A DOMAIN DECOMPOSITION METHOD FOR THE POLAR FACTORIZATION OF VECTOR FIELDS

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ABSTRACT. We recall results on the convergence of an overlapping domain decomposition method for the Polar Factorization of vector valued functions and the numerical algorithm obtained via a consistent discretization of this problem. We present numerical results for this method.

1. INTRODUCTION

We are interested in the following non linear problem :

Let $\Omega \subset \mathbb{R}^d$ be a convex bounded open set, let $u \in \mathcal{L}^p(\Omega, \mathbb{R}^d)$, find the unique rearrangement of u on Ω in the form $\nabla\psi$, ψ convex. This problem (occurring in meteorological modeling [7]) has been extensively studied by Brenier [4] as the Polar Factorization of vector fields and generalizes the concept of monotone rearrangement of real functions [5].

We proposed a Domain Decomposition method to solve this problem :

According to a geometrical condition we call the "convex overlapping" principle, we decompose Ω in two convex overlapping subdomains and alternately compute on each subdomain the Polar Factorization of the rearrangements given by the previous steps. This strategy provides an iterative procedure (u provides the initialization) which reduces our problem to a sequence of subproblems. We established the convergence of this Schwarz alternating method for the Polar Factorization. The proof (detailed in [2]) does not rely on results for Elliptic non-linear equations obtained by Lions [6] but rather on continuity properties of the Polar Factorization [4].

A discretization of the Polar Factorization problem was introduced in [3] under the form of an assignment problem (consistency of this problem is proved in [2]). In the spirit of the continuous algorithm, a suitable decomposition is defined and an analogous domain decomposition method has been proposed to solve the discrete version of the Polar Factorization. As a consequence of the degeneracy produced by the discretization, the discrete algorithm is not always convergent. However we prove in [2] that the sequence of outputs produced by the algorithm for each level of discretization tends to $\nabla\psi$ as we refine the discretization. We generalize this method to multi-subdomains decomposition and obtain a numerical algorithm.

Section 2 rapidly describe the Polar Factorization and its discretization, and presents the algorithm. Section 3 contains the numerical study.

2. POLAR FACTORIZATION THEORY

2.1. The Polar Factorization and its discretization.
Let $\Omega \subset \mathbb{R}^d$ be a convex bounded open set, $u \in \mathcal{L}^p(\Omega, \mathbb{R}^d)$ non degenerate in the sense that $mes(u^{-1}(E)) = 0$ for each Lebesgue negligible subset E. Then, there is a unique Polar Factorization

$$(1) \qquad\qquad u = \nabla \psi \circ s$$

with $\psi \in \mathcal{W}^{1,p}(\Omega, \mathbb{R})$ convex and $s \in \mathcal{S}$, the set of measure preserving mappings $(\mathcal{S} = \{s, \int_\Omega f(s(x))\, dx = \int_\Omega f(x)\, dx, \; \forall f \in \mathcal{C}(\overline{\Omega})\})$.
The domain Ω is mapped into itself by s and $\nabla \psi$ is a rearrangement of u on Ω.
Moreover s is characterized by the following optimization problem :

$$(2) \qquad\qquad \int_\Omega u(x).s(x)\, dx = \max_{t \in \mathcal{S}} \int_\Omega u(x).t(x)\, dx$$

See [4] for more on Polar Factorization.

Following [3] we introduce (x_i^n) $i = 1..n(= N^2)$ the lattice points of a regular $(N \times N)$ square grid discretizing a square Ω and (u_i^n) a discretization of u on this grid. We define $\mathcal{S}^n$ to be the set of permutations of the first n integers. Restricting the measure preserving mappings to the class of permutations of identical cells of center (x_i^n) covering Ω and approximating u by (u^n), the piecewise constant vector field of value u_i^n on the corresponding cell, we obtain from (2) the following assignment problem :
Find $\sigma \in \mathcal{S}^n$ such that

$$(3) \qquad\qquad \sum_i x_i^n.u_{\sigma(i)}^n = \max_{\xi \in S^n} \sum_i x_i^n.u_{\xi(i)}^n.$$

This is called the discrete Polar Factorization problem and $(u_{\sigma(i)}^n)$ is the discrete optimal rearrangement of (u_i^n). It converges with n to $\nabla \psi$ in $\mathcal{L}^1$ [2].

2.2. Domain decomposition technique for the Polar Factorization.
The convergence of a Schwarz alternating method with overlapping subdomains for problem (1) has been established under the geometrical "convex overlapping" condition (i.e. "every segment joining the subdomains has a non-zero length intersection with the overlap"). An analogous domain decomposition algorithm for problem 3 has been proposed and studied : Set

$$\omega_1 \cup \omega_{12} \cup \omega_2 = \Omega, \; \omega_1 \cup \omega_{12} = \Omega_1, \; \omega_{12} \cup \omega_2 = \Omega_2,$$

a domain decomposition of Ω (see figure 1) verifying the "convex overlapping" condition and note O_j^n the set of indices i such that $x_i^n \in \Omega_j$ for $j = 1, 2$.
Note that $O_1^n \cup O_2^n = O^n$, the set of all indices and $O_1^n \cap O_2^n$ is non empty.
$\mathcal{S}_j^n$ is the set of permutations of indices in O_j^n $(j = 1, 2)$.

