Contemporary Mathematics
Volume 157, 1994

# Self–adaptive Coupling of Mathematical Models and/or Numerical Methods

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## 1. Introduction

In aerodynamics, the description of viscous flows at a high Reynolds number may require the simultaneous use of different models. Indeed, in this case large parts of the flow can be considered inviscid, thus they can appropriately be described by a simplified set of equations (the Euler equations). Usually, the viscous effects are relevant just close to physical boundaries and around shocks, where the more complex Navier–Stokes equations should be used. Though it is a common practice to solve the full Navier–Stokes equations in the whole fluid domain, we believe that a very promising alternative consists in coupling the Euler and the Navier–Stokes models in a self–adaptive way. In such a way, each set of equations (which are of different mathematical nature) can be discretized by the most appropriate numerical scheme. The local choice between the two sets of equations is not fixed a priori, but it is part of the model itself. As mentioned before, suitable interface conditions are required to ensure a smooth transition between the two models.

Based on these considerations, a model of viscous/inviscid coupling, known as "$\chi$–formulation" was introduced in [**BCR**]. The basic idea is as follows. Suppose that, in the fluid domain $\Omega \subset \mathbf{R}^n$, the complete set of (steady) viscous equations is written as

$$\nu \mathbf{D}(u) + \mathbf{C}(u) = f, \tag{1.1}$$

where $\mathbf{D}(u)$ and $\mathbf{C}(u)$ stand for differential operators describing the viscous interaction and the advective phenomena respectively, and $\nu$ is a number which scales the magnitude of the two effects. Clearly, (1.1) is supplemented by the appropriate boundary conditions. The $\chi$–formulation consists of replacing (1.1) by

$$\nu \chi(\mathbf{D}(u)) + \mathbf{C}(u) = f, \tag{1.2}$$

1991 Mathematics Subject Classification. Primary 35B25, 65N55. Secondary 76R00.

This paper is in final form and no version of it will be submitted for publication elsewhere.

where $\chi$ is a smooth, monotone function which vanishes where the size of its argument is below a prescribed threshold $\delta$, and basically concides with the identity elsewhere. With such a definition, equation (1.2) coincides with (1.1) where the $\chi$–function is the identity (i.e., where the viscous terms $\mathbf{D}(u)$ cannot be ignored), while (1.2) reduces to the inviscid model

$$\mathbf{C}(u) = f \tag{1.3}$$

where $\chi \equiv 0$ (i.e., where the viscous terms are negligeable). The two regions in which the domain $\Omega$ is partitioned by the $\chi$–function are not fixed a priori, but they are automatically determined by the solution $u$ of the $\chi$–equation (1.2). Moreover, the smoothness of the $\chi$–function allows a smooth coupling between the viscous and inviscid solutions. Finally, the threshold $\delta$ can be chosen as a function of $\nu$ in such a way that the solution of (1.2) is as close as we want to the solution of (1.1).

For a scalar advection–diffusion equation, the $\chi$–problem (1.2) has been investigated in [**BCR**] and [**R**] for Dirichlet boundary conditions and smooth domains, and in [**CR1**] for mixed boundary conditions and domains with corners. An analysis of finite elements approximations has been carried out in [**Ri**], while the practical implementation of the $\chi$–method for the Burgers equation and the conical Navier–Stokes equations is discussed in [**AC1**], [**AC2**]. Pironneau and co–workers (see [**AP**], [**AGP**], [**AMP**]) have considered other implementations of the $\chi$–function for the viscous/inviscid coupling. Moreover, they applied the $\chi$–method in the coupling of other flow models, such as the potential/rotational models and the Stokes/Navier–Stokes equations.

