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A Three-field Domain Decomposition Method

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Abstract. We present a three-field formulation for second order linear elliptic problems well suited for parallel implementation. Great generality is allowed on the choice of discretizations: different methods can be used from one subdomain to another. A preconditioner is also presented, very useful to deal with non symmetric problems and unstructured subdomain decompositions.

1. Introduction

The aim of this paper is to present a three-field formulation for linear elliptic problems which is particularly well suited for domain decomposition methods. The formulation is inspired by the hybrid formulation of Tong [10] for elasticity problems, the main difference being that we work here at the macro-element ($\equiv$subdomain) level instead of working at the element level. The effect of this is that we obtain a new formulation of the continuous problem which can then be discretized in many different ways, including the possibility of using different methods (or the same method with different meshes) from one subdomain to another. To perform this, proper compatibility conditions, often not straightforward, have to be satisfied. We circumvent this problem by introducing a stabilized formulation (inspired by [3]) which essentially consists in adding suitable terms that do not alter the consistency of the scheme. In this way, we get a fairly general framework in which most of the domain decomposition methods using non overlapping subdomains can be reinterpreted. We also propose a preconditioner that seems particularly appealing for dealing with non symmetric problems and unstructured decompositions.

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2. The three-field formulation

Let us consider, for the sake of simplicity, a polygonal domain $\Omega \subset \mathbb{R}^2$ split into a finite number of polygonal subdomains Ω_k $(k = 1, .., K)$. Let

$$\Omega = \overline{\bigcup_k \overset{\circ}{\Omega}_k} \quad ; \quad \Gamma_k = \partial\Omega_k \quad ; \quad \Sigma = \bigcup_k \Gamma_k. \tag{2.1}$$

Let A be a linear elliptic operator of the form

$$Au = \sum_i \{\sum_j (-\frac{\partial}{\partial x_j}(a_{ij}(x)\frac{\partial u}{\partial x_i} + b_j(x)u)) + c_i(x)\frac{\partial u}{\partial x_i}\} + d(x)u. \tag{2.2}$$

We assume that the coefficients a_{ij}, b_j, c_i, d belong to $L^\infty(\Omega)$ and are smooth in each Ω_k, and we consider the bilinear forms associated with A in each Ω_k, that is,

$$\begin{aligned} &for\ u,\, v \in\ H^1(\Omega_k): \\ &a_k(u,\, v) := \int_{\Omega_k} \{\sum_i (\sum_j (a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + b_j u \frac{\partial v}{\partial x_j}) + c_i \frac{\partial u}{\partial x_i} v) + duv\}\, dx. \end{aligned} \tag{2.3}$$

We also set, for $u,\, v \in \prod_k H^1(\Omega_k)$

$$a(u,\, v) := \sum_k a_k(u,\, v); \tag{2.4}$$

for the sake of simplicity we also assume that there exists a constant $\alpha > 0$ such that

$$a(v,\, v) \geq \alpha ||v||^2_{H^1(\Omega)} \qquad \forall v \in H^1_0(\Omega). \tag{2.5}$$

From now on we are going to use the following notation: $(.,.)$ will be the usual inner product in $L^2(\Omega)$; for $k = 1, ...K$, $(.,.)_k$ will be the inner product in $L^2(\Omega_k)$ and $< .,. >_k$ will be the inner product in $L^2(\Gamma_k)$ (or, when necessary, the duality pairing between $H^{-\frac{1}{2}}(\Gamma_k)$ and $H^{\frac{1}{2}}(\Gamma_k)$). Similarly, we will use $||.||_s$ for the $H^s(\Omega)$ norm, and $||.||_{s,k}$, $|||.|||_{s,k}$ for the $H^s(\Omega_k)$ and $H^s(\Gamma_k)$ norms respectively $(k = 1, ..., K)$. Let us now introduce the spaces that will be used in our macro-hybrid formulation. For $k = 1, ..., K$ we set

$$V_k := H^1(\Omega_k) \qquad ; \qquad M_k := H^{-\frac{1}{2}}(\Gamma_k). \tag{2.6}$$

We then define

$$V := \prod_k V_k \qquad ; \qquad M := \prod_k M_k, \tag{2.7}$$

and

$$\Phi := \{\varphi \in L^2(\Sigma) \ : \ \exists v \in H_0^1(\Omega) \text{ with } \varphi = v_{|\Sigma}\} \equiv H_0^1(\Omega)_{|\Sigma}, \tag{2.8}$$

with the obvious norms

$$\| v \|_V^2 = \sum_k ||v^k||_{1,k}^2 \qquad (v \in V;\ v = (v^1, ..., v^K)); \tag{2.9}$$

$$\| \mu \|_M^2 = \sum_k |||\mu^k|||_{-\frac{1}{2},k}^2 \qquad (\mu \in M;\ \mu = (\mu^1, ..., \mu^K)); \tag{2.10}$$

$$\| \varphi \|_\Phi = \inf\{ \ \| v \|_1 \mid v \in H_0^1(\Omega),\ v_{|\Sigma} = \varphi\}. \tag{2.11}$$

