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Domain Decomposition Methods with Local Fourier Basis for Parabolic Problems *

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Abstract

Two basic spectral domain decomposition techniques are developed and compared for the solution of time-dependent nonlinear partial differential equations. The particular feature of both techniques is that they take advantage, in an explicit way, of the locality of parabolic equations. In the first approach the matching of elemental solutions in subdomains is performed by using boundary Green's functions. The explicit matching relations are derived which require data exchange between only neighbouring subdomains. In the second approach the solution of multidomain problem is constructed as a superposition of local overlapping solutions which satisfy zero boundary conditions in the extended subdomains. This approach presents a generalization of Overlapping Domain Decomposition method [4]. For space discretization a pseudo-spectral method is used with local trigonometric bases, supported on a range of a subdomain. The computational algorithm involves an appropiate projection procedure for the smooth decomposition of the source function.

1 Introduction

Domain decomposition (DD) methods become an efficient tool for parallelizing numerical algorithms, since each subdomain can be assigned to different processor and computed independently. Then, matching of different elemental solutions is necessary for obtaining a smooth global solution. This is a crusial step for DD algorithms because it introduces overheads in the implementation on a parallel computer and may, in extreme cases, reduce or eliminate the gains of parallelization. Therefore, the main objective of any DD method is to reduce,

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as much as possible, the amount of data transfer and communication among subdomains.

In this paper we describe two parallel algorithms, based on spectral multidomain techniques, for solving time-dependent equations

$$\frac{\partial u}{\partial t} = \Delta u + N(u,u) + G(x,t), \qquad x \in \Omega \subset R^2, \tag{1.1}$$

(this type of equations is widely used in Computational Fluid Dynamics as they model the complete Navier-Stokes system [2, 6] or they are involved as one split computational step in time-splitting methods [5]). A novel feature of these algorithms is, that they exploit the local behaviour of solutions to minimize the communication of processors.

After semi-implicit time discretization the evolutionary problem is converted to a system of linear second order elliptic equations for the unknown variables u^n at the new time step $t_n = n\tau$:

$$\Delta u^n - \lambda^2 u^n = f^{n-1}, \tag{1.2}$$

where the function $f^n = f(u^n, x; \lambda)$ in the RHS is evaluated on a previous time step, and parameter $\lambda \propto 1/\tau$. By using appropriate projection procedures, the source function f is partitioned in a smooth way. Then the spectral Fourier method with a locally supported trigonometric basis is employed to construct the particular elemental solutions.

In the first approach (section 2) the particular solutions in the subdomains are computed with arbitrary boundary conditions at the interfaces. The continuity of the overall solution is achieved by using a proper combination of the homogeneous solutions in each subdomain. These functions appear well localized near the interfaces, which enables the matching relations to be decoupled in a way that only communication between contiguous subdomains is required in the matching step.

In a second approach (section 3) we modify the overlapping DD method, suggested in [4], to be compatible with spetcral methods. This approach is based on the rapid decay of the solutions of singularly perturbed elliptic operators, resulting from a semi-implicit time discretization of parabolic time-dependent problems. The solution to the original problem is constructed as a superposition of local overlapping solutions which decay to zero at a sufficiently large distance from the location of the source. The advantage of such approach is that it does not require the matching of elemental solutions and utilizes the data exchange only between neighbouring subdomains. We demonstrate the capability of both local spectral methods to resolve high gradient in solution with a good accuracy. We also compare our results and those that were obtained by the Overlapping DD method in [4] with a stepwise partitioning of the source function.

2 Parabolic Domain Decomposition (PDD) Method

To illustrate the basic technique, we consider the one-dimensional version of equation (1.2)

$$u'' - \lambda^2 u = f(x,t), \qquad x \in [0, \mathcal{L}], \tag{2.1}$$

where f is a known function. The interval $\mathcal{L}$ is divided into S equal subdomains of length $l = \mathcal{L}/S$.

The algorithm consists of two main steps. At first we construct the particular solutions in each subdomain independently, applying the arbitrary boundary conditions at the interfaces. For a space discretization we use a Local Fourier Basis (LFB) [1]. The advantage of this basis is that the matrices of differential operators in a transformed space are diagonal (in contrast with full matrices for the Chebyshev or Legendre bases). At the same time, nonperiodic functions are represented by the rapidly converging series, i.e. this basis can be efficiently used for solving differential equations in multidomain regions. After this step the overall solution appear as a piecewise function with the jumps across the interfaces.