The $\chi$–formulation induces in a natural way a domain decomposition method. Indeed, from a practical point of view, solving the $\chi$–equation (1.2) in the whole domain would be even more expensive than solving the original equation (1.1). Instead, it is natural to solve equation (1.2) only where the $\chi$–term is non-zero; elsewhere, equation (1.2) coincides with the inviscid equation (1.3). The domain decomposition can be accomplished as follows. Denote by $Z \subset \Omega$ the set of points where $\chi(\mathbf{D}(u))$ is zero. Let us choose an open set $\Omega_i$ strictly contained in $Z$, and let us set $\Omega_v := \Omega \setminus \bar{\Omega}_i$. Then, equation (1.2) splits into the couple of equations

$$\mathbf{C}(u_i) = f \qquad \text{in } \Omega_i, \tag{1.4}$$

$$-\nu\chi(\mathbf{D}(u_v)) + \mathbf{C}(u_v) = f \qquad \text{in } \Omega_v, \tag{1.5}$$

where $u_i = u_{|\Omega_i}$ and $u_v = u_{|\Omega_v}$. These equations should be supplemented by appropriate conditions on the interface $\Gamma := \partial\Omega_i \cap \partial\Omega_v$ such that *i)* the resulting problems in $\Omega_i$ and $\Omega_v$ are well–posed and *ii)* the coupling of the two problems is equivalent to the original problem (1.2). For equation (1.5) in $\Omega_v$, the natural condition is suggested by the fact that, by the definition of $\Omega_i$, the $\chi$–function vanishes on $\Gamma$. Hence, equation (1.2) is equivalent on $\Gamma$ to $\mathbf{C}(u) = f$, which we consider as an oblique derivative condition to be attached to equation

(1.5). The splitting (1.4)–(1.5) suggests a way of solving the $\chi$–problem (1.2), via an iterative procedure which alternatively solves problem (1.4) and (1.5). The matching on $\Gamma$ is of hyperbolic/hyperbolic type, even if heterogeneous equations are solved in the two subdomains. A further feature of the domain decomposition method based on the $\chi$–formulation is the possibility of automatic detection of $Z$ as a part of the iterative process, with consequent adaptive adjustment of the interface position $\Gamma$. This feature has been succesfully implemented in the solution of Burgers' equation in [**AC1**]; its theoretical analysis will be discussed elsewhere.

The present paper is a summary of the main results contained in [**CR1**] and [**CR2**]. We refer to those papers for the detailed proofs and for further results.

## 2. Statement of the Problem

Consider the following convection–diffusion problem:

$$\tag{2.1} \begin{cases} -\nu\Delta u + \mathbf{a}\cdot\nabla u + bu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Here $\Omega \subset \mathbf{R}^2$ is a bounded domain, whose boundary is a piecewise $C^{1,1}$ curvilinear polygon, with no cusps and no angles larger than $\pi$; the diffusion coefficient $\nu > 0$ is constant; $\mathbf{a} \in [C^{1,1}(\bar\Omega)]^2$, $b \in C^0(\bar\Omega)$ satisfy the condition

$$\tag{2.2} -\frac{1}{2}\nabla\cdot\mathbf{a} + b \geq \alpha > 0 \quad \text{in } \Omega \quad \text{for some positive constant } \alpha;$$

the data satisfy the regularity assumptions $f \in L^2(\Omega)$, $g$ is the trace on $\partial\Omega$ of a function in $H^2(\Omega)$. It is well known (see, e.g., [**G**]) that with these hypotheses problem (2.1) admits a unique solution in $H^2(\Omega)$.

The "$\chi$–formulation" introduced in [**BCR**] consists of replacing (2.1) with the modified problem

$$\tag{2.3} \begin{cases} -\nu\chi(\Delta u) + \mathbf{a}\cdot\nabla u + bu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where the monotone function $\chi : \mathbf{R} \to \mathbf{R}$ is defined by

$$\tag{2.4} \begin{aligned} \chi(s) &= \begin{cases} 0 & 0 \leq s \leq \delta - \sigma \\ \dfrac{\delta}{\sigma}(s-\delta) + \delta & \delta - \sigma < s < \delta \\ s & s \geq \delta \end{cases} \\ \chi(s) &= -\chi(-s) \qquad\qquad\qquad s < 0, \end{aligned}$$

$\delta$, $\sigma$ being two fixed parameters satisfying $0 < \delta$, $0 < \sigma < \delta$.