For every f, say, in $L^2(\Omega)$, we can now consider the following two problems:

$$\begin{cases} \text{find } w \in H_0^1(\Omega) \text{ such that} \\ a(w, v) = (f, v) \qquad \forall v \in H_0^1(\Omega) \end{cases} \tag{2.12}$$

and

$$\begin{cases} \text{find } u \in V,\ \lambda \in M \text{ and } \psi \in \Phi \text{ such that} \\ \text{i)} \quad a(u, v) - \sum_k < \lambda^k, v^k >_k = \ (f, v) \qquad \forall v \in V \\ \text{ii)} \quad \sum_k < \mu^k, \psi - u^k >_k = \ 0 \qquad \forall \mu \in M \\ \text{iii)} \quad \sum_k < \lambda^k, \varphi >_k = 0 \qquad \forall \varphi \in \Phi. \end{cases} \tag{2.13}$$

Theorem 1 For every $f \in L^2(\Omega)$, both problems (2.12) and (2.13) have a unique solution. Moreover we have

$$u^k = w \qquad \text{in } \Omega_k \quad (k = 1, ..., K), \tag{2.14}$$

$$\lambda^k = \frac{\partial w}{\partial n_A^k} \qquad \text{on } \Gamma_k \quad (k = 1, ..., K), \tag{2.15}$$

$$\psi = w \qquad \text{on } \Sigma \tag{2.16}$$

where $\partial w / \partial n_A^k$ is the outward conormal derivative (of the restriction of w to Ω_k) with respect to the operator A.

Proof We refer to [6] for the proof. ∎

It is very important, for applications to domain decomposition methods, to remark explicitly that the first two equations of (2.13) can be written as

$$\begin{cases} a_k(u^k, v^k) - < \lambda^k, v^k >_k = \ (f, v^k)_k \qquad \forall v^k \in V_k, \quad \forall k \\ < u^k, \mu^k >_k = < \psi, \mu^k >_k \qquad \forall \mu^k \in M_k, \quad \forall k. \end{cases} \tag{2.17}$$

In particular, for all fixed k, assuming f and ψ as data, (2.17) is the variational formulation of the Dirichlet problem

$$\begin{cases} Au^k &= f \qquad \text{in } \Omega_k, \\ u^k &= \psi \qquad \text{on } \Gamma_k, \end{cases} \tag{2.18}$$

where the boundary condition is imposed by means of a Lagrange multiplier (that finally comes out to be $\lambda^k \equiv \partial u^k/\partial n_A^k$) as in Babuška [1]. Hence, for f and ψ given, the resolution of the first two equations of (2.13) amounts to the resolution of K independent Dirichlet problems.

Problem (2.13) can now be approximated in many different ways. Choosing V_h, M_h, and Φ_h finite dimensional subspaces of V, M, Φ, we can consider the discretized problem

$$\begin{cases} \text{find } u_h \in V_h,\ \lambda_h \in M_h \text{ and } \psi_h \in \Phi_h \text{ such that} \\ \text{i)}\quad a(u_h, v) - \sum_k < \lambda_h^k, v^k >_k = (f, v) \qquad \forall v \in V_h \\ \text{ii)}\quad \sum_k < \mu^k, \psi_h - u_h^k >_k = 0 \qquad \forall \mu \in M_h \\ \text{iii)}\quad \sum_k < \lambda_h^k, \varphi >_k = 0 \qquad \forall \varphi \in \Phi_h. \end{cases} \tag{2.19}$$

As an example, assume, for the sake of simplicity, that we have a global decomposition $\mathcal{T}_h$ of Ω into finite elements ω (say, triangles), which is compatible with the macro-element subdivision (2.1) (in other words, for every ω in $\mathcal{T}_h$ and for every Ω_k, the symmetric difference $(\Omega_k \cup \omega) \setminus (\Omega_k \cap \omega)$ has zero measure). In this case, taking a finite element approximation $\tilde{V}_h$ of $H^1(\Omega)$, one can set

$$\begin{cases} V_h^k = \tilde{V}_{h|\Omega_k} \quad ; \quad V_h = \prod V_h^k \quad ; \quad \tilde{V}_h^\circ = \tilde{V}_h \bigcap H_0^1(\Omega) \\ \Phi_h = \tilde{V}_{h|\Sigma}^\circ \quad ; \quad M_h^k = (V_h^k{}_{|\Gamma_k})' \ ; \quad M_h = \prod M_h^k. \end{cases} \tag{2.20}$$