In the second step, we match the elemental solutions by using the boundary Green's functions (solutions of a corresponding homogeneous problem).

2.1 Spectral LFB Technique for Solving Non-Periodic PDE's

We consider the solution of (2.1) in one of the subdomains $x \in (a,b) \subset (0,\mathcal{L})$. The computational algorithm involves a smoothing procedure near the interfaces to localize the functions in a smooth manner and the application of the spectral Fourier method with the locally supported trigonometric basis.

To implement smoothing, we introduce a bell function $B(x)$, supported on an extended interval
$a_1 < a < b < b_1$:

$$\begin{array}{ll} B^2(x) + B^2(2\bar{a} - x) = 1 & x \in (a_1, a) \\ B(x) = 1 & x \in [a, b] \\ B^2(x) + B^2(2\bar{b} - x) = 1 & x \in (b, b_1) \\ B(x) = 0 & x < a_1,\ x > b_1 \end{array} \tag{2.2}$$

where $\bar{a} = (a+a_1)/2$, $\bar{b} = (b+b_1)/2$. Inside the subdomain this function is equal to $B = 1$ and smoothly decays outwords over a distance $2\epsilon = b_1 - b = a - a_1$. Some specific forms of $B(x)$ were tested in [3].

The smoothing $\tilde{f}$ of the function f appears as a "folding" across the lines $\bar{a}$ and $\bar{b}$ (see Fig.1):

$$\tilde{f} \equiv B \cdot f = \mathcal{F}_{\bar{a}}\mathcal{F}_{\bar{b}} f(x) = B(x)f(x) - B(2\bar{a} - x)f(2\bar{a} - x) - B(2\bar{b} - x)f(2\bar{b} - x) \tag{2.3}$$

(the "folded" function $\tilde{f}$ is defined in $[\bar{a}, \bar{b}]$; the second term is "switched on" only on the interval $x \in (a_1, a)$ and the third term- on the interval $x \in (b. b_1)$. respectively). The extra pieces of the function f, required for the smoothing,

are provided by overlapping of neighbouring subdomains over a range of 4ϵ. On the interval $x \in (a, b)$ we have $\tilde{f} = f$.

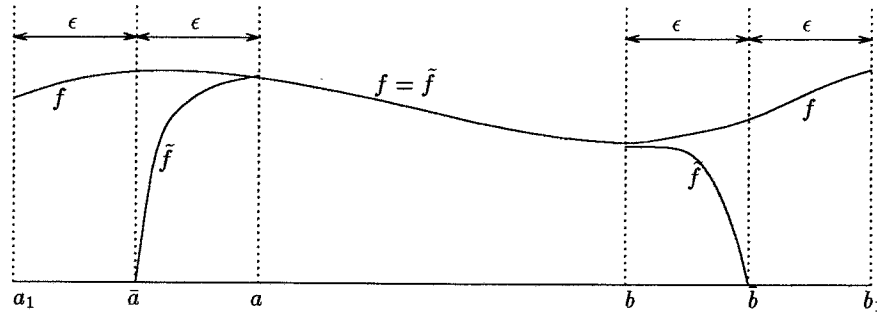

Figure 1: The folding operation

The smoothing procedure keeps the function $\tilde{f}$ highly continuous at $x = a, b$. Form (2.3) we see that in the vicinity of the points $x = \bar{a}$, $x = \bar{b}$ the function $\tilde{f}(x)$ is odd and thus all even derivatives $\tilde{f}^{(2r)}(\bar{a}) = \tilde{f}^{(2r)}(\bar{b}) = 0$, $r = 0, 1, ...$ After an antisymmetric reflection across the point $x = \bar{b}$ (or $x = \bar{a}$) we get a smooth periodic function, which can be represented by a rapidly converging Fourier series.

After that we apply the pseudo-spectral Fourier method to the problem (2.1) with the "smoothed" source function $\tilde{f}$ instead of f. The corresponding solution $\tilde{p}$ will coincide with the solution p to the original problem on the interval $x \in (a, b)$, where the projection procedure does not distort $f(x)$.