Further assumptions are required for the well-posedness of the $\chi$–problem. Precisely, we make the following hypotheses:

$$b \geq b_0 > 0 \quad \text{for some constant } b_0; \tag{2.5}$$

there exists a function $\phi_1 \in W^{2,\infty}(\Omega)$ such that

$$\Delta\phi_1 \leq 0 \quad \text{in } \bar{\Omega}, \qquad \mathbf{a} \cdot \nabla\phi_1 \geq \alpha_0 \quad \text{in } \bar{\Omega} \quad \text{for some constant } \alpha_0 > 0. \tag{2.6}$$

Problem (2.3) was first investigated in [**BCR**] and in [**R**], where existence and uniqueness of a solution, as well as further properties of it, were established. In [**CR1**], similar results were obtained for a more general problem, allowing for mixed Dirichlet/oblique derivative boundary conditions and less stringent assumptions on the geometry and on the data. For the sake of clarity, we prefer to discuss the domain decomposition technique on the simple problem (2.3); however, all the subsequent results hold in the most general situation considered in [**CR1**].

Let us now make the basic assumption that there exists a subdomain of $\Omega$ where $\chi(\Delta u)$ is identically zero, so that equation (2.3) is of hyperbolic type therein. More precisely, let us suppose that $f \in C^0(\bar{\Omega})$ and $g$ is the trace of a function in $W^{2,p}(\Omega)$ for some $p > 2$. According to [**BCR**, Thm. 3.3], $u \in C^1(\bar{\Omega})$ and $\chi(\Delta u)$ is continuous in $\bar{\Omega}$. Then, let us set $Z = \{x \in \Omega : \chi(\Delta u)(x) = 0\}$ and let us assume that the interior of $Z$ is non empty. Let us denote by $\Omega_i$ a fixed open set contained in $Z$, and let us set $\Omega_v := \Omega \setminus \bar{\Omega}_i$. In this way we have decomposed $\Omega$ into two disjoint open subdomains $\Omega_i$ and $\Omega_v$. Let us denote by $\Gamma := \partial\Omega_i \cap \partial\Omega_v$ the common interface between the subdomains. Technical assumptions on $\Gamma$ will be made precise in the sequel of the paper. Let us set $u_i := u_{|\Omega_i}$ and $u_v := u_{|\Omega_v}$. Then, $u_i$ satisfies the reduced equation

$$\mathbf{a} \cdot \nabla u_i + b u_i = f \quad \text{in } \Omega_i, \tag{2.7}$$

whereas $u_v$ satisfies the complete $\chi$–equation

$$-\nu\chi(\Delta u_v) + \mathbf{a} \cdot \nabla u_v + b u_v = f \quad \text{in } \Omega_v. \tag{2.8}$$

We now supplement (2.7) and (2.8) with appropriate boundary and interface conditions such that: *i)* the resulting problems in $\Omega_i$ and $\Omega_v$ are well–posed, and *ii)* the coupling of the two problems is equivalent to the original $\chi$–problem (2.3).

To this end, let us denote by $\mathbf{n}_i$ and $\mathbf{n}_v$ the outgoing normals to the boundary of $\Omega_i$ and $\Omega_v$, respectively, and define the "inflow" boundary of $\Omega_i$ as

$$\partial\Omega_i^- := \{x \in \partial\Omega_i : \mathbf{a}(x) \cdot \mathbf{n}_i(x) < 0\}. \tag{2.9}$$

The conditions on the common interface $\Gamma$ of the subdomains are derived by looking at $u$ on $\Gamma$. Firstly, recalling that $u \in C^1(\bar{\Omega})$ we get in particular that

$$u \text{ is } C^0 \text{ across } \Gamma; \tag{2.10}$$

secondly, taking into account that $\chi(\Delta u) = 0$ in a neighborhood of $\Gamma$, we get from (2.3)

$$\mathbf{a} \cdot \nabla u + bu = f \quad \text{on } \Gamma. \tag{2.11}$$

The previous identity makes sense in $C^0(\Gamma)$. Setting $\Gamma^- = \partial\Omega_i^- \cap \Gamma$, $\Gamma^+ = \Gamma \backslash \Gamma^-$, we conclude that $u_i$ and $u_v$ respectively solve the two coupled boundary value problems:

