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An Additive Schwarz Algorithm for Piecewise Hermite Bicubic Orthogonal Spline Collocation

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ABSTRACT. An overlapping domain decomposition, additive Schwarz, conjugate gradient method is presented for the solution of the linear systems which arise when orthogonal spline collocation with piecewise Hermite bicubics is applied to the Dirichlet problem for Poisson's equation on a rectangle.

1. Introduction

Consider the Dirichlet problem for Poisson's equation

$$-\Delta u = f(x,y) \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where $\Omega = (0,1) \times (0,1)$ and $\partial\Omega$ is its boundary.

Let $h = 1/N_h$ and let $\{t_k^h\}_{k=0}^{N_h}$ be a partition of $[0,1]$ such that $t_k^h = kh$. In the following, this partition is referred to as the fine grid partition of $[0,1]$. Let $\mathcal{M}_h(0,1)$ be the space of piecewise Hermite cubics on $[0,1]$ defined by

$$\mathcal{M}_h(0,1) = \{v \in C^1[0,1] : v|_{[t_{k-1}^h, t_k^h]} \in P_3, k = 1, \dots, N_h\},$$

where P_3 denotes the set of all polynomials of degree ≤ 3, and set

$$V^h = \mathcal{M}_h^0(0,1) \otimes \mathcal{M}_h^0(0,1), \tag{1.2}$$

where

$$\mathcal{M}_h^0(0,1) = \{v \in \mathcal{M}_h(0,1) : v(0) = v(1) = 0\}.$$

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Let $\{\xi_l^h\}_{l=1}^{2N_h}$ be the set of Gauss points in $(0,1)$, where

$$\xi_{2k-1}^h = t_{k-1}^h + h(3-\sqrt{3})/6, \qquad \xi_{2k}^h = t_{k-1}^h + h(3+\sqrt{3})/6, \qquad k = 1,\dots,N_h,$$

and let $\mathcal{G}^h$ be the set of Gauss points in Ω, namely,

$$\mathcal{G}^h = \left\{(x,y) : x,y \in \{\xi_l^h\}_{l=1}^{2N_h}\right\}. \tag{1.3}$$

The fine grid piecewise Hermite bicubic orthogonal spline collocation (OSC) problem corresponding to (1.1) consists in finding $u_h \in V^h$ such that

$$-\Delta u_h(\xi) = f(\xi), \qquad \xi \in \mathcal{G}^h. \tag{1.4}$$

It was shown in [**6**] that there is a unique solution of (1.4) and that $\|u - u_h\|_{H^j(\Omega)} = O(h^{4-j})$, $j = 0, 1$, if $u \in H^{6-j}(\Omega)$.

The OSC solution u_h satisfying (1.4) can be computed in $O(N_h^2 \log N_h)$ arithmetic operations by the direct fast Fourier transform (FFT) solver of [**1**], which is well-suited for parallel computation. In the present paper, we discuss a parallel iterative domain decomposition method for solving (1.4) which is based on dividing Ω into a number of overlapping squares. Our method is an additive Schwarz (AS) conjugate gradient (CG) algorithm with coarse grid and arbitrary overlap, and it involves solving independent fine grid OSC subproblems at each iteration step. Each of these subproblems can be solved by the FFT solver of [**1**]. The presentation and analysis of the OSC AS CG method are based on [**3**], where a general framework of AS methods for solving a variational equation was developed. It should be noted that in [**2**] Fourier analysis was used to investigate the convergence of the classical Schwarz alternating algorithm with two and three overlapping subrectangles for the solution of (1.4).

The paper is organized as follows. The description and analysis of the OSC AS CG method is given in Section 2 and its implementation is discussed in Section 3. Results of preliminary numerical experiments are reported in Section 4.

2. Additive Schwarz Conjugate Gradient Method

Following [**3**], we first decompose Ω into overlapping squares and introduce corresponding spaces, bilinear forms, and operators, all of which are essential for the variational formulation of AS methods.

Let $H = 1/N_H$ be an integer multiple of h and let $\{t_i^H\}_{i=0}^{N_H}$ be the "coarse grid" partition of $[0,1]$ such that $t_i^H = iH \in \{t_k^h\}_{k=0}^{N_h}$. Assume that each coarse grid square $\Omega_{ij} = (t_{i-1}^H, t_i^H) \times (t_{j-1}^H, t_j^H)$, $1 \le i,j \le N_H$, is extended to a square $\Omega'_{ij} \subset \Omega$ so that $\partial\Omega'_{ij}$ does not cut through any fine grid square $(t_{k-1}^h, t_k^h) \times (t_{l-1}^h, t_l^h)$, $1 \le k,l \le N_h$, and so that $\operatorname{dist}(\partial\Omega_{ij} \setminus \partial\Omega, \partial\Omega'_{ij} \setminus \partial\Omega) \ge \delta > 0$, where $\delta \le H$.

