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Uniform Convergence Estimates for Multigrid V-cycle Algorithms with Less than Full Elliptic Regularity

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ABSTRACT. In this paper, we provide uniform estimates for V-cycle algorithms with one smoothing on each level. This theory is based on some elliptic regularity but does not require a smoother interaction hypothesis (sometimes referred to as a strengthened Cauchy Schwarz inequality) assumed in other theories. Thus, it is a natural extension of the full regularity V-cycle estimates provided by Braess and Hackbusch in [2].

1. Introduction.

In this paper, we provide some new convergence estimates for multigrid algorithms. In recent years, there have been many advances in the understanding of multigrid algorithms (e.g., see [1]–[3], [5], [6], [9], [11], [12], [15], [17]–[19]). Two apparently different analytical approaches have been developed. Historically, the first used a two level error recurrence and proceeded to develop estimates for the multilevel case by repeated application (cf. [1], [12]). The second approach expands the fine grid error in a product which reflects the effect of every coarser grid [6]. This approach has been effectively applied even without the use of explicit regularity assumptions for the underlying differential equation.

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As far as we know, the verification of the hypotheses for the first approach mentioned above requires the use of regularity properties for the underlying partial differential equation. The simplest example is that of a second order elliptic operator L on a domain Ω in R^n with Dirichlet boundary conditions. Let u be the solution of

$$\begin{aligned} Lu &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Regularity results of the form

$$\|u\|_{1+\alpha} \leq C\,\|f\|_{-1+\alpha} \tag{1.2}$$

are required to apply the earlier theories. Here $\|\cdot\|_s$ denotes the norm in the Sobolev space of order s (cf. [13]), C is a positive constant and $\alpha \in (0,1]$. The shift α is not arbitrary but depends on the smoothness of the coefficients defining L as well as the boundary $\partial\Omega$. Early works on multigrid showed that the W-cycle multigrid algorithm applied to discretizations of (1.1) converged for any α with a uniform rate of reduction provided that sufficiently many smoothings were used on each grid level. Subsequently, Braess and Hackbush [2] obtained uniform convergence estimates for multigrid V-cycle and W-cycle algorithms (with one smoothing iteration per level) applied to discretizations of (1.1) if the coefficients and the domain were such that (1.2) held with $\alpha = 1$. Estimates for V-cycle algorithms with $\alpha < 1$ and one smoothing per iteration using the first analysis were provided in [3], [11] but only with deterioration depending on the number of levels.

The second type of theory does not necessarily depend on explicit regularity estimates. Earlier results of this type were given in [10], [16] and provided regularity free estimates for the two level case. Extensions to the multilevel case required different techniques and were provided in [6]. The analysis given in [6] gave a second technique for proving multigrid estimates even though the estimates developed there were not independent of the number of levels. The theory showed the crucial relation between the multigrid algorithm and its additive counterpart, e.g. the algorithm described in [7].

Subsequently, new techniques were being developed to analyze additive multigrid algorithms. In particular, [14] provided uniform estimates for the additive method of [7] by using Besov space techniques. Concurrently, [20] proved a uniform upper estimate for the additive algorithm by utilizing special properties of the application of finer grid smoothers on coarser grid functions.

The work of [20] motivated the analysis of [5] which provided uniform estimates for V-cycle algorithms with one smoothing per level in many applications with $\alpha < 1$. It was also shown in [5] that the uniform lower estimate for the additive algorithm given by [14] could be proved provided that some elliptic regularity held, i.e., (1.2) holds with $\alpha > 0$. The analysis in [5] depended on smoothing interaction conditions similar to those proved in [20] for the standard application.

The purpose of this paper is to show that uniform multigrid estimates for the V-cycle algorithm with one smoothing per iteration can be proved under the assumption of (1.2) without the smoothing interaction condition. Thus, the results of this paper generalize those of [2] to the case when (1.2) only holds for $0 < \alpha < 1$. The techniques also extend to the case of local mesh refinements as defined in [7].