$$\mathbf{a} \cdot \nabla u_i + bu_i = f \quad \text{in } \Omega_i, \tag{2.12.1}$$

$$u_i = g \quad \text{on } \partial\Omega_i^- \cap \partial\Omega, \tag{2.12.2}$$

$$u_i = u_v \quad \text{on } \Gamma^-, \tag{2.12.3}$$

and

$$-\nu\chi(\Delta u_v) + \mathbf{a} \cdot \nabla u_v + bu_v = f \quad \text{in } \Omega_v, \tag{2.13.1}$$

$$u_v = g \quad \text{on } \partial\Omega_v \cap \partial\Omega, \tag{2.13.2}$$

$$u_v = u_i \quad \text{on } \Gamma^+, \tag{2.13.3}$$

$$\mathbf{a} \cdot \nabla u_v + bu_v = f \quad \text{on } \Gamma^-. \tag{2.13.4}$$

Note that (2.13.4) is an oblique–derivative condition which is (formally) equivalent to imposing the vanishing of $\chi(\Delta u_v)$ on $\Gamma^-$. This is consistent with our philosophy of placing the viscous/inviscid interface within the region where the $\chi$–function vanishes. Problem (2.12) is a hyperbolic problem in $\Omega_i$ with Dirichlet conditions specified on the inflow part of $\partial\Omega_i$. Problem (2.13) is a non–linear elliptic problem on $\Omega_v$, with mixed boundary conditions on $\partial\Omega_v$.

It can be shown that solving the coupled problems (2.12)–(2.13) and then gluing together the two solutions $u_i$ and $u_v$ is equivalent to solve the original problem (2.3). For all details, we refer to [**CR2**], Section 2.

## 3. Solving the Coupled Problems by an Iterative Method

In this Section, we will study a simple iterative algorithm to solve problem (2.3) using sub–problems (2.12) and (2.13). Precisely, we propose to alternate between the $\chi$-viscous and inviscid subdomains as follows: given $u_v^0$ on $\Gamma^-$, for $n = 1, 2, \ldots$ define $u_i^n$ in $\Omega_i$ as the solution of

$$\begin{cases} \mathbf{a} \cdot \nabla u_i^n + bu_i^n = f & \text{in } \Omega_i, \\ u_i^n = g & \text{on } \partial\Omega_i^- \cap \partial\Omega, \\ u_i^n = u_v^{n-1} & \text{on } \Gamma^-, \end{cases} \tag{3.1}$$

and define $u_v^n$ in $\Omega_v$ as the solution of

$$\begin{cases} -\nu\chi(\Delta u_v^n) + \mathbf{a} \cdot \nabla u_v^n + bu_v^n = f & \text{in } \Omega_v, \\ u_v^n = g & \text{on } \partial\Omega_v \cap \partial\Omega, \\ u_v^n = u_i^n & \text{on } \Gamma^+, \\ \mathbf{a} \cdot \nabla u_v^n + bu_v^n = f & \text{on } \Gamma^-. \end{cases} \tag{3.2}$$

Note that the exchange of information between the two subproblems is of the same type as the one used in a hyperbolic/hyperbolic coupling: each problem receives Dirichlet data from the other subdomain through the part of the interface where the flow is coming in. From now on, we assume that $\Gamma$ is a piecewise $C^{1,1}$ curve with no cusps, which does not intersect $\partial\Omega$ forming cusps; besides, each connected component of $\Gamma^-$ is a $C^{1,1}$ curve. Moreover, we make two assumptions on the behaviour of the coefficients $\mathbf{a}$ and $b$ on $\Gamma$. To this end, let $\mathbf{n}$ denote the unit vector normal to $\Gamma$ pointing inside $\Omega_i$, let $\boldsymbol{\tau} = (-n_2, n_1)$ be the tangent vector, and let $\mathbf{a} = a_n\mathbf{n} + a_\tau\boldsymbol{\tau}$ be the corresponding splitting of $\mathbf{a}$ on $\Gamma$. We assume that the vector field $\mathbf{a}$ is nowhere tangent to $\Gamma$, precisely

