converge to the exact solution when $h_i \to 0$ and the following estimate is valid:

$$||u - u_h||_{W_2^1(\Omega)} \leq C \ h \ ||f||_{L_2(\Omega)},$$

where the constant $C > 0$ does not depend on u, u_h, h_i, h, f, and $h_i < h$ are typical mesh sizes in Ω.

Let us consider now the problem (1.3), where $\tilde{\Omega} \subset \Omega$. We consider the decomposition of Ω into rectangles $\{\Omega_i\}_{i=1}^{I+1}$ and introduce in $\{\Omega_i\}$ triangular meshes. We associate each mesh node from Ω_i, $\partial\Omega_i$ (which does not belong to $\cup_{i=1}^{I}\gamma_i$) with piece-wise linear Courant function $\omega_i^{(k)}$ and on $\{\gamma_i\}$ we introduce again functions of the form (2.1). We suppose also that the boundary $\partial\tilde{\Omega}$ of the domain coincides with the mesh line. As before, we look for an approximate solution in the following form

$$u_h = \sum_k {\sum_i}' a_i^{(k)}\omega_i^{(k)} + \sum_{i=1}^{N^{(\gamma)}} a_i^{(\gamma)}\omega_i^{(\gamma)} \tag{2.6}$$

where the prime "′" means that we sum up in the values i which correspond to the mesh nodes from $\cup_k(\tilde{\Omega}_k \cup \partial\tilde{\Omega}_k)$ but do not belong to $\{\gamma_k\}$ (i.e. $(x_i, y_i) \in \cup_k(\tilde{\Omega}_k \cup \partial\tilde{\Omega}_k)$, $(x_i, y_i) \notin \{\gamma_k\}$). The unknown coefficients are derived from a system of the form (2.4)

Approximation properties of basis functions used here are given by the following lemma.

LEMMA 2.2 [13]. *For any function $v \in W_2^2(\tilde{\Omega})$ there exists a function v_h of the form (2.6) such that: $||v - v_h||_{W_2^1(\tilde{\Omega})} \leq C\ h\ ||v||_{W_2^2(\tilde{\Omega})}$, $h_i < h$, $\forall i$, where $C=$ const does not depend on h, v.*

Using the statement of this lemma it can be proved that if the solution of the problem (1.3) belongs to $W_2^2(\tilde{\Omega})$ then approximate solutions $\{u_{h_i}\}$ converge to the exact solution u when $h_i < h \to 0$ and the following estimate is valid: $||u - u_h||_{W_2^1(\tilde{\Omega})} \leq C\ h\ ||u||_{W_2^2(\tilde{\Omega})}$, where the constant C does not depend on h_i, h, u.

3. Properties of matrices

Let us consider the system of equations arisen while solving the problem (1.1) with the use of modified basis functions given in Section 2. We will assume below that $N^{(\gamma_i)} = N^{(\gamma)}/I \equiv N_0^{(\gamma)}$.

LEMMA 3.1 [6,13]. *If for solving the problem (1.1) we use modified basis functions given above then for the matrix $\mathcal{A} = \widehat{A}_\gamma - \sum_{k=1}^{I+1}\widehat{L}_k\widehat{A}_k^{-1}\widehat{U}_k$ the following inequalities are valid:*

$$0 < C_1 < \frac{(\mathcal{A}\vec{b}, \vec{b})_2}{(\widehat{A}_\gamma^{(1)}\vec{b}, \vec{b})_2} \leq C_2 < \infty, \tag{3.1}$$

where symmetric, positive definite matrix $\widehat{A}_\gamma^{(1)}$ is the block- diagonal one with elements $(\widehat{A}_\gamma^{(1)})_{ij} = (\nabla\omega_j^{(\gamma)}, \nabla\omega_i^{(\gamma)})$, each block of $\widehat{A}_\gamma^{(1)}$ is a tri-diagonal matrix and C_1, C_2 are the constants from (1.2).