Let $\overline{\Omega}_h$ be the set of all fine grid nodal points in $\overline{\Omega}$, that is,

$$\overline{\Omega}_h = \left\{(x,y) : x,y \in \{t_k^h\}_{k=0}^{N_h}\right\}.$$

With each Ω'_{ij}, we associate the space

$$V_{ij} = \{v \in V^h : v|_{\partial\Omega'_{ij}} = 0, v = v_x = v_y = v_{xy} = 0 \text{ at all } (x,y) \in \overline{\Omega}_h \setminus \overline{\Omega}'_{ij}\}.$$

It is important to note that if $v \in V_{ij}$, then, in general, $v \neq 0$ in a narrow strip of width h around Ω'_{ij}. We also introduce the space $V_{00} = V^H$, where V^H is defined by (1.2) with h replaced by H, and note that

$$V^h = V_{00} + \sum_{i,j=1}^{N_H} V_{ij}.$$

Let b and b_{ij}, $1 \leq i,j \leq N_H$, and b_{00} be the bilinear forms on $V^h \times V^h$, $V_{ij} \times V_{ij}$, $1 \leq i,j \leq N_H$, and $V_{00} \times V_{00}$ defined respectively by

$$b(v,w) = \langle -\Delta v, w\rangle_{\mathcal{G}^h}, \qquad \langle z, w\rangle_{\mathcal{G}^h} = (h^2/4) \sum_{\xi \in \mathcal{G}^h} (zw)(\xi),$$

$$b_{ij}(v,w) = \langle -\Delta v, w\rangle_{\mathcal{G}_{ij}}, \qquad \langle z, w\rangle_{\mathcal{G}_{ij}} = (h^2/4) \sum_{\xi \in \mathcal{G}_{ij}} (zw)(\xi), \qquad 1 \leq i,j \leq N_H,$$

$$b_{00}(v,w) = \langle -\Delta v, w\rangle_{\mathcal{G}_{00}}, \qquad \langle z, w\rangle_{\mathcal{G}_{00}} = (H^2/4) \sum_{\xi \in \mathcal{G}_{00}} (zw)(\xi),$$

where $\mathcal{G}_{ij} = \mathcal{G}^h \cap \Omega'_{ij}$, $1 \leq i,j \leq N_H$, $\mathcal{G}_{00} = \mathcal{G}^H$ and $\mathcal{G}^H$ is defined by (1.3) with h replaced by H.

In the following, C and C_i, $i = 1, 2$, denote generic positive constants that are independent of h, H, and δ.

Equation (3.2) in [**4**] and equations (2.7) and (2.8) in [**6**] imply that b is a symmetric bilinear form on $V^h \times V^h$ and that

$$C_1 \|v\|^2_{H^1(\Omega)} \leq b(v,v) \leq C_2 \|v\|^2_{H^1(\Omega)}, \qquad v \in V^h.$$

Hence V^h can be regarded as a Hilbert space with $b(\cdot,\cdot)$ as an inner product. It also follows that b_{ij} is a symmetric positive definite bilinear form on $V_{ij} \times V_{ij}$.

Let T_{ij}, $1 \leq i,j \leq N_H$, and $i = j = 0$, be the linear operator from V^h into V_{ij} such that

$$b_{ij}(T_{ij}v, w) = b(v,w), \qquad w \in V_{ij},$$

and let T be the linear operator from V^h into V^h given by

$$T = T_{00} + \sum_{i,j=1}^{N_H} T_{ij}.$$

The following theorem is the main theoretical result presented in this paper.

THEOREM 2.1. *The operator T is self-adjoint on V^h with respect to the inner product $b(\cdot,\cdot)$. Moreover,*

$$C_1 (1 + H/\delta)^{-1} b(v,v) \leq b(Tv,v) \leq C_2\, b(v,v), \qquad v \in V^h. \tag{2.1}$$

The proof of Theorem 2.1, which will be given in detail elsewhere, follows from the three key inequalities in the general theory of AS methods [**3**]. These inequalities can be verified, in turn, using the following, easy to prove, lemmas.