$$|\mathbf{a}\cdot\mathbf{n}| \geq \beta > 0 \quad \text{on } \Gamma \tag{3.3}$$

for some constant $\beta > 0$. Furthermore, we make the following coerciveness assumption on $\Gamma^-$:

$$a_n + \nu\frac{\eta}{\mu_n} - \frac{\nu}{2}\frac{\partial}{\partial\boldsymbol{\tau}}\left(\frac{\mu_\tau}{\mu_n}\right) \geq -\gamma\sqrt{\alpha\nu} \qquad \text{on } \Gamma^-, \tag{3.4}$$

with $\gamma := \sqrt{2}/C_*^2C_p$ (here, $C_*$ is the norm of the inclusion $H^{1/2}(\Omega) \hookrightarrow L^2(\Gamma^-)$ and $C_p$ is the Poincaré constant defined by $\|v\|_{H^1(\Omega)} \leq C_p\|\nabla v\|_{L^2(\Omega)}$, for each $v \in H^1(\Omega)$ such that $v = 0$ on $\Sigma_D$).

REMARK 3.1. Assumption (3.3) on the transversality of $\mathbf{a}$ with respect to $\Gamma$ enforces some geometrical conditions on $\Gamma$. For instance, it implies that $\Gamma^+$ and $\Gamma^-$ should meet forming a corner point. However, we recall that the curve $\Gamma$ which separates $\Omega_i$ and $\Omega_v$ is completely at our choice, provided it lies in the region $Z$ where $\chi(\Delta u)$ vanishes. Therefore, it is always possible to adjust locally the shape of $\Gamma$ in order to match the transversality condition with respect to the vector field $\mathbf{a}$. ∎

In the next two subsections, we will examine separately subproblems (2.12) and (2.13), which form the two stages of the iterative method. We start with a short analysis of problem (2.12).

**3.1 The Hyperbolic Problem in $\Omega_i$.** In this subsection, we consider the following problem:

$$\begin{cases} \mathbf{a}\cdot\nabla u_i + bu_i = f & \text{in } \Omega_i \\ u_i = g & \text{on } \partial\Omega\cap\partial\Omega_i^- \\ u_i = \phi & \text{on } \Gamma^-. \end{cases} \tag{3.1.1}$$

Setting $Au = \mathbf{a}\cdot\nabla u$, we introduce the space $D(A) := \{u \in L^2(\Omega) : \mathbf{a}\cdot\nabla u \in L^2(\Omega)\}$ endowed with the graph norm. We look for a solution of (3.1.1) in this space. Recalling that our assumption on $\Gamma$ implies that $\mathbf{a}\cdot\mathbf{n}$ is bounded away from zero therein, one can prove the following result ([**B**, Theorem 2.3]).

THEOREM 3.1.1. *Let $g \in L^2(\partial\Omega \cap \partial\Omega_i^-)$ and $\phi \in L^2(\Gamma^-)$. Then there exists a unique solution $u_i \in D(A)$ of problem* (3.1.1). *Furthermore, $u_i|_{\partial\Omega_i^+} \in L^2(\partial\Omega_i^+)$, and the following estimate holds:*

$$\|u\|_{D(A)}+\|u\|_{L^2(\partial\Omega_i^+)} \leq C(\|f\|_{L^2(\Omega_i)}+\|g\|_{L^2(\partial\Omega\cap\partial\Omega_i^-)}+\|\phi\|_{L^2(\Gamma^-)}). \quad \blacksquare \tag{3.1.2}$$

The previous Theorem allows us to define the following continuous affine operator (depending on $f$ and $g$)

$$\begin{aligned} T_i : L^2(\Gamma^-) &\longrightarrow L^2(\Gamma^+) \\ \phi \quad &\longmapsto \quad u_i|_{\Gamma^+}. \end{aligned} \tag{3.1.3}$$