REMARK. The explicit appearance of $\widehat{A}_\gamma^{(1)}$ is presented in [13].

Consider the problem (1.3), the system of modified basis functions introduced to approximate this problem and the matrix $\hat{A}_\gamma^{(2)}$ with elements $(\hat{A}_\gamma^{(2)})_{ij} = (\nabla\omega_j^{(\gamma)}, \nabla\omega_i^{(\gamma)})_{L_2(\tilde{\Omega})} + \int_{\tilde{\Omega}\cap\gamma} \omega_j^{(\gamma)}\omega_i^{(\gamma)} d\gamma$. The following statement is valid.

LEMMA 3.2. *There exist positive constants C_3, C_4 which do not depend on $h, N_0^{(\gamma)}$ and such that*

$$0 < C_3 < \frac{(\mathcal{A}\vec{b}, \vec{b})_2}{(\hat{A}_\gamma^{(2)}\vec{b}, \vec{b})_2} \le C_4 < \infty, \quad \forall \vec{b} \not\equiv \vec{0}. \tag{3.2}$$

4. Domain decomposition algorithms

As we have noted before, the domain decomposition algorithms can be considered as realizations of one or another iterative process applied to the equation (2.5).

Let us consider the problem (1.1) and write down the iterative process of minimal corrections method [12] with the use of the preconditioner $\widehat{B} \equiv \widehat{A}_\gamma^{(1)}$:

$$\vec{a}^0 = 0,\ \vec{\xi}^j = \mathcal{A}\vec{a}^j - \vec{F},\ \vec{G}^j = \widehat{B}^{-1}\vec{\xi}^j,\ \ \vec{a}^{j+1} = \vec{a}^j - \tau_j\vec{G}^j, \tag{4.1}$$

$$\tau_j = \frac{(\mathcal{A}\vec{G}^j, \vec{G}^j)_2}{(\widehat{B}^{-1}\mathcal{A}\vec{G}^j, \mathcal{A}\vec{G}^j)_2},\ j = 0, 1, \ldots$$

The domain decomposition algorithm base on the iterative process (4.1) consists of the following steps.

Step 0 (preliminary step; $j = 0$):

$$\widehat{A}_k\vec{g}_k = \vec{f}_k,\ k = 1,..,I+1,\ \ \vec{F} = \vec{f}_\gamma - \sum_{k=1}^{I+1} \widehat{L}_k\vec{g}_k,\ \ \vec{\xi}^0 = -\vec{F},\ \vec{a}^0 = 0 \tag{4.2}$$

Step 1:

$$\widehat{A}_\gamma^{(1)}\vec{G}^j = \vec{\xi}^j,\ \widehat{A}_k\vec{g}_k = \widehat{U}_k\vec{G}^j,\ \ k = 1,..,I+1,\ \ \vec{p}^j = \widehat{A}_\gamma\vec{G}^j - \sum_{k=1}^{I+1} \widehat{L}_k\vec{g}_k \tag{4.3}$$

Step 2:

$$\vec{g} = (\widehat{A}_\gamma^{(1)})^{-1}\vec{p}^j,\ \ \tau_j = \frac{(\vec{p}^j, \vec{G}^j)_2}{(\vec{g}, \vec{p}^j)_2},\ \ \vec{a}^{j+1} = \vec{a}^j - \tau_j\,\vec{G}^j,\ \ \vec{\xi}^{j+1} = \vec{\xi}^j - \tau_j\vec{p}^j. \tag{4.4}$$

Then we return to *Step 1* with new $\vec{a}^{j+1}$, $\vec{\xi}^{j+1}$. We repeat this procedure up to the step j_0 when the norm of the vectors $\vec{\xi}^{j_0}$ is sufficiently small. Then we proceed to the last step.

Step 3 (the last step): Find the other unknowns:

$$\widehat{A}_k \vec{a}_k^{j_0} = -\widehat{L}_k \vec{a}^{j_0} + \vec{f}_k, \quad k = 1, ..., I+1. \tag{4.5}$$

Now using the estimates (3.1) and known results of the theory of iterative processes we can conclude that the following theorem holds true.