The accuracy of the soothing procedure (2.3) depends on the number of collocation points N_ϵ on the interval of smoothing 2ϵ. If the density of points is sufficiently large, a high resolution can be obtained for a relatively small ratio ϵ/l. Thus, the overhead due to overlapping of subdomains, nedeed for the smoothing, becomes insignificant as the number of collocation points increases.

2.2 Matching Procedure

The algorithm, described above is performed concurrently in all subdomains. The local solutions $p^{(n)}$ and their first derivatives will have, in general, jumps on the interfaces. To match those solutions we use a proper combination of two boundary Green's functions $e^{-\lambda x}$ and $e^{\lambda(x-l)}$, which are the homogeneous solutions for (2.1).

We represent a continuous solution in each subdomain as a superposition of a particular and homogeneous parts:

$$u = \bigcup_{s=1}^{S} u^{(s)},\ u^{(s)} = p^{(s)} + A_s e^{\lambda(x-l)} + B_s e^{-\lambda x},\ x \in [0, l], \tag{2.4}$$

The unknown coefficients A_s, B_s are determined so that the continuity conditions $u^{(s)} = u^{(s+1)}$, $(u^{(s)})' = (u^{(s+1)})'$ on the interfaces $x = sl$ are satisfied.

Thus, we get a system of matching relations

$$A_s = A_{s+1}e^{-\lambda l} - \alpha_s, \quad B_{s+1} = B_s e^{-\lambda l} - \beta_s$$

$$\alpha_s = \frac{1}{2}(\frac{\delta'_s}{\lambda} + \delta_s), \quad \beta_s = \frac{1}{2}(\frac{\delta'_s}{\lambda} - \delta_s),$$

$$\delta_s = U_p^{(s)}(l) - U_p^{(s+1)}(0), \quad \delta'_s = U_p'^{(s)}(l) - U_p'^{(s+1)}(0) \quad s = 1, 2, \ldots, S-1 \quad (2.5a)$$

The corresponding relations for the two extreme subdomains depend on the global boundary conditions. For example, in the case of the Dirichlet boundary conditions $u(0) = U_0$, $u(L) = U_L$:

$$A_1 e^{-\lambda l} + B_1 = \alpha_0, \; A_S + B_S e^{-\lambda l} = -\beta_S$$

$$\alpha_0 = U_0 - p^{(1)}(0), \quad \beta_S = p^{(S)}(l) - U_L \quad (2.5b)$$

This system can be resolved explicitly with respect to one of the coefficients, say B_1:

$$B_1 = \frac{1}{1 - e^{-2S\lambda l}} \left[\sum_{s=0}^{S-1} e^{-s\lambda l}(\alpha_s + \beta_{S-s} e^{-S\lambda l}) \right], \quad (2.6)$$

Then by using the recurrance relations (2.5a) we can determine all the other coefficients as functions of the jumps δ_s, δ'_s on the interfaces. Thus, instead of overall coupling of collocation points which is inherent in the spectral Fourier method, the communication is reduced to exchanging of interface data. Note that the explicit matching relations (2.5) do not depend on a specific form of nonlinearity or forcing term in (1.1), but only on the type of the boundary conditions [3].

The communication among the processors remains global, because all the interfaces are involved in the matching relations. The decisive simplification can be made by the following observation. Our auxiliary matching functions $e^{-\lambda x}, e^{\lambda(x-l)}$ decay exponentially away from the interfaces. If the parameter λ is large enough (the time step τ of a time-marching scheme is small), those functions will be well localized near the interfaces, and all the matching relations (2.5) become uncoupled:

$$A_s = -\alpha_s, \; B_s = -\beta_s \quad (2.7)$$

(the coefficients A_s, B_s depend on the jumps only at one corresponding interface). In words, instead of global interaction among all the subdomains, only local communication between the contiguous subdomains is important. This is the main idea of the Parabolic Domain Decomposition (PDD) approach.

3 Spectral Overlapping Domain Decomposition method

In the second approach, we exploit again, but in a different way, the local properties of elliptic operator, resulting from a semi-implicit time discretization of the parabolic equation (1.1). This approach generalizes the spectral implementation of the Overlapping Domain Decomposition (ODD) method, introduced in [4]. Hereafter, we outline the main idea of the ODD method and then describe its modified smooth version.