In order to establish further results that will be needed in the sequel, let us introduce some more notation. To this end, note that since $\mathbf{a} \in [C^{1,1}(\bar{\Omega}_i)]^2$, it is possible to extend it in $\mathbf{R}^2$ in such a way that the integral curves of $\mathbf{a}$

$$\begin{cases} \dfrac{\partial}{\partial t}\gamma(t,x) & = \mathbf{a}(\gamma(t,x)) \qquad x \in \bar{\Omega}_i \\ \gamma(0,x) & = x \end{cases}$$

are defined for all $t \in \mathbf{R}$ (see, e.g., [**A**, pag. 92]). Since $\mathbf{a}$ is non–singular, the Poincaré–Bendixson Theorem guarantees that the integral curves of the field stay in $\bar{\Omega}_i$ only for a bounded set of times. Therefore, for each $x \in \Gamma^+$, let $t^-(x) < 0$ be the maximum of negative times such that $\xi := \gamma(t^-(x), x) \in \partial\Omega_i$. It is easily seen that indeed $\xi$ belongs to the closure of $\partial\Omega_i^-$, say $\mathrm{cl}(\partial\Omega_i^-)$. Let us define the mapping $\Xi : \Gamma^+ \longrightarrow \mathrm{cl}(\partial\Omega_i^-)$ by $\Xi(x) = \xi$, and let us set

$$d_0 := \inf\{\mathrm{dist}(x, \Xi(x)) : x \in \Gamma^+, \Xi(x) \in \Gamma^-\}. \tag{3.1.4}$$

In other words, $d_0$ measures the smallest length of the characteristic curves in $\Omega_i$ joining a point of $\Gamma^+$ to a point of $\Gamma^-$. In Section 3.3 we will analyze the way the behaviour of the iterative method (3.1)–(3.2) depends on $d_0$.

We state now a particular form of the maximum principle that will be needed in the sequel (see Thm. 3.1.2 in [**CR2**]).

THEOREM 3.1.2. *Let $\phi_1, \phi_2 \in L^2(\Gamma^-)$ be such that $\phi_1 - \phi_2 \in L^\infty(\Gamma^-)$. Then $T_i(\phi_1 - \phi_2) \in L^\infty(\Gamma^+)$ and*

$$\|T_i(\phi_1-\phi_2)\|_{L^\infty(\Gamma^+)} \leq e^{-b_0 d_0/\|\mathbf{a}\|_{L^\infty(\Omega_i)}} \|\phi_1-\phi_2\|_{L^\infty(\Gamma^-)}. \quad \blacksquare \tag{3.1.5}$$

COROLLARY 3.1.1. *If $w$ is the solution of problem* (3.1.6) *then*

$$\|w\|_{L^\infty(\Omega_i)} \leq \|\phi_1-\phi_2\|_{L^\infty(\Gamma^-)}. \quad \blacksquare$$

We recall that the solution of a hyperbolic problem (even of the simplest type (3.1.1)) may develop discontinuities despite the smoothness of the inflow data and the coefficients of the operator, if the inflow boundary $\partial\Omega_i^-$ is not connected. Such singularities may also accumulate if, for instance, there is an infinite number of connected components of $\partial\Omega_i^-$. Therefore, even if we assume that the inflow data $\phi$ is regular, we can guarantee no more than $L^\infty$–regularity for the outflow value $T_i(\phi)$. Thus, we must be able to solve the $\chi$–elliptic subproblem with a Dirichlet data having less than $H^{1/2}$–regularity. In the next subsection, we will be able to weaken the regularity of the Dirichlet data for the $\chi$–elliptic problem down to mere $L^2$–integrability, apart from an arbitrarily small neighborhood of the points of $\Gamma$ in which the boundary conditions change of type. Therefore, we will now make an extra assumption on the relationship between the vector field $\mathbf{a}$ and the interface $\Gamma$. We ask that

$$\text{(3.1.6)} \qquad \begin{array}{l} \text{for all } x \in \bar\Gamma^+ \cap \bar\Gamma^- \text{ there exists a neighborhood } \mathcal{N} \text{ in } \bar\Gamma^+ \\ \text{which is mapped diffeomorphically by } \Xi \text{ onto an open set in } \partial\Omega_i^-. \end{array}$$

Assumption (3.1.6) precisely prevents the accumulation of infinitely many discontinuities of $T_i(\phi)$ at the points of $\bar\Gamma^+ \cap \bar\Gamma^-$. Indeed, it is easily seen that the following consequence of (3.1.6) holds.