THEOREM 4.1. *If we use the domain decomposition algorithm (4.2)-(4.5) to solve (1.1) then for the approximate solution to the problem $u_h^{j_0}$ obtained after j_0 iterations the following estimate is valid*

$$\left|\left|u_h - u_h^{j_0}\right|\right|_{\overset{\circ}{W}{}_2^1(\Omega)} \leq C \cdot \left(\frac{C_2 - C_1}{C_2 + C_1}\right)^{j_0} \cdot \left|\left|u_h - u_h^0\right|\right|_{\overset{\circ}{W}{}_2^1(\Omega)}, \tag{4.6}$$

where the constant C does not depend on mesh parameters, and C_1, C_2 are the constants from (1.2).

Consider the problem (1.3). To solve the system (2.4) in this case we use an algorithm based on the iterative locally-optimal three-steps method [12] with the use of the preconditioner $\widehat{B} = \widehat{A}_\gamma^{(2)}$:

$$\vec{a}^{n+1} = \alpha_{n+1}\vec{a}^n + (1 - \alpha_{n+1})\vec{a}^{n-1} - \alpha_{n+1}\tau_{n+1}\widehat{B}^{-1}\vec{\xi}^n, \tag{4.7}$$

$$\vec{\xi}^{n+1} = \alpha_{n+1}\vec{\xi}^n + (1 - \alpha_{n+1})\vec{\xi}^{n-1} - \alpha_{n+1}\tau_{n+1}\vec{p}^n,$$

where

$$\vec{\xi}^n = \mathcal{A}\vec{a}^n - \vec{F},\ \vec{p}^n = \mathcal{A}\widehat{B}^{-1}\vec{\xi}^n, \quad \vec{a}^1 = \vec{a}^0 - \tau_1\widehat{B}^{-1}\vec{\xi}^0,\ \vec{\xi}^1 = \vec{\xi}^0 - \tau_1\vec{p}^0,$$

and the coefficients are definied as follows:

$$\alpha_{n+1} = \frac{(b_n - c_n)c_n - e_n f_n}{(d_n - e_n)f_n - (b_n - c_n)^2}, \quad n = 1, 2, ..., \quad \alpha_1 = 1$$

$$\tau_{n+1} = \frac{b_n}{g_n} + \frac{1 - \alpha_{n+1}}{\alpha_{n+1}}\frac{c_n}{f_n}, \quad n = 0, 1, 2, ...$$

$$b_n = (\mathcal{A}\vec{\omega}^n, \vec{\omega}^n)_2, \quad c_n = (\mathcal{A}\vec{\omega}^n, \vec{\omega}^{n-1})_2, \tag{4.8}$$

$$d_n = (\vec{\omega}^n, \vec{\xi}^n - \vec{\xi}^{n-1})_2, \quad e_n = (\vec{\omega}^{n-1}, \vec{\xi}^n - \vec{\xi}^{n-1})_2,$$
$$f_n = (\widehat{B}^{-1}\mathcal{A}\vec{\omega}^n, \mathcal{A}\vec{\omega}^n)_2, \quad \text{where } \omega^n = \widehat{B}^{-1}\vec{\xi}^n.$$

In this case the convergence rate of iterative domain decomposition algorithm with the use of locally optimal process is estimated as follows:

$$||\mathcal{A}\vec{a}^j - \vec{f}||_{\widehat{D}} \leq \frac{2\cdot\rho^j}{1+\rho^{2j}}||\mathcal{A}\vec{a}^0 - \vec{f}||_{\widehat{D}} \to 0, \quad j = 1, ..., \tag{4.9}$$

where $\widehat{D} = \mathcal{A}^*\widehat{B}^{-1}\mathcal{A}$, $\rho = (1-\sqrt{\theta})/(1+\sqrt{\theta})$, the constant $\theta = C_3/C_4 < 1$ does not depend on mesh sizes and C_3, C_4 are the constants from (3.2).