The remainder of this paper consists of two sections. In the first, we define the multigrid algorithms following the development in [3]. In Section 3, we present the multigrid analysis.

2. The multigrid algorithms.

In this section, following [3], we describe the symmetric multigrid algorithms. We mention some basic recurrence relations which play major roles in the analysis of the methods and are proved elsewhere. For convenience, the algorithms are developed in an abstract Hilbert space setting. The results most naturally apply to finite element multigrid

algorithms but can also be applied to certain formulations of finite difference multigrid algorithms. These applications are discussed in [3], [6] and [7].

Let us assume that we are given a nested sequence of finite dimensional vector spaces

$$M_1 \subset M_2 \subset \ldots \subset M_j.$$

In addition, let $A(\cdot,\cdot)$ and $(\cdot,\cdot)$ be symmetric positive definite bilinear forms on M_j. The norm corresponding to $(\cdot,\cdot)$ will be denoted $\|\cdot\|$. We shall develop multigrid algorithms for the solution of the problem: Given $f \in M_j$, find $v \in M_j$ satisfying

$$A(v,\phi) = (f,\phi) \qquad \text{for all } \phi \in M_j. \tag{2.1}$$

To define the multigrid algorithms, we shall define auxiliary operators. For $k = 1,\ldots,j$, define the operator $A_k : M_k \mapsto M_k$ by

$$(A_k w,\phi) = A(w,\phi) \qquad \text{for all } \phi \in M_k. \tag{2.2}$$

The operator A_k is clearly symmetric (in both the $A(\cdot,\cdot)$

and $(\cdot,\cdot)$ inner products) and positive definite. Also define the projectors $P_k : M_j \mapsto M_k$ and operators $Q_k : M_j \mapsto M_k$ by

$$A(P_k w,\phi) = A(w,\phi) \qquad \text{for all } \phi \in M_k,$$

and

$$(Q_k w,\phi) = (w,\phi) \qquad \text{for all } \phi \in M_k.$$

To introduce smoothing into the multigrid algorithms, we shall use "generic" smoothing operators $R_k : M_k \mapsto M_k$, for $k = 2,\ldots,j$. Examples of these operators in standard applications are given in [4]. The properties which they satisfy will be discussed in the subsequent analysis. We set $R_1 = A_1^{-1}$, i.e., we solve on the coarsest space. In addition, we set R_k^t to be the adjoint of R_k with respect to the inner product $(\cdot,\cdot)$.

The simplest symmetric V-cycle multigrid operator $B_k : M_k \mapsto M_k$ is defined as follows.

ALGORITHM 2.1. *Set* $B_1 = A_1^{-1}$. *Assume that* B_{k-1} *has been defined and define* $B_k g$ *for* $g \in M_k$ *as follows:*

(1) *Set* $x^1 = R_k^t g$.
(2) *Define* $x^2 = x^1 + B_{k-1}\big[Q_{k-1}(g - A_k x^1)\big]$.
(3) *Set* $B_k g = x^2 + R_k(g - A_k x^2)$.

The standard multigrid algorithm is often defined as an iterative process. It is shown in [5] that the reduction matrix associated with this process is equal to $I - B_j A_j$ where B_j is given by Algorithm 2.1. Because of this relation, results which we later provide for Algorithm 2.1 immediately imply contraction estimates for the corresponding multigrid process. Moreover, the evaluation of the operator Q_{k-1} is avoided in the implementation of Step 2 by the suitable choice of the smoother R_k (cf., [4]).

Let $T_1 = P_1$ and $T_k = R_k A_k P_k$ for $k = 2, \ldots, j$. It was shown in [6] that the error reduction operator associated with Algorithm 2.1 (the standard multigrid algorithm) can be written

$$(I - B_j A_j) = (I - T_j)(I - T_{j-1}) \ldots (I - T_2)(I - T_1) \\ (I - T_2^*) \ldots (I - T_{j-1}^*)(I - T_j^*). \tag{2.3}$$

Here, $T_k^* = R_k^t A_k P_k$ which is the adjoint of T_k with respect to the inner product $A(\cdot,\cdot)$. Identity (2.3) depends upon the assumption that the subspaces are imbedded and that one form is used to define the operators on all levels (see (2.2)). Equality (2.3) holds on the space M_j.