PROPOSITION 3.1.1. *Define the Frechét space*

$$L^2_T(\Gamma^+) := \{v \in L^2(\Gamma^+) : \exists \text{ a neighborhood } \mathcal{N} \text{ of } T \text{ in which } v_{|\mathcal{N}} \in H^{1/2}(\mathcal{N})\}.$$

*Then the operator $T_i$ defined in* (3.1.3) *maps $H^{1/2}(\Gamma^-)$ into $L^2_T(\Gamma^+)$.* ∎

**3.2 The $\chi$–elliptic Problem in $\Omega_v$.** In this subsection, we consider the following problem:

$$\text{(3.2.1)} \qquad \begin{cases} -\nu\chi(\Delta u_v) + \mathbf{a}\cdot\nabla u_v + b u_v = f & \text{in } \Omega_v, \\ u_v = g & \text{on } \partial\Omega_v \cap \partial\Omega, \\ u_v = \psi & \text{on } \Gamma^+, \\ \mathbf{a}\cdot\nabla u_v + b u_v = f & \text{on } \Gamma^-, \end{cases}$$

where $\psi$ is a $L^\infty$–function on $\Gamma^+$ which has $H^{1/2}$ regularity on a neighborhood of $\partial\Gamma^+$. In the iterative procedure, $\psi$ will be the trace on $\Gamma^+$ of the solution of the hyperbolic problem (3.1.1). As we have seen above, $\psi$ may not belong to $H^{1/2}(\Gamma^+)$.

It is proven in Thms. 3.2.1 and 3.2.2 of [**CR2**] that if $\psi \in L^2_T(\Gamma^+)$, then we have existence and uniqueness of a solution $u_v$ of problem (3.2.1), in the "very weak" sense (see, e.g., [**N**]). Furthermore, we can define the continuous nonlinear operator (depending on $f$ and $g$)

$$\begin{aligned} T_v : L^2_T(\Gamma^+) &\longrightarrow H^{1/2}(\Gamma^-) \\ \psi \quad &\longmapsto \quad u_v|_{\Gamma^-}. \end{aligned}$$

Finally, if $\psi_1, \psi_2 \in L^\infty(\Gamma^+) \cap L^2_T(\Gamma^+)$, then $T_v(\psi_1 - \psi_2) \in L^\infty(\Gamma^-)$ and

$$\|T_v(\psi_1 - \psi_2)\|_{L^\infty(\Gamma^-)} \leq \|\psi_1 - \psi_2\|_{L^\infty(\Gamma^+)} \tag{3.2.2}$$

We are now ready to analyze the iterative procedure for solving problem (2.12)–(2.13).

**3.3 The Convergence of the Iterative Method.** Consider the iterative scheme (3.1)–(3.2). Assume that $u_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ and set $u_v^0 = u_0$ on $\Gamma^-$. First of all, let us note that the sequence $(u_i^n, u_v^n)$ $(n \geq 1)$ is well defined. Indeed, if $u_v^{n-1}|_{\Gamma^-} \in H^{1/2}(\Gamma^-)$, then by Theorem 3.1.1 and Proposition 3.1.1, Problem (3.1) has a unique solution $u_i^n$ such that $u_i^n|_{\Gamma^+} \in L^2_T(\Gamma^+)$. Then, by the results of Subsection 3.2, Problem (3.2) has a unique solution $u_v^n$ such that $u_v^n|_{\Gamma^-} \in H^{1/2}(\Gamma^-)$.

Let us define $u^n \in L^2(\Omega)$ as

$$u^n(x) = \begin{cases} u_i^n(x) & x \in \Omega_i \\ u_v^n(x) & x \in \Omega_v \end{cases}$$

(note that $u_v^n = u_i^n$ on $\Gamma^-$). The maximum principles for the hyperbolic and $\chi$-elliptic problems give the following error estimate.