To shorten the consideration we will not write down here realization steps of the algorithm (4.7), (4.8).

Let $u_h^{j_0}$ be an approximate solution to the original problem after j_0 iterations. Then using (4.9) we conclude that the following statement holds.

THEOREM 4.2. *If we use the algorithm (4.7)-(4.8) to solve (2.4) then*

$$||u_h - u_h^{j_0}||_{W_2^1(\tilde{\Omega})} \leq C\cdot\frac{\rho^{j_0}}{1+\rho^{2j_0}}||u_h - u_h^0||_{W_2^1(\tilde{\Omega})}$$

where the constant C does not depend on u_h, j_0, mesh parameters, and ρ has the form:

$$\rho = \frac{(1-\sqrt{C_3/C_4})}{(1+\sqrt{C_3/C_4})}.$$

The considered modification of iterative decomposition algorithm with the use of locally optimal three-step process in comparison with three-step conjugate direction method has higher numerical stability [3] though in this case calculation formulae for iterative parameters have more complicated form. Moreover, locally optimal three-step process is applicable when the only restriction is satisfied: the matrix of the system is positive defined.

5. Numerical example and remarks

Firstly we'll make some remarks: 1). Let us notice that the meshes in $\{\Omega_i\}$ should not be compatible on the interface between subdomains. 2). In $\{\Omega_i\}$ we can use other basis functions (bilinear functions and so on). 3). The proposed method of the construction of the modified basis functions can be used in the problems with mixed boundary conditions, in some 3D-problems and in the problems with more complicated domain Ω (in the last case we can use the well-known technique of the coordinate transformation ([7],[8])). 4). We can associate γ with the following modified basis functions: we introduce on γ some mesh and some piece-wise polynomial functions extended to subdomains in a harmonic (or "generalized-harmonic") manner (for example, by expansion in a series of Poincaré-Steklov fundamental functions, [6]). To illustrate some of these remarks let us consider the following numerical example.

Consider the problem given by

$$-div(\epsilon\nabla u) = f \ \ in\ \Omega;\ u = u_{(\Gamma)}\ on\ \Gamma_D;\ \frac{\partial u}{\partial N} = 0\ on\ \Gamma_N \tag{5.1}$$

or by the following generalized statement: find $u \in W_2^1(\Omega),\ u = u_{(\Gamma)}$ such that the relationship

$$a(u,v) = \sum_{k=1}^{2}\epsilon_k \int_{\Omega_k}\sum_{j=1}^{2}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_j}\mathrm{d}\Omega = (f,v)\ \forall v \in W_2^1(\Omega),\ v\,|_{\Gamma_D} = 0 \tag{5.2}$$

holds, where the domain $\Omega = \Omega_1 \cup \gamma \cup \Omega_2$ is presented on Fig.1, $\epsilon = \{\epsilon_k = const > 0\ \ in\ \Omega_i,\ i = 1,2\}$, $f \in L_2(\Omega)$, $u_{(\Gamma)}$ is some given function of $W_2^1(\Omega)$.

Let us introduce in $\{\Omega_i\}$ some (different) rectangle meshes and consider bilinear basis functions $\{\omega_i^{(k)}\}$. On γ let us consider (for simplicity) the uniform mesh $\{x_i = i\cdot h\}$ of the step h, piece-wise linear functions $\{\tilde{\omega}_i^{(\gamma)}(x)\}$ and modified basis functions, defined as follows:

$$\omega_i^{(\gamma)}(x,y) = \Big\{\sum_{j=0}^{n}\alpha_{j,i}^{(k)}\cdot\phi_j^{(k)}(x,y)\ in\ \Omega_k,\ k = 1,2\Big\} \tag{5.3}$$