LEMMA 2.1. *Let Ω_γ be the inside boundary strip in Ω of width γ. Then*

$$\|v\|^2_{L^2(\Omega_\gamma)} \leq C\,\gamma^2\,\|\nabla v\|^2_{L^2(\Omega)}, \qquad v \in H^1_0(\Omega).$$

LEMMA 2.2. *Let $\tilde{V}^h = \mathcal{M}_h(0,1) \otimes \mathcal{M}_h(0,1)$, and let*

$$\|v\|^2_{L^2_h(\Omega)} = h^2 \sum_{(x,y)\in\overline{\Omega}_h} \left[v^2 + (hv_x)^2 + (hv_y)^2 + (h^2 v_{xy})^2\right](x,y), \qquad v \in \tilde{V}^h.$$

Then

$$C_1\,\|v\|_{L^2_h(\Omega)} \leq \|v\|_{L^2(\Omega)} \leq C_2\,\|v\|_{L^2_h(\Omega)}, \qquad v \in \tilde{V}^h.$$

We are now in a position to describe the OSC AS CG method of this paper. Let $u_h \in V^h$ be the solution of (1.4) and let $g_h = Tu_h$. (It is important to note that it is possible to compute g_h even though u_h is unknown.) Consider the problem of finding $v_h \in V^h$ such that

$$Tv_h = g_h. \tag{2.2}$$

Since T is an invertible operator, it follows that (1.4) and (2.2) have the same unique solution. The AS method for (1.4) consists in solving (2.2), rather than (1.4), by an appropriate iterative method. Theorem 2.1 implies that the CG method in the Hilbert space V^h, with the inner product $b(\cdot,\cdot)$, can be applied to (2.2). It follows from (2.1) and well-known results on the CG method (see, for example, [**5**]) that the rate of convergence of the proposed AS CG algorithm will depend on $\sqrt{H/\delta}$ but it will be independent of h. Moreover, if $\mathrm{dist}(\partial\Omega_{ij} \setminus \partial\Omega, \partial\Omega'_{ij} \setminus \partial\Omega)$, $1 \leq i,j \leq N_H$, is proportional to H, then the the rate of convergence will also be independent of the overlap.

3. Implementation

An implementation of the AS CG method requires, among other things, the computation of

$$w_{ij} = T_{ij}v, \qquad v \in V^h, \tag{3.1}$$

$1 \leq i,j \leq N_H$, and $i = j = 0$. In the following, we show that computing w_{ij} for $1 \leq i,j \leq N_H$ amounts to solving a fine grid OSC subproblem on Ω'_{ij} with a modified right-hand side. To this end, for $0 \leq k \leq N_h$, let $\phi_{2k}, \phi_{2k+1} \in \mathcal{M}_h(0,1)$ be such that

(3.2)

$$\phi_{2k}(t^h_l) = \delta_{k,l}, \quad \phi'_{2k}(t^h_l) = 0, \quad \phi_{2k+1}(t^h_l) = 0, \quad \phi'_{2k+1}(t^h_l) = \delta_{k,l}, \quad 0 \leq l \leq N_h,$$

where $\delta_{k,l}$ is the Kronecker delta. Assume that $\Omega'_{ij} = (t^h_{k_1}, t^h_{k_2}) \times (t^h_{l_1}, t^h_{l_2})$. For $2k_1 + 1 \leq m \leq 2k_2$, let $\theta^x_m \in \mathcal{M}^0_h(t^h_{k_1}, t^h_{k_2})$ be such that

$$\theta^x_m(\xi_n) = \delta_{m,n}, \qquad 2k_1 + 1 \leq n \leq 2k_2. \tag{3.3}$$

(For the existence and uniqueness of θ^x_m see [4] and [6].) Similarly, for $2l_1 + 1 \leq n \leq 2l_2$, let $\theta^y_n \in \mathcal{M}^0_h(t^h_{l_1}, t^h_{l_2})$ be such that

$$\theta^y_n(\xi_m) = \delta_{n,m}, \qquad 2l_1 + 1 \leq m \leq 2l_2. \tag{3.4}$$