3.1 Overlapping Domain Decomposition (ODD) Method

Consider the 1-D elliptic equation

$$H(x)u \equiv u_{xx} - \lambda^2 u = f(x) \qquad \text{in } \Omega = (-\frac{\mathcal{L}}{2}, \frac{\mathcal{L}}{2}),$$
$$u(-\mathcal{L}/2) = U_L, \qquad u(\mathcal{L}/2) = U_R, \tag{3.1}$$

on a two-domain region $\Omega = \Omega_1 \bigcup \Omega_2$. Instead of solving (3.1) in each particular domain $\Omega_{1,2}$ with the posterior matching at the interface $x = 0$, we consider the following problems:

$$Hu_1 = f_1, \text{ in } \Omega \qquad u_1(-\mathcal{L}/2) = U_L,\ u_1(\mathcal{L}/2) = 0,$$
$$Hu_2 = f_2, \text{ in } \Omega \qquad u_2(-\mathcal{L}/2) = 0,\ u_2(\mathcal{L}/2) = U_R, \tag{3.2}$$

where

$$f_1 = \begin{cases} f, & \text{in } \Omega_1 \\ 0 & \text{in } \Omega_2 \end{cases} \qquad f_2 = \begin{cases} 0, & \text{in } \Omega_1 \\ f & \text{in } \Omega_2 \end{cases}$$

Since at each particular point $x \in \Omega$ the sum $f_1 + f_2 = f$ then the solution of the linear problem (3.1) may be written as a superposition $u = u_1 + u_2$.

Due to the locality of the Green's function of the operator $H(x)$, the solution u_1 will decay exponentially in the region Ω_2 faraway from the boundary $x = 0$. The same is true for the solution u_2 in the region Ω_1. This enable us to impose zero boundary conditions at the artificial interfaces $x = \pm\epsilon$ and solve the following problems

$$H\tilde{u}_1 = \tilde{f}_1, \text{ in } \tilde{\Omega}_1 \qquad u_1(-\mathcal{L}/2) = U_L,\ u_1(\epsilon) = 0,$$
$$H\tilde{u}_2 = \tilde{f}_2, \text{ in } \tilde{\Omega}_2 \qquad u_2(-\epsilon) = 0,\ u_2(\mathcal{L}/2) = U_R,$$
$$\tilde{f}_1 = \begin{cases} f, & \text{in } \Omega_1 \\ 0 & \text{in } \Omega_{12} \end{cases} \qquad \tilde{f}_2 = \begin{cases} 0, & \text{in } \Omega_{21} \\ f & \text{in } \Omega_2 \end{cases} \tag{3.3}$$

where $\tilde{\Omega}_1 = \Omega_1 \bigcup \Omega_{12}$, $\tilde{\Omega}_2 = \Omega_2 \bigcup \Omega_{21}$, the length of extensions $\Omega_{12} = \Omega_{21} = \epsilon$. The function $\tilde{u} = \tilde{u}_1 + \tilde{u}_2$ will approximate the solution of the original problem (3.1) with accuracy $\varepsilon \approx (C/\lambda)e^{-\epsilon\lambda}$.

When the time step τ is small, the parameter $\lambda \propto 1/\sqrt{\tau}$ is large so that only a small overlapping $\Omega_{12} + \Omega_{21} = 2\epsilon$ is required to assure a prescribed accuracy ε.

3.2 Smooth Modification of the ODD (SODD) Method

An appropriate spatial discretization of the auxiliary problems (3.3) is either finite-difference or finite-element. The direct application of any spectral method to these problems with a piecewise source functions $\tilde{f}_{1,2}$ would give a poor precision. To make the spectral implementation accurate, we must decompose the source function $f(x)$ in a smooth manner.