where

$$\alpha_{j,i}^{(k)} = \int_{x_{i-1}}^{x_{i+1}}\tilde{\omega}_i^{(\gamma)}(x)\cdot\phi_j^{(k)}(x,0)\mathrm{d}x,$$

$$\alpha_{0,i}^{(1)} = \frac{h}{\sqrt{A}},\ \alpha_{j,i}^{(1)} = \frac{4}{h}\sqrt{\frac{2}{A}}sin^2(\frac{j\pi h}{2A})\cdot(\frac{A}{j\pi})^2\cdot cos(\frac{j\pi x_i}{A}),$$

$$\phi_0^{(1)}(x,y) = \frac{b_1 - y}{\sqrt{A}\cdot b_1},\ \phi_j^{(1)}(x,y) = \sqrt{\frac{2}{A}}\cdot cos(\frac{j\pi x}{A})\cdot\frac{sh(j\pi(b_1-y)/A)}{sh(j\pi b_1/A)},$$

....

Let us notice that if $n = \infty$ in (5.3) then $\{\omega_i^{(\gamma)}\}$ are the solutions of the following problems:

$$\Delta\omega_i^{(\gamma)} = 0\ in\ \Omega_k;\ \frac{\partial\omega_i^{(\gamma)}}{\partial n_k} = 0\ on\ \Gamma_N\cap\partial\Omega_k;$$

$$\omega_i^{(\gamma)} = \tilde{\omega}_i^{(\gamma)} \ at \ y = 0; \quad \omega_i^{(\gamma)} = 0 \quad at \ y = b_1, \ b_2 \quad \{k = 1, 2\}.$$

Here $\{\phi_j^{(k)}\}$ are the fundamental functions of Poincaré-Steklov operators associated with $\{\Omega_k\}$ (see, [6]).

The approximate solution to the problem (5.2) we look for in the form

$$u_h = u_{(\Gamma)} + \sum_{k=1}^{2} \sum_{j=1}^{N_k} a_j^{(k)} \omega_j^{(k)} + \sum_{i=1}^{N^{(\gamma)}} a_i^{(\gamma)} \omega_i^{(\gamma)}$$

where coefficients $\{a_j^{(k)}\}$, $\{a_i^{(\gamma)}\}$ are determined by the method (2.3). It is easy to see that here $\widehat{U}_k = 0$, $\widehat{L}_k = 0$, $k = 1, 2$ (i.e. the matrix of (2.4) is the block-diagonal one) and therefore we can solve (2.4) "in explicit form": $\vec{a}_k = \widehat{A}_k^{-1} \vec{f}_k$, $\vec{a}_\gamma = \widehat{A}_\gamma^{-1} \vec{f}_\gamma$, $k = 1, 2$.

Let us consider some numerical results.

Exp.1 Let us take: $\epsilon_1 = \epsilon_2 = 1$, $b_1 = 1$, $b_2 = -1$, $A = 1$, $a_1 = 0.3$, $a_2 = 0.7$. The exact solution is $u(x, y) = \{cos \frac{\pi y}{2b_1} \ in \ \Omega_1; \ 1 \ in \ \Omega_2 \ \}$. In TABLE 1 the error $\xi = u - u_h$ as a function of n (see (5.3)) is presented.

	$n = 5$	$n = 10$	$n = 15$
x_1	-0.07835	-0.01638	-0.01283
x_2	0.03134	-0.01357	-0.01197
x_3	-0.05390	-0.01430	-0.01466
x_4	-0.04319	-0.01688	-0.01505
x_5	-0.01323	-0.01633	-0.01428
x_6	0.05545	-0.01627	-0.01453
x_7	-0.01873	-0.01306	-0.01365
x_8	0.01300	-0.01219	-0.01080
x_9	-0.01045	-0.01376	-0.01111

TABLE 1.

From experiments we could conclude that to have the approximate solutions of $(0.2 \div 2.2)$ % - accuracy it is sufficiently to take $n \simeq 10$. On the Fig.2 the section of $\omega_5^{(\gamma)}(x, y)$ is presented.