It is easy to check that with these choices the solution of (2.19) is nothing else but the standard finite element approximation of the solution of (2.12) by means of the subspace $\tilde{V}_h^\circ$. In a more general case, it is clear that suitable inf-sup conditions have to be assumed for V_h, M_h and Φ_h in order to ensure stability and optimal error bounds for the discrete problems (2.19) (see [5]). However, it is possible to stabilize (2.19), for general discretizations, by adding proper stabilizing terms "à la Hughes" (see [3] or, more generally, [9]). In order to do this, let us assume, as before, that we have a global decomposition $\mathcal{T}_h$ compatible with the macro-element subdivision (2.1). The decomposition $\mathcal{T}_h$ induces then, in a natural way, finite element decompositions of each Ω_k, of each Γ_k, and of Σ. For the sake of simplicity we shall write $\sum_{\omega(k)}$ and $\sum_{\sigma(k)}$ for the sum over those elements ω (resp. σ) belonging to Ω_k (resp. Γ_k). We shall also denote by h_ω and h_σ the diameter of ω and σ, respectively. As far as the degrees of the polynomials are concerned, we allow the maximum generality; the degree can also change from one macro-element to another. Since $V_h \subset V$, the functions $v_h \in V_h$ must be continuous in Ω_k. Similarly, the functions $\varphi_h \in \Phi_h$ will also be continuous on Σ, while no continuity is required on functions $\mu_h \in M_h$.

Remark 2.1: Our assumptions are much more restrictive than necessary. In principle we can easily adapt these ideas to more general subspaces, even allowing different Galerkin methods (Fourier, spectral, wavelets etc.) from one Ω_k to another. However, as we shall see, the notation (more than the actual implementation) is already cumbersome in our simplified case, and would become too heavy in a more general one. ∎

Note that, if u, λ, ψ is the solution of (2.13), all the following equalities hold

$$\sum_k \sum_{\omega(k)} h_\omega^2 (Au^k - f, Av^k)_{L^2(\omega)} \equiv 0, \tag{2.21}$$

$$\sum_k \sum_{\sigma(k)} h_\sigma (\lambda^k - \partial u^k/\partial n_A^k, \mu^k - \partial v^k/\partial n_A^k)_{L^2(\sigma)} \equiv 0, \tag{2.22}$$

$$\sum_k \sum_{\sigma(k)} h_\sigma (u^k - \psi, \varphi)_{H^1(\sigma)} \equiv 0, \tag{2.23}$$

for any choice of piecewise smooth v, μ, φ. In view of that, we can substitute problem (2.19) with the following expanded form

$$\begin{aligned}
&\text{find } (u_h,\ \lambda_h,\ \psi_h) \in V_h \times M_h \times \Phi_h \text{ such that}\\
&\sum_k \{a_k(u_h^k, v_h^k) - < \lambda_h^k, v_h^k >_k + \sum_{\omega(k)} h_\omega^2 (Au_h^k, Av_h^k)_{L^2(\omega)} + \qquad (2.24)\\
&- \sum_{\sigma(k)} h_\sigma[(\lambda_h^k - \partial u_h^k/\partial n_A^k, \partial v_h^k/\partial n_A^k)_{L^2(\sigma)} + (u_h^k - \psi_h, v_h^k)_{H^1(\sigma)}]\} =\\
&= \sum_k \sum_{\omega(k)} (f, v_h^k + h_\omega^2 A v_h^k)_{L^2(\omega)} \qquad \forall v_h
\end{aligned}$$

$$\sum_k \{< \mu_h^k, u_h^k - \psi_h >_k + \sum_{\sigma(k)} h_\sigma (\lambda_h^k - \partial u_h^k/\partial n_A^k, \mu_h^k)_{L^2(\sigma)}\} = 0 \quad \forall \mu_h \tag{2.25}$$

$$\sum_k \left\{ < \lambda_h^k, \varphi_h >_k + \sum_{\sigma(k)} h_\sigma (u_h^k - \psi_h, \varphi_h)_{H^1(\sigma)} \right\} = 0 \qquad \forall \varphi_h. \tag{2.26}$$

Problem (2.24)-(2.26) has clearly a unique solution. We point out explicitly that the regularized formulation (2.24)-(2.26) is still well suited for parallel implementation. Indeed, for ψ_h and f given, the resolution of (2.24)-(2.26) amounts to the resolution of K independent problems, each of them being a Dirichlet problem with Lagrange multipliers treated with a variant of [3]. For an analysis of the convergence of (2.24)-(2.26) we refer to [7], [2].