Clearly, for m and n such that $2k_1 + 1 \leq m \leq 2k_2$ and $2l_1 + 1 \leq n \leq 2l_2$,

$$\begin{aligned} \theta^x_m(x) &= \sum_{k \in S_{k_1,k_2}} \alpha_{m,k}\, \phi_k(x), \quad x \in [t^h_{k_1}, t^h_{k_2}], \\ \theta^y_n(y) &= \sum_{l \in S_{l_1,l_2}} \beta_{n,l}\, \phi_l(y), \quad y \in [t^h_{l_1}, t^h_{l_2}], \end{aligned} \tag{3.5}$$

where

$$S_{i_1,i_2} = \{2i_1 + 1, \ldots, 2i_2 - 1, 2i_2 + 1\}.$$

Using (3.3)-(3.5), it is shown in Appendix A that for $1 \leq i, j \leq N_H$, w_{ij} of (3.1) satisfies

$$\Delta w_{ij}(\xi) = \Delta v(\xi) + z_{ij}(\xi), \qquad \xi \in \mathcal{G}_{ij}, \tag{3.6}$$

where

$$\begin{aligned} z_{ij}(\xi_m, \xi_n) = \\ &\sum_{r=1}^{2}\sum_{s=1}^{2} \alpha_{m,2k_r+1}\beta_{n,2l_s+1} \sum_{k=k'_r-1}^{k'_r} \sum_{l=l'_s-1}^{l'_s} \Delta v(\xi_k, \xi_l)\phi_{2k_r+1}(\xi_k)\phi_{2l_s+1}(\xi_l) \\ &+ \sum_{r=1}^{2} \alpha_{m,2k_r+1} \sum_{k=k'_r-1}^{k'_r} \Delta v(\xi_k, \xi_n)\phi_{2k_r+1}(\xi_k) \\ &+ \sum_{s=1}^{2} \beta_{n,2l_s+1} \sum_{l=l'_s-1}^{l'_s} \Delta v(\xi_m, \xi_l)\phi_{2l_s+1}(\xi_l), \end{aligned}$$

and where $j'_i = 2(j_i + i - 1)$. If $k_1, l_1 = 0$ or $k_2, l_2 = N$, then the corresponding sums with respect to k and l are absent in z_{ij}. It should be noted that the modification z_{ij} on the right-hand side of (3.6) involves values of Δv at the Gauss points in the strip of width h around Ω'_{ij}. This is a consequence of the fact that functions in V_{ij} are different from zero in such a strip. For fixed $1 \leq i, j \leq N_H$, the coefficients α and β appearing in z_{ij} can be computed once and for all before starting the iteration process in $O(\mu_{ij})$ operations, where μ_{ij} is the number of unknowns in w_{ij}. Each fine grid OSC subproblem (3.6) can be solved, in turn, by the FFT algorithm of [1] at a cost of $O(\mu_{ij} \log \mu_{ij})$ operations. The computation of w_{00} of (3.1) is equivalent to solving the coarse grid OSC problem with stepsize H. As in the case of fine grid OSC subproblems, the solution of the coarse grid

OSC problem involves certain coefficients that can be computed before starting the iteration process at a cost of $O(\mu_{00})$ operations, where $\mu_{00} = O(H^{-2})$ is the number of unknowns in w_{00}. The setting up of the right-hand side in the coarse grid OSC problem requires $O(\mu_{00}^{3/2} + \mu)$ operations, where $\mu = O(h^{-2})$ is the number of unknowns in u_h. Hence, if H is at least as large as $O(h^{2/3})$, then the cost of one iteration in the OSC AS CG method is $O(\mu \log \mu)$ operations.

It should be noted that, if $u_h^{(0)}$ is a starting approximation in the OSC AS CG method, then there is no problem with computing $Tu_h^{(0)} - g_h = T(u_h^{(0)} - u_h)$ in the first step of the iteration process since $\Delta u_h(\xi)$ is known, by (1.4), at each Gauss point $\xi \in \mathcal{G}^h$.

4. Numerical Results

The purpose of the numerical experiments was to examine the rate of convergence of the OSC AS CG method. Problem (1.1) with the exact solution

$$u(x, y) = 10x(1 - x)y(1 - y)$$

was discretized using OSC with piecewise Hermite bicubics, resulting in (1.4) with the solution $u_h = u$. The unit square Ω was divided into two overlapping subrectangles $\Omega'_{11} = (0, 2/3) \times (0, 1)$ and $\Omega'_{21} = (1/3, 1) \times (0, 1)$. For different values of the stepsize $h = 1/N_h$, the AS CG method with the starting approximation $u_h^{(0)} = 0$ was used to compute successive iterates $u_h^{(n)}$ converging to u_h. Since the local Hermite basis functions of (3.2) were used to represent u_h, the number of unknown parameters in u_h was $4N_h^2$. Table 1 presents the discrete maximum norm errors

$$\varepsilon_{N_h}^{(n)} = \max_{(x,y)\in\overline{\Omega}_h} |(u_h - u_h^{(n)})(x, y)|.$$