Exp.2 Consider the data: $b_1 = 1$, $b_2 = 0.05$, $A = 1$, $a_1 = 0.3$, $a_2 = 0.7$, $f = \{1 \ in \ \Omega_1^{(1)}; \ 0 \ in \ \Omega \setminus \Omega_1^{(1)}\}$, $\epsilon_1 = \epsilon_2 = 1$, $u_{(\Gamma)} = 0$. On Fig.3 the numerical solution calculated using the modified basis functions is presented.

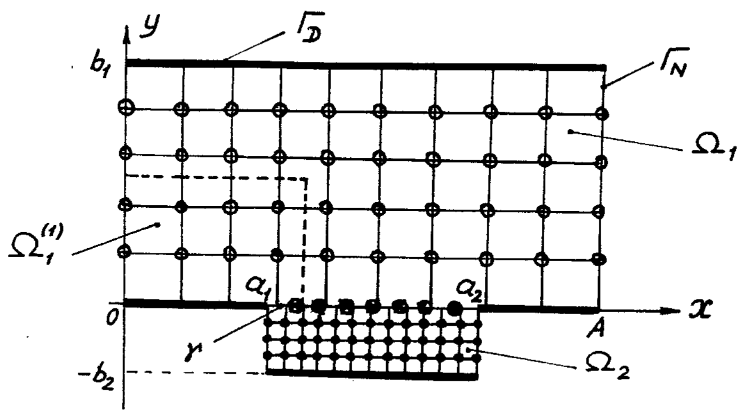

FIG.1 THE DOMAIN Ω

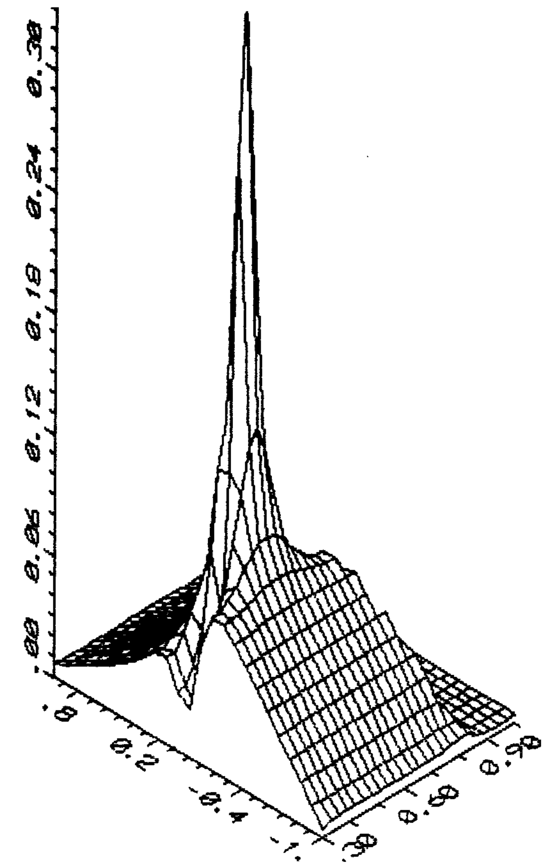

FIG.2 MODIFIED BASIS FUNCTION $\omega_5^{(\gamma)}$

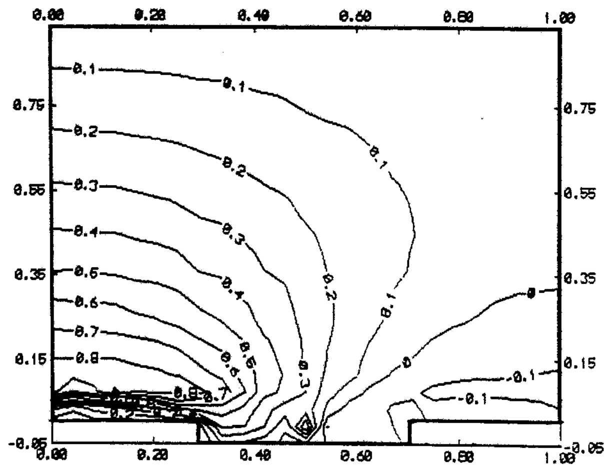

FIG.3 THE NUMERICAL SOLUTION u_h.