THEOREM 3.3.1. *Under all the assumptions of Sections 1 and 3, the function $u^n$ belongs to $L^\infty(\Omega)$ and satisfies the estimate*

$$\|u - u^n\|_{L^\infty(\Omega)} \leq \left[e^{-b_0 d_0/\|\mathbf{a}\|_{L^\infty(\Gamma^-)}}\right]^{n-1} \|u - u^0\|_{L^\infty(\Gamma^-)}. \tag{3.3.1}$$

PROOF: See Thm. 3.3.1 in [**CR2**]. ∎

REMARK 3.3.1. The convergence of the iterative scheme depends on the value of $d_0$, which can be either 0, or a strictly positive number, or even $+\infty$. When $d_0 > 0$, convergence can be achieved in some cases after a finite number of iterations. Conversely, if $d_0 = 0$, the iterative method may not converge at all in the maximum norm. A remedy consists of a local overlapping of $\Omega_i$ over $\Omega_v$ around the points of $\bar{\Gamma}^+ \cap \bar{\Gamma}^-$. A more elegant remedy is the self-adaptive adjustment of the interface $\Gamma$ during the iterations. We refer to [**CR2**] for more details. ∎

## REFERENCES

[**AP**] Y. Achdou and O. Pironneau, *The $\chi$-method for the Navier–Stokes equations*, Note C.R.A.S. **I 168** (1991).

[**AGP**] Y. Achdou, R. Glowinski and O. Pironneau, *Tuning the mesh of a mixed method in $\psi, \omega$ for Navier–Stokes equations*, I.N.R.I.A. Report (1991).

[**AMP**] Y. Achdou, B. Mohammadi and O. Pironneau, *The $\chi$–method for flows with boundary layers and separation*, in Proceedings of the 10th International Conference on Computing Methods in Applied Sciences and Engineering (Ed. R. Glowinski) (1992), 621–632.

[**AC1**] R. Arina and C. Canuto, *A self adptive domain decomposition for the viscous/inviscid coupling. I. Burgers equation*, I.A.N.–C.N.R. Report 719, to appear in J. Comp. Physics (1989).

[**AC2**] R. Arina and C. Canuto, *The viscous–inviscid coupling via a self–adaptive domain decomposition technique*, in Proceedings of the 12th International Conference on Numerical Methods in Fluid Dynamics (Ed. K. W. Morton) (1990), 162–166.

[**A**] A. Avez, "Differential Calculus," John Wiley and Sons, 1986.

[**B**] C. Bardos, *Problèmes aux limites pour les équations aux dérivées partielles du premier ordre a coefficients réels; théoremes d'approximation; application a l'équation de transport*, Ann. Scient. Éc. Norm. Sup. **IV-3** (1970), 185-233.

[**BCR**] F. Brezzi, C. Canuto and A. Russo, *A self–adaptive formulation for the Euler/Navier–Stokes coupling*, Comp. Meth. in Appl. Mech. and Eng. **73** (1989), 317–330.

[**CR1**] C. Canuto and A. Russo, *A viscous-inviscid coupling under mixed boundary conditions*, to appear in $M^3AS$ - Math. Mod. Meths. Appl. Sci. (1992).

[**CR2**] C. Canuto and A. Russo, *On the elliptic–hyperbolic coupling. I: the advection–diffusion equation via the $\chi$–formulation*, to appear in $M^3AS$ - Math. Mod. Meths. Appl. Sci. (1993).

[**G**] P. Grisvard, "Elliptic Boundary Value Problems in Non–Smooth Domains," Pitman, London, 1985.

[**N**] J. Necas, "Les Méthodes Directes en Théorie des Équations Elliptiques," Masson, Paris, 1967.

[**Ri**] A. Rizza, *Numerical analysis of a self–adptive formulation for the viscous/inviscid coupling*, Calcolo **27** (1990), 219–250.

[**R**] A. Russo, *Uniqueness results for a class of nonlinear problems*, Appl. Math. Lett. **4** (1991), 39–42.

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