We are interested in solving the problem (3.1) on the interval Ω splitted into S subdomains of length $l = \mathcal{L}/S$. Let us introduce the bell function $B(x)$ having the following properties:

$$B(x) + B(x \pm l) = 1, \qquad \sup B \subset (-\epsilon, l + \epsilon),$$

$$B(-\epsilon) = B(l+\epsilon) = 0,$$
$$x \in (\epsilon, l-\epsilon) \qquad B(x) = 1 \tag{3.5}$$

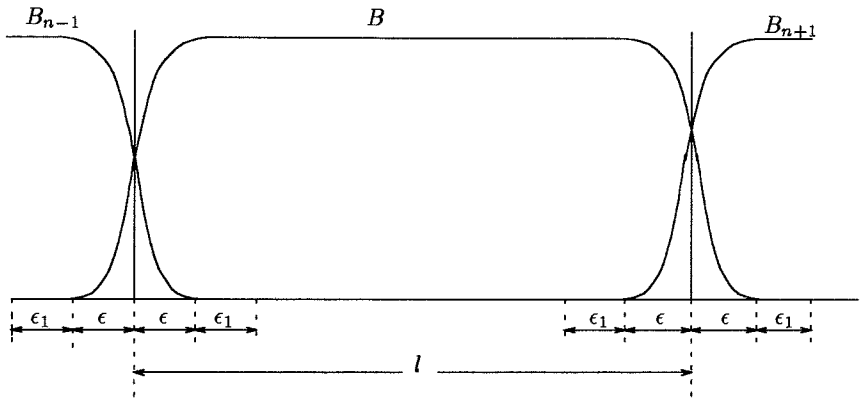

Figure 2: Sequence of overlapping bells

The whole interval Ω is covered by the overlapping bells $B_n(x)$, $B_{n\pm1}(x) = B_n(x \pm l)$ as on Fig.2.

Following the idea of the ODD approach, we define a compound bell $\bar{B}$ such that

$$\bar{B} = \begin{cases} 0, & -l-\epsilon-\epsilon_1 < x < -l-\epsilon \\ B(x), & -l-\epsilon \le x \le l+\epsilon \\ 0, & l+\epsilon < x < l+\epsilon+\epsilon_1 \end{cases} \tag{3.6}$$

Using a sequence of bells $\bar{B}_n$, we can decompose the source function into local overlapping pieces $f(x) = \sum_{s=1}^{S} B_n(x) f(x)$. The solution to the linear problem (2.1) in the whole domain can be viewed as a superposition of the corresponding local solutions $u = \sum_{s=1}^{S} u^n$, $Hu^s = \tilde{f}^s$, $\tilde{f}^s = B_s f$. In contrast with a piecewise extention (3.3), each local source function $B_s f$ is highly smooth at the joint points $x = -\epsilon, l+\epsilon$, so that one of the spectral methods can be applied for solving the local problems in the subdomains.

We illustrate the accuracy of both PDD and SODD local approaches for the inner layer solution $u(x) = \frac{1}{2}(tanh 20x+1)$, $x \in (-1,1)$, which has a steep profile near the origin. This solution satisfies the equation (3.1) with an appropriate source function $f(x)$ and Dirichlet boundary conditions. Table 1 presents the maximum numerical error at the parameters $S = 2, N/l = 128$ and several values of the overlapping interval $\epsilon_T = \epsilon + \epsilon_1$.

The error becomes smaller as the parameter λ increases. At the same range of the overlapping ϵ_T the PDD method ensure higher accuracy, than the SOOD method. When ϵ_T contracts, the precision of the SODD method deteriorates much faster in comparisone with the PDD approach.

The last column gives the error, obtained by the ODD method with a piecewise partitioning of the source function (3.3) and using finite-differences for a

ϵ_T/l	λ^2	ε_{PDD}	ε_{SODD}	ε_{ODD}
1/2	100	5.2 (-8)	4.1 (-3)	5.0 (-3)
	800	4.4 (-16)	8.4 (-7)	4.7 (-4)
	1600	5.4 (-16)	4.2 (-9)	3.6 (-4)
1/4	400	1.8 (-12)	3.6 (-3)	3.3 (-3)
	1600	1.7 (-13)	2.8 (-5)	3.6 (-4)
	3200	2.7 (-13)	5.6 (-7)	2.5 (-4)
1/8	400	5.4 (-9)	4.2 (-2)	4.1 (-2)
	1600	8.0 (-9)	3.5 (-3)	2.5 (-3)
	3200	8.0 (-9)	4.7 (-4)	2.8 (-4)

Table 1: Comparison of the numerical errors for the PDD, SODD and ODD methods

space discretization. It is seen that the accuracy of this method is substentially inferior to both spectral algorithms.

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