We alternately rearrange on each subset of points defined by O_1^n and O_2^n by solving subproblems of form (3). We proceed as follow :

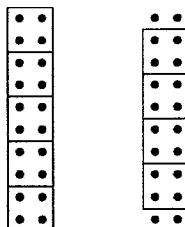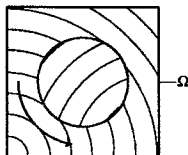Initialize $v_i^n = u_i^n$ for all i.

FIGURE 3. Decomposition of the subproblems.

3. TESTING THE ALGORITHM

In order to test the algorithm we take the identity field on $[0, L] \times [0, L]$, gradient of the simple convex surface $\frac{x^2+y^2}{2}$ ((x,y) are the space coordinate), and perturb it by rotating the values of this field on a disk in the domain (see figure 4). Then we

FIGURE 4. Field lines of Identity perturbed by the $\frac{\pi}{2}$ rotation of a disk.

discretize it on a regular grid and input it to our algorithm. The monotony of the vector field (convexity for the potential) will be locally violated at the boundary of the disk but not inside the disk. This problem is a difficult one for the basic steps of our algorithm are local and still it will have to globally rotate the disk to recover the solution.

We present results obtained for different angles of rotation and compare them to those obtained with a routine solving exactly the general assignment problem (we will refer to this routine as AP and to ours as DD). We use a 10×10 grid as the AP routine is a lot time consuming.

Our algorithm can be seen as a successive displacement on the grid of "particles" (by the local permutations) on which the vector field values are attached. So we represent by scaled arrows the move of the particles from the initial to the final state. For each test, we use the same scaling for the arrows when representing the solutions of the different algorithms. We expect to recover the inverse rotation to the initial perturbation which would restore the monotony of the vector field. In order to avoid the algorithm to be stuck in a cycle of permutations for which the global criterium is non increasing (our stop test), we choose heuristically to permute randomly the values of the vector field on the grid before the application of the algorithm. The results for different angles of rotation are displayed for AP and DD in figure 5. Except for a slight difference in the π and $\frac{\pi}{4}$ case, the results are identical. The time required on a workstation for these simulations go from 0.25 to 0.3s for DD against 2 to 11s for AP (the best algorithms for the assignment problem are in $O(n^3)$ at the worst and $O(n^2 log n)$ on average [1]).
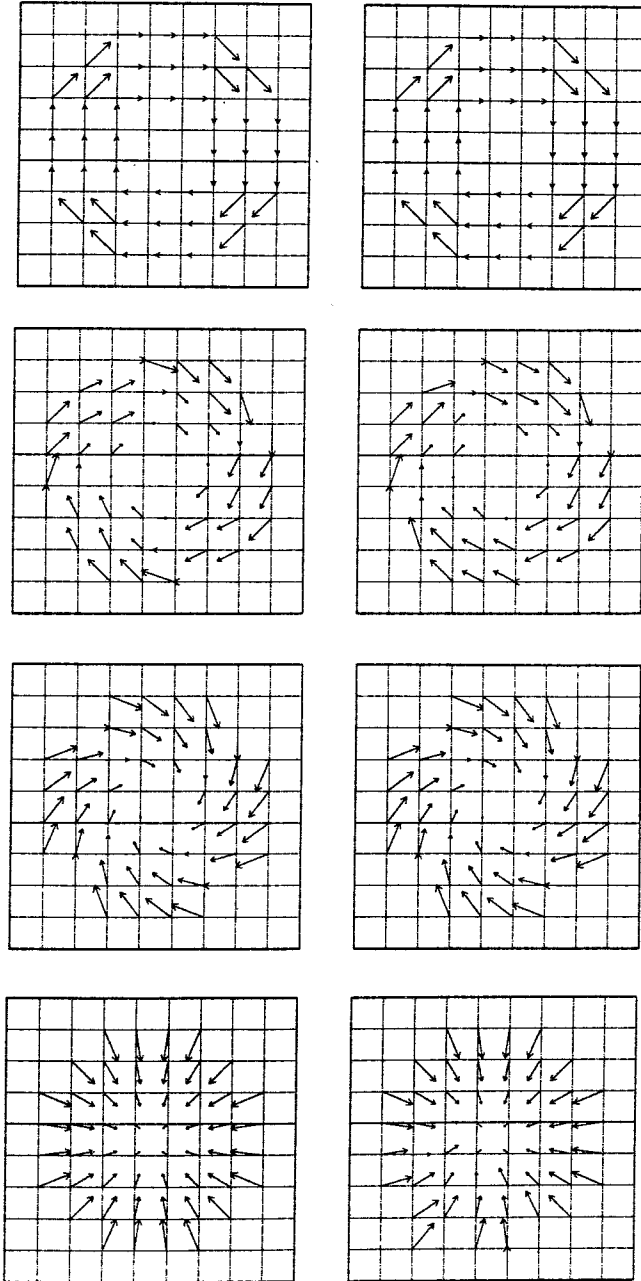
FIGURE 5. AP (left) and DD (right) solutions for different angles of rotation : from top to bottom $\frac{\pi}{8}$, $\frac{\pi}{4}$, $\frac{\pi}{2}$, π.
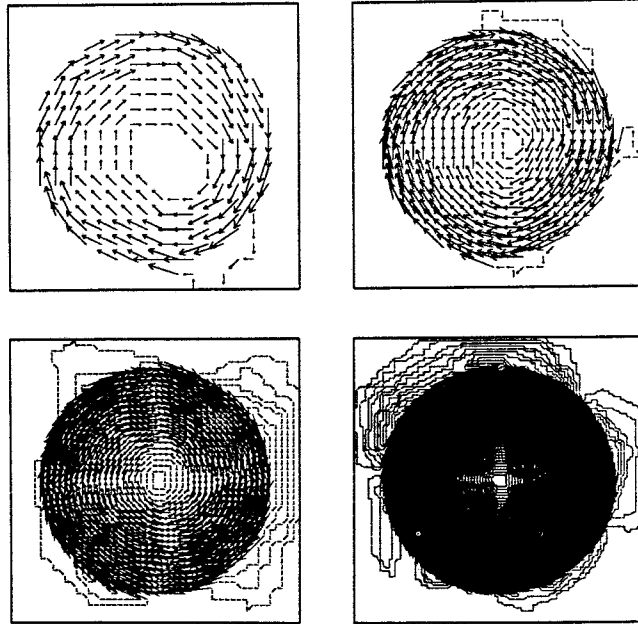
FIGURE 6. DD solutions for a $\frac{\pi}{8}$ angle of rotation and different values of n : 400, 900, 2500, 10000