The purpose of this paper is to provide estimates for the multigrid contraction number, i.e., estimates for the quantity δ satisfying the inequality

$$|A((I - B_j A_j)u, u)| \le \delta A(u, u) \qquad \text{for all } u \in M_j. \tag{2.4}$$

Let $E_j = (I - T_j) \ldots (I - T_2)(I - T_1)$. By (2.3), δ is square of the norm of the operator E_j^* or equivalently E_j, i.e.,

$$\delta = \sup_{v \in M_j} \frac{A(E_j v, E_j v)}{A(v, v)}.$$

3. Multigrid analysis

We prove the main theorem of the paper in this section. Uniform convergence estimates for the multigrid V-cycle algorithm will be shown to hold provided that a number of abstract conditions are satisfied. Let λ_k denote the largest eigenvalue of the operator A_k and for any real s, let $\|\cdot\|_s$ denote the norm on M_j defined by

$$\|v\|_s^2 = (A_j^s v, v).$$

Note that

$$\|v\|_1 = A(v,v)^{1/2} \quad \text{and} \quad \|v\|_0 = \|v\|.$$

We require that

(C.1) There exists positive constants $\gamma_1 \le \gamma_2 < 1$ such that

$$\gamma_1 \lambda_{k+1} \le \lambda_k \le \gamma_2 \lambda_{k+1}.$$

(C.2) There exists $\alpha \in (0,1]$ and positive constants C_1 and C_2 such that

$$(A_k^{1-\alpha} Q_k u, Q_k u) \le C_1 \|u\|_{1-\alpha}^2, \tag{3.1}$$

and

$$\|(I - P_{k-1})u\|_{1-\alpha}^2 \le C_2 \lambda_k^{-\alpha} \|u\|_1^2 \qquad \text{for all } u \in M_j. \tag{3.2}$$

The first condition above is a natural assumption which is often satisfied provided that $A(\cdot,\cdot)$ and $(\cdot,\cdot)$ are suitably scaled. In many applications, the operator A_k gives rise to a scale of norms on M_k which are uniformly equivalent to a scale of Sobolev norms. Thus, (3.1) often follows from results concerning boundedness of the projector Q_k with respect to appropriate Sobolev norms. Note that if (3.1) holds for $\alpha = 0$, then it follows for $\alpha \in (0,1)$ by interpolation. Inequality (3.2) is often used in the proof of the so called "regularity and approximation" assumption (cf. [3]).

The first uniform convergence estimates for the V-cycle algorithm were due to [2] and held only in the case of $\alpha = 1$, i.e., if full regularity and approximation or, equivalently, the "approximation property" of [12] was satisfied. Alternatively, uniform convergence estimates have been obtained for the multigrid V-cycle algorithm under additional assumptions concerning the interaction of the multilevel spaces (see, Theorem 3.2 and (3.5) of [5]). This interaction property is sometimes referred to as a strengthened Cauchy Schwarz inequality. Our theorem below shows that uniform estimates for the V-cycle algorithm hold with less than full regularity without any additional assumptions concerning the interaction of the approximation spaces. Thus, it provides a generalization to the result of [2] to the case of $\alpha < 1$.

The final assumptions which we shall impose are on the smoothing operator and are typical. Let $K_k = I - R_k A_k$ and $K_k^* = (I - R_k^t A_k)$. For $k = 2, \dots, j$, we assume that R_k satisfies

(R.1) There is a constant $C_R \ge 1$ which does not depend on k such that the smoothing procedure satisfies

$$\frac{\|u\|^2}{\lambda_k} \le C_R(\bar{R}_k u, u) \qquad \text{for all } u \in M_k. \tag{3.3}$$

Here $\bar{R}_k = (I - K_k^* K_k) A_k^{-1}$. Note that (3.3) holds with $C_R = 1$ for $k = 1$ since $\bar{R}_1 = A_1^{-1}$.

