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# NONITERATIVE DOMAIN DECOMPOSITION FOR SECOND ORDER HYPERBOLIC PROBLEMS

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ABSTRACT. The solutions of damped second order hyperbolic problems can have smooth components which decay slowly and rough components which decay quickly. If the behavior of the solution is of interest on the time scale of the slowly-decaying modes, then implicit time stepping methods may be more efficient than explicit methods.

We formulate and analyze a Galerkin method for approximating the solutions of second order hyperbolic problems. This method involves domain decomposition in its formulation rather than as a means of solving the elliptic problems that result at each time step when a usual implicit method is used.

## 1. INTRODUCTION

Here we present and analyze a domain decomposition method for a damped second order hyperbolic problem. This method is closely related to the explicit/implicit conservative Galerkin method that we developed for parabolic problems [1]. We exhibit a "conservation of energy" principle for this explicit/implicit method, and give a simple convergence analysis based on this bound.

The work to take a time step has two phases. The first is an explicit calculation that defines a function on the interdomain boundaries; this calculation involves a small amount of communication between adjacent subdomains. The second phase involves solving a collection of subdomain problems which are completely independent of each other. The explicit step induces a time step limitation in terms of a spatial parameter, but the constraint is less confining than it would be for an explicit method based on the same function spaces.

## 2. DOMAIN DECOMPOSITION METHOD

Suppose that $u = u(x, y, t)$ is a smooth solution of the differential equation

$$u_{tt} + du_t - \Delta u + ku = f \text{ on } \Omega \times [0, T] = Q, \tag{1}$$

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here $d$ and $k$ are nonnegative constants and $\Omega = (0,1)\times(0,1)$. Also suppose that $u$ satisfies the boundary and initial conditions

$$u(x,y,0) = u_0(x,y) \text{ on } \Omega, \tag{2}$$

$$u_t(x,y,0) = u_1(x,y) \text{ on } \Omega, \tag{3}$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \times [0,T], \tag{4}$$

where $\nu$ is the direction perpendicular to the boundary of $\Omega$. Here $f, u_0$, and $u_1$ are given functions, and we want to compute an approximation to $u$.

The domain $\Omega$ is the disjoint union of the following sets:

$$\begin{aligned} \Gamma &= \{(\frac{1}{2}, y) : 0 < y < 1\}, \\ \Omega_1 &= \{(x,y) \in \Omega : 0 < x < \frac{1}{2}\}, \\ \Omega_2 &= \{(x,y) \in \Omega : \frac{1}{2} < x < 1\}. \end{aligned}$$

If $v$ is a pair of functions, one defined on $\Omega_1$ and one defined on $\Omega_2$, we will identify $v$ with a function defined on $\Omega_1 \cup \Omega_2$. Note that, for $v \in H^1(\Omega_1)\times H^1(\Omega_2)$, $u$ satisfies

$$(u_{tt} + du_t, v) + a(u,v) + \int_\Gamma u_x[v]dy = (f,v), \tag{5}$$

where

$$\begin{aligned} a(\psi,\eta) &= a_1(\psi,\eta) + a_2(\psi,\eta), \\ a_i(\psi,\eta) &= \int_{\Omega_i} \nabla\psi\cdot\nabla\eta + k\psi\eta \, dxdy, \\ [v](y) &= v\left(\frac{1}{2}+0,y\right) - v\left(\frac{1}{2}-0,y\right), \\ (\psi,\eta) &= \int_{\Omega_1\cup\Omega_2} \psi\eta dxdy. \end{aligned}$$

The above relation will be discretized to give a Galerkin method for approximating $u$. We will use the following notation for $\triangle t > 0$:

$$\begin{aligned} \psi^s &= \psi(s\triangle t), \\ \psi^{s,\frac{1}{2}} &= (\psi^{s+\frac{1}{2}} + \psi^{s-\frac{1}{2}})/2, \\ \partial\psi^s &= (\psi^{s+\frac{1}{2}} - \psi^{s-\frac{1}{2}})/\triangle t, \\ \delta\psi^s &= (\psi^{s+1} - \psi^{s-1})/(2\triangle t), \\ \partial^2\psi^s &= (\psi^{s+1} - 2\psi^s + \psi^{s-1})/(\triangle t)^2, \\ \psi^{s;\theta} &= \theta\psi^{s+1} + (1-2\theta)\psi^s + \theta\psi^{s-1}. \end{aligned}$$

Let $\mathcal{M}_i$ be a finite dimensional subspace of $H^1(\Omega_i)$, and let $\mathcal{M}$ be the space of functions $v$ defined on $\Omega_1\cup\Omega_2$ such that $v$ restricted to $\Omega_i$ is in $\mathcal{M}_i, i = 1,2$, i.e.,

$\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$. Suppose that $U^{n+\frac{1}{2}} \in \mathcal{M}$ for $n = -1, 0, 1, \dots$, and suppose that for $n \geq 0$ and $v \in \mathcal{M}$,

$$(6) \qquad (\partial^2 U^{n+\frac{1}{2}} + d\delta U^{n+\frac{1}{2}}, v) + a(U^{n+\frac{1}{2};\frac{1}{2}}, v) + b(U^{n+\frac{1}{2}}, v) = (f^{n+\frac{1}{2};\frac{1}{2}}, v),$$

where the bilinear form $b$ is defined by

$$\begin{aligned} \phi_2(x) &= \max(0, 1 - |x|), \\ \phi_{2,H}(x) &= \phi_2\left(\left(x - \frac{1}{2}\right)/H\right)/H. \\ B(\psi)(y) &= -\int_0^1 \phi'_{2,H}(x)\psi(x,y)dx, \\ b(\psi, \eta) &= \int