We concentrate on the $\frac{\pi}{8}$ case and increase progressively the number of points of the discretization. It provides a numerical tool to verify the consistency of the algorithm studied in [2] and to compute an empirical cost. The results are displayed in figure 6. We see that the move of "particles" approximate better and better the rotation of a disk in spite of some parasitic movements. The algorithm requires an average of $Cn^{1\,6}$ operations which therefore confirm the $O(n^2)$ cost.

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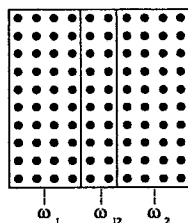*E-mail address*: brenier@dmi.ens.fr

FIGURE 1. Ω decomposed and the points (x_i^n) of its discretization.

Step a
Solve $\sum_{i \in O_1^n} x_i^n . v_{\sigma^n(i)}^n = \max_{\xi^n \in S_1^n} \sum_{i \in O_1^n} x_i^n . v_{\xi^n(i)}^n$.
Update $v_i^n = v_{\sigma^n(i)}^n$ *for* $i \in O_1^n$.

Step b
Solve $\sum_{i \in O_2^n} x_i^n . v_{\sigma^n(i)}^n = \max_{\xi \in S_2^n} \sum_{i \in O_2^n} x_i^n . v_{\xi^n(i)}^n$.
Update $v_i^n = v_{\sigma^n(i)}^n$ *for* $i \in O_2^n$.

The global criterium $(\sum_{i \in O^n} x_i^n . v_i^n)$ is increasing. We iterate step a and b until that increase stops. Although convergence of the iterative method is not guaranteed for fixed n, we established the convergence with n of the discrete vector field (v_i^n) to the rearrangement $\nabla \psi$ of u in $\mathcal{L}^1$. Domain decomposition for the Polar Factorization of vector fields is detailed in [2].

2.3. The multi-subdomains algorithm. We decompose a $n = N \times N$ points grid in two steps. We solve subproblems on a decomposition of $2 \times N$ vertical subproblems that are alternately shifted horizontally (figure 2) until the global criterium is stationnary. Each of these subproblems is solved by the same technique
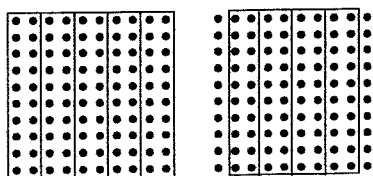

FIGURE 2. Decompositions of Ω.

(see figure 3) thus reducing the problem to a sequence of 2×2 subproblems which can be treated explicitly. We tested this algorithm with satisfactory (though not exact) results. We conjecture and it has been observed numerically that the necessary number of iterations to converge for a grid which largest dimension is N is of the same order N. Then we obtain a $O(n^2)$ cost for a double decomposition such as the one described here. Note that all the subproblems which domains do not overlap can be solved in parallel. This reduces the cost of our method to $O(n)$ on a massively parallel computer.