(R.2) There is a constant $\theta < 2$ not depending on k such that

$$A(T_k v, T_k v) \le \theta A(T_k v, v) \qquad \text{for all } v \in M_k. \tag{3.4}$$

Conditions under which Jacobi and Gauss-Seidel smoothing operators satisfy Assumptions (R.1) and (R.2) were provided in [4]. They are valid for many standard applications.

We can now state and prove the theorem for estimating δ in (2.4).

THEOREM 3.1. *Assume that (C.1), (C.2), (R.1) and (R.2) hold. Then (2.4) holds for $\delta < 1$ not depending on j.*

Before proving the above theorem, we prove the following lemma.

LEMMA 3.1. *Assume that (C.1) and (C.2) hold. Then*

$$\sum_{k=1}^{j} \|(P_k - Q_k)u\|_1^2 \le C_1 C_2 \Big(\frac{\gamma_2^{\alpha/2}}{1-\gamma_2^{\alpha/2}} \Big)^2 \|u\|_1^2 \qquad \text{for all } u \in M_j.$$

PROOF. Let u be in M_j. By the definition of λ_k and (C.2),

$$\begin{aligned}
\sum_{k=1}^{j} \|(P_k - Q_k)u\|_1^2 &= \sum_{k=1}^{j} (A_k(P_k - Q_k)u, (P_k - Q_k)u) \\
&\le \sum_{k=1}^{j} \lambda_k^{\alpha} (A_k^{1-\alpha}(P_k - Q_k)u, (P_k - Q_k)u) \\
&= \sum_{k=1}^{j} \lambda_k^{\alpha} (A_k^{1-\alpha} Q_k (P_k - I)u, Q_k(P_k - I)u) \\
&\le C_1 \sum_{k=1}^{j} \lambda_k^{\alpha} \|(I - P_k)u\|_{1-\alpha}^2 .
\end{aligned}$$

Using the identity

$$(I - P_k) = \sum_{l=k+1}^{j} (P_l - P_{l-1})$$

and the Schwarz inequality gives

$$\sum_{k=1}^{j} \|(P_k - Q_k)u\|_1^2$$
$$\le C_1 \sum_{k=1}^{j} \sum_{l=k+1}^{j} \sum_{m=k+1}^{j} \lambda_k^{\alpha} \|(P_l - P_{l-1})u\|_{1-\alpha} \|(P_m - P_{m-1})\|_{1-\alpha} .$$

It is easy to see that $(P_l - P_{l-1})$ is a projector and hence (C.2) implies that

$$\sum_{k=1}^{j} \|(P_k - Q_k)u\|_1^2$$
$$\begin{aligned}
&\le C_1 C_2 \sum_{k=1}^{j} \sum_{l=k+1}^{j} \sum_{m=k+1}^{j} \Big(\frac{\lambda_k^2}{\lambda_l \lambda_m} \Big)^{\alpha/2} \|(P_l - P_{l-1})u\|_1 \|(P_m - P_{m-1})u\|_1 \\
&\le C_1 C_2 \sum_{k=1}^{j} \sum_{l=k+1}^{j} \sum_{m=k+1}^{j} \gamma_2^{(l+m-2k)\alpha/2} \|(P_l - P_{l-1})u\|_1^2 .
\end{aligned}$$

We used (C.1) and the arithmetic geometric mean inequality to derive the last inequality above. Summing over m and subsequently changing the order of summation gives

$$\begin{aligned}\sum_{k=1}^{j}\|(P_k-Q_k)u\|_1^2 &\le C_1C_2\Big(\frac{\gamma_2^{\alpha/2}}{1-\gamma_2^{\alpha/2}}\Big)\sum_{l=2}^{j}\|(P_l-P_{l-1})u\|_1^2\sum_{k=1}^{l-1}\gamma_2^{(l-k)\alpha/2}\\ &\le C_1C_2\Big(\frac{\gamma_2^{\alpha/2}}{1-\gamma_2^{\alpha/2}}\Big)^2\sum_{l=2}^{j}\|(P_l-P_{l-1})u\|_1^2\\ &\le C_1C_2\Big(\frac{\gamma_2^{\alpha/2}}{1-\gamma_2^{\alpha/2}}\Big)^2 A(u,u).\end{aligned}$$

This completes the proof of the lemma.