6. Conclusion

Let us make some remarks on the domain decomposition methods with modified basis functions that we have proposed here.

Firstly let us enlight some advantages of these methods. In several problems if we use modified basis functions we can diagonalize (by blocks) the algebraic system of equations (i.e. we have $\widehat{L}_k = 0$, $\widehat{U}_k = 0$) and solve the problem by inverting each block-diagonal matrix. In other cases (when $\widehat{L}_k \neq 0$, $\widehat{U}_k \neq 0$) the matrix $\widehat{A}^{(1)}_{(\gamma)}$, constructed using modified basis functions, may be used as an effective preconditioner. Therefore, in some problems with variable coefficient and complicated boundaries it is possible to construct domain decomposition methods which convergence rate does not depend on the mesh size. Let us notice that domain decomposition methods using modified basis functions can be interpreted as the well known Fourier' method in the problem with a domain partitioned into simple shape subdomains. Although the latter situation occurs in many cases of real interest, this geometric limitation is one of the main draw backs of our approach. Note also that in our opinion the method presented in this paper can fully exploit the theory of special functions. The author wishes to thank Prof. A. Quarteroni and Dr. S. Nepomnyaschikh for fruitful discussions and remarks.

REFERENCES

1. G.I. Marchuk and V.I. Agoshkov, *Introduction to Projection-Difference Methods*, Nauka, Moscow, (1980) (in Russian).

2. V.I. Agoshkov and V.I. Lebedev, Poincaré-Stekelov operators and domain decomposition algorithms in variational problems. In: *Computer Process and Systems*, vol. 2, Nauka, Moscow, (1985) (in Russian).

3. S.N. Buleev, Programme realization of domain decomposition algorithms. Preprint No. 244, Dept. Numer. Math., USSR Acad. Sci., Moscow, (1990) (in Russian).

4. S.N. Buleev and V.I. Agoshkov, The investigation of some domain decomposition algorithms. In: *Adjoint Equations and Perturbation Algorithms in Problems of Mathematical Physics.* Dept. Numer. Math., USSR Acad. Sci., Moscow, (1989), 91-111 (in Russian).

5. V.I. Agoshkov and V.I. Lebedev, Variational algorithms of domain decomposition method. Preprint No. 54, Dept. Numer. Math., USSR Acad. Sci., Moscow, (1983) (in Russian).

6. V.I. Agoshkov, Domain decomposition method: some results of theory and applications. In: *Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations*, (Ed. R. Glowinski and oth.), SIAM, Philadelphia, (1991), 3-18.

7. S.G. Miklin and Kh.L. Smolitskii, *Approximate Methods for Solving Differential and Integral Equations.* Nauka, Moscow, (1965) (in Russian).

8. L.S. Xanthis and C. Schwab, The method of Arbitrary Lines. C.R.Sci. Paris, t.312, Serie I, (1991), 181-187.

9. A. Quarteroni, Domain decomposition and parallel processing for the numerical solution of partial differential equations. Surv. Math. Ind., **1** (1991), 75-118.

10. L.D. Marini, A. Quarteroni, A relaxation procedure for domain decomposition methods using finite elements. Numer. Math. **55** (1989), 575-598.

11. F. Saleri, Domain decomposition for finite elements and their implementation on a vector multiprocessor. Impact Comput. Sci. Engineer. **2** (1990), 99-12.

12. A.A. Samarskj and E.S. Nikolaev, *Methods for Solving Difference Schemes.* Nauka, Moscow, (1978) (in Russian).

13. V.I. Agoshkov, S.N. Buleev, Domain Decomposition Methods Using Modified Basis Functions. (to appear).

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