Table 1. Errors $\varepsilon_{N_h}^{(n)}$ in the OSC AS CG method.

n	0	2	4	6	8
$\varepsilon_6^{(n)}$	.6	$.7 \times 10^{-1}$	$.5 \times 10^{-3}$	$.5 \times 10^{-4}$	$.1 \times 10^{-5}$
$\varepsilon_{48}^{(n)}$	.6	$.6 \times 10^{-1}$	$.4 \times 10^{-3}$	$.4 \times 10^{-4}$	$.9 \times 10^{-6}$

As expected, the results obtained show that the rate of convergence of the OSC AS CG method is independent of the fine grid stepsize h.

Out of curiosity we also tested the OSC AS CG method without modification, that is, OSC subproblems (3.6) with $z_{ij} = 0$ were solved at each iteration. The corresponding results are presented in Table 2.

Table 2. Errors $\varepsilon_{N_h}^{(n)}$ in the OSC AS CG method with $z_{ij} = 0$.

n	0	2	4	6	8
$\varepsilon_6^{(n)}$	.6	$.4 \times 10^{-1}$	$.1 \times 10^{-1}$	$.6 \times 10^{-3}$	$.1 \times 10^{-3}$
$\varepsilon_{48}^{(n)}$	.6	$.2 \times 10^{-1}$	$.8 \times 10^{-3}$	$.2 \times 10^{-4}$	$.1 \times 10^{-5}$

On comparing corresponding entries of Tables 1 and 2, we see that both OSC AS CG methods, with and without modification, performed equally well for h sufficiently small.

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Appendix A.

To verify (3.6), we note by (3.1) that

$$\langle \Delta w_{ij}, z\rangle_{\mathcal{G}_{ij}} = \langle \Delta v, z\rangle_{\mathcal{G}^h}, \qquad z \in V_{ij}.$$

Since $\{\phi_k\}_{k\in S_{k_1,k_2}} \otimes \{\phi_l\}_{l\in S_{l_1,l_2}}$ is a basis for V_{ij},

$$\langle \Delta w_{ij}, \phi_k\phi_l\rangle_{\mathcal{G}_{ij}} = \langle \Delta v, \phi_k\phi_l\rangle_{\mathcal{G}^h}, \qquad k \in S_{k_1,k_2}, \quad l \in S_{l_1,l_2}.$$

Therefore, for m, n such that $2k_1 + 1 \le m \le 2k_2$ and $2l_1 + 1 \le n \le 2l_2$,

(A.1)
$$\langle \Delta w_{ij}, \alpha_{m,k}\beta_{n,l}\phi_k\phi_l\rangle_{\mathcal{G}_{ij}} = \langle \Delta v, \alpha_{m,k}\beta_{n,l}\phi_k\phi_l\rangle_{\mathcal{G}_{ij}} + \langle \Delta v, \alpha_{m,k}\beta_{n,l}\phi_k\phi_l\rangle_{\mathcal{G}^h\setminus\mathcal{G}_{ij}},$$

where $\alpha_{m,k}$ and $\beta_{n,l}$ are the coefficients of (3.5). Summing both sides of (A.1) over all $k \in S_{k_1,k_2}$ and all $l \in S_{l_1,l_2}$, noting by (3.5) that

$$\sum_{k\in S_{k_1,k_2}} \sum_{l\in S_{l_1,l_2}} \alpha_{m,k}\beta_{n,l}\phi_k(x)\phi_l(y) = \theta_m^x(x)\theta_n^y(y), \qquad x \in [t_{k_1}^h, t_{k_2}^h], \quad y \in [t_{l_1}^h, t_{l_2}^h],$$

and using (3.3), (3.4), we get

(A.2)
$$\Delta w_{ij}(\xi_m, \xi_n) = \Delta v(\xi_m, \xi_n) + (4/h^2)\langle \Delta v, (\sum_{k\in S_{k_1,k_2}} \alpha_{m,k}\phi_k)(\sum_{l\in S_{l_1,l_2}} \beta_{n,l}\phi_l)\rangle_{\mathcal{G}^h\setminus\mathcal{G}_{ij}}.$$

Finally, the expression for z_{ij} is obtained from the second term on the right-hand side of (A.2).

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