We next prove Theorem 3.1.

PROOF. Let $E_0=I$ and for $k=1,2,\ldots,$ set

$$E_k=(I-T_k)E_{k-1}. \tag{3.5}$$

Let u be an arbitrary vector of M_j.

To prove (2.4), it suffices to show that

$$A(u,u)\le C[A(u,u)-A(E_ju,E_ju)].$$

In the above and subsequent inequalities, C denotes a generic positive constant which will possibly take on different values in different occurances. It was shown in the proof of Theorem 3.2 of [5] that

$$A(u,u)-A(E_ju,E_ju)=\sum_{k=1}^{j}A((2I-T_k)E_{k-1}u,T_kE_{k-1}u). \tag{3.6}$$

Consequently, it suffices to show that

$$A(u,u)\le C\sum_{k=1}^{j}A((2I-T_k)E_{k-1}u,T_kE_{k-1}u), \tag{3.7}$$

with C independent of j.

It is shown in the proof of Theorem 1 of [6] that

$$\begin{aligned}A(u,u)&=\sum_{k=2}^{j}A\left(E_{k-1}u,(Q_k-Q_{k-1})u\right)\\ &\quad+\sum_{k=2}^{j-1}A\left(T_kE_{k-1}u,(I-Q_k)u\right)+A((2I-T_1)u,T_1u).\end{aligned} \tag{3.8}$$

For the first sum of (3.8), we have

$$
\begin{aligned}
(3.9)\qquad \left|\sum_{k=2}^{j} A\left(E_{k-1}u, (Q_k - Q_{k-1})u\right)\right| &= \left|\sum_{k=2}^{j} (A_k P_k E_{k-1}u, (Q_k - Q_{k-1})u)\right| \\
&\le \sum_{k=2}^{j} \|A_k P_k E_{k-1}u\| \; \|(Q_k - Q_{k-1})u\|.
\end{aligned}
$$

We have not assumed approximation properties for Q_k however it follows from [8] and (3.2) that for $u \in M_j$,

$$\|(I - Q_k)u\| = \inf_{v \in M_k} \|u - v\| \le C\lambda_k^{-1/2} \|u\|_1.$$

Consequently, using (C.1),

$$
\begin{aligned}
\lambda_k^{1/2} \|(Q_k - Q_{k-1})u\| &= \lambda_k^{1/2} \|(I - Q_{k-1})(Q_k - Q_{k-1})u\| \le C \|(Q_k - Q_{k-1})u\|_1 \\
&\le C\Big(\|(P_k - Q_k)u\|_1 + \|(P_{k-1} - Q_{k-1})u\|_1 + \|(P_k - P_{k-1})u\|_1 \Big).
\end{aligned}
$$

Summing and applying Lemma 3.1 gives

$$
\begin{aligned}
(3.10)\qquad \sum_{k=2}^{j} \lambda_k \|(Q_k - Q_{k-1})u\|^2 &\le C\Big[A(u,u) + \sum_{k=2}^{j} A((P_k - P_{k-1})u, u)\Big] \\
&\le CA(u,u).
\end{aligned}
$$

Using the Schwarz inequality, (3.9) and (3.10) yields

$$
\begin{aligned}
&\left|\sum_{k=2}^{j} A(E_{k-1}u,(Q_k - Q_{k-1})u)\right| \\
&\qquad \le CA^{1/2}(u,u)\left(\sum_{k=2}^{j} \lambda_k^{-1}\|A_k P_k E_{k-1}u\|^2\right)^{1/2}.
\end{aligned}
$$