3. A preconditioner and numerical results

We already pointed out that the first two equations of (2.13) can be interpreted as K independent Dirichlet problems, one in each macro-element. We also noted that the same will be true for every reasonable discretization of (2.13), as for instance (2.19) with the choice (2.20), or directly (2.24)-(2.26). If we now eliminate from the first two equations of (2.13) the unknowns u and λ as functions of ψ and f, and substitute in the third equation, we end up with a single equation in the unknown ψ of the type

$$\mathcal{S}\psi = \mathcal{G}. \tag{3.1}$$

For the discrete case the procedure is essentially the same, possibly, as for instance for (2.24)-(2.26), with more difficult notation. In any case, we end up with a problem of the form

$$\mathcal{S}_h\psi_h = \mathcal{G}_h, \tag{3.2}$$

where the computation of $\mathcal{G}_h$ and, for any given φ_h, of $\mathcal{S}_h\varphi_h$ involves the solution of K independent Dirichlet problems, one for each subdomain.

Remark 3.1 In the case of a finite element discretization such as (2.20) in (2.19), $\mathcal{S}_h$ is nothing but the classical Schur complement [8]. ∎

In order to solve (3.2) with, say, conjugate gradient method, one faces two difficulties. The first one is due to the fact that $\mathcal{S}_h$ is the discretization of the operator $\mathcal{S}$, which is linear and continuous from Φ to Φ' (and, in particular, is a pseudo-differential operator of order one) [6]. Hence, we have to expect that $\mathcal{S}_h$ has a spectral radius of order $\mathcal{O}(1/h)$. We therefore need a preconditioner acting, very roughly speaking, as the inverse of a first order operator. Unfortunately, it is easy to devise nice symmetric positive definite second order operators, but is much less easy to do it with first order ones. See for instance [4] and the references therein for several interesting attempts in this direction. The second difficulty is the possible lack of symmetry. If the operator A in (2.2) is symmetric, and a standard finite element approximation as (2.20) is used in (2.19), then $\mathcal{S}_h$ will be symmetric. However, if either A is not symmetric or a more general (finite element) discretization is used, so that the stabilized form (2.24)-(2.26) has to be employed, $\mathcal{S}_h$ will not be symmetric. A possible way out [6] is to precondition (3.2) in the form

$$\mathcal{S}_h^* T_h^{-1} \mathcal{S}_h \psi_h = \mathcal{S}_h^* T_h^{-1} \mathcal{G}_h. \tag{3.3}$$

where T_h is a discrete second order differential operator, given for instance by

$$(T_h \psi_h, \varphi_h) = \sum_k \sum_{\sigma(k)} (\psi_h, \varphi_h)_{H^1(\sigma)}. \tag{3.4}$$

Note that, for any given φ_h, the computation of $\mathcal{S}_h^* \varphi_h$ amounts to the discrete resolution of K independent Dirichlet problems for the adjoint operator A^*.

We conclude this section with some numerical results, obtained using piecewise linear finite element discretizations as (2.20) in (2.19). In Table 1 we report the results for the problem $\Delta u = 0$ in $(0,1) \times (0,1)$, $u = 0$ on the boundary. In Table 2 the non symmetric case is considered: $-\Delta u + \beta u_x = 0$ in $(0,1) \times (0,1)$, $u = 0$ on the boundary, and $\beta = 10$. In both cases the solution of (3.2) is obtained with the preconditioned conjugate gradient method, where the preconditioner for the Schur complement $\mathcal{S}_h$ is taken as in (3.3)-(3.4). Subdivisions into $K = 4$ and $K = 16$ square subdomains are reported; uniform finite element decompositions into right triangles of mesh size h are used in each subdomain. We report the average error reduction per iteration (AER) and the number of iterations (NIT) necessary to reduce the initial residual by a factor of 10^{-4} in the maximum norm. The tables clearly indicate that the number of iterations and the average error reduction depend on the number of subdomains but not on the mesh parameter.

h	NIT, $K=4$	AER, $K=4$	NIT, $K=16$	AER, $K=16$
1/10	8	.22	12	.47
1/20	9	.26	13	.50
1/30	9	.27	13	.49
1/40	9	.28	13	.49
1/50	9	.28	14	.52

Table 1. The symmetric case

h	NIT, $K=4$	AER, $K=4$	NIT, $K=16$	AER, $K=16$
1/10	7	.25	14	.52
1/20	9	.28	16	.56
1/30	10	.30	16	.56
1/40	10	.30	17	.58
1/50	10	.30	18	.58

Table 2. The non symmetric case

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