By (R.1),

$$
\begin{aligned}
\lambda_k^{-1}\|A_k P_k E_{k-1}u\|^2 &\le C_R A((I - K_k^* K_k)P_k E_{k-1}u, P_k E_{k-1}u) \\
&= C_R A((2I - T_k)E_{k-1}u, T_k E_{k-1}u).
\end{aligned}
$$

Combining the above inequalities gives

$$
\begin{aligned}
(3.11)\qquad &\left|\sum_{k=2}^{j} A(E_{k-1}u,(Q_k - Q_{k-1})u)\right| \\
&\qquad \le CA^{1/2}(u,u)\left(\sum_{k=2}^{j} A((2I - T_k)E_{k-1}u, T_k E_{k-1}u)\right)^{1/2}.
\end{aligned}
$$

For the second sum of (3.8), by Lemma 3.1,

$$
\begin{aligned}
&\left|\sum_{k=2}^{j-1} A\left(T_k E_{k-1} u, (I-Q_k)u\right)\right| \\
(3.12)\qquad &\leq \left(\sum_{k=2}^{j-1} A(T_k E_{k-1}u, T_k E_{k-1}u)\right)^{1/2} \left(\sum_{k=2}^{j-1} \|(P_k - Q_k)u\|_1^2\right)^{1/2} \\
&\leq C\left(\sum_{k=2}^{j-1} A(T_k E_{k-1}u, T_k E_{k-1}u)\right)^{1/2} A(u,u)^{1/2}.
\end{aligned}
$$

It follows from (R.2) that

$$
A(T_k E_{k-1}u, T_k E_{k-1}u) \leq \frac{\theta}{2-\theta} A((2I - T_k)E_{k-1}u, T_k E_{k-1}u). \tag{3.13}
$$

Combining (3.8), (3.11)–(3.13) gives that $A(u,u)$ is bounded by

$$
CA^{1/2}(u,u)\Big(\sum_{k=2}^{j-1} A((2I-T_k)E_{k-1}u, T_k E_{k-1}u) + A((2I-T_1)u, T_1 u)\Big)^{1/2}
$$

from which (3.7) easily follows. This completes the proof of the theorem.

REMARK 3.1. We considered the case of one smoothing in Algorithm 2.1 since it is the most interesting algorithm. Uniform convergence rates for algorithms involving more smoothings easily follow from the above analysis (see [6]) but do not guarantee any improvement in convergence over the case of one smoothing. Similar estimates hold in the case of algorithms with more than one correction step, e.g., the W-cycle algorithm.

REMARK 3.2. The above techniques easily extend to provide uniform estimates in the case of locally refined meshs as developed for the second order application in [7]. Rather than try to extend the abstract formalism of this section, we will restrict our discussion to the local mesh refinement application of [7]. The assumptions (C.1) and (C.2) are only required to hold for the larger spaces $\bar{M}_k$ (and analogous operators $\bar{A}_k$, $\bar{Q}_k$, and $\bar{P}_k$) defined by uniform refinement on the entire domain. Note that the analogous version of Lemma 3.1 holds on these larger spaces. Smoothing subspaces $\tilde{M}_k$ are defined to be the functions in $\bar{M}_k$ with support in the k'th refinement region (as in [7]). The k'th multigrid space is defined to be the sum of the smoothing subspaces up to k. In the local refinement application, $R_k : M_k \mapsto \tilde{M}_k$ and is such that $R_k = R_k\tilde{Q}_k$ where $\tilde{Q}_k$ denotes the $(\cdot,\cdot)$ projector onto $\tilde{M}_k$. One replaces Q_k in (3.8) by the operator $\mathcal{Q}_k$ used in the proof of Theorem 5.1 of [5]. The remainder of the proof of Theorem 3.1 follows easily from the inequality

$$
\|(I-\mathcal{Q}_k)u\| \leq C\left\|(I-\bar{Q}_k)u\right\| \qquad \text{for all } u \in M_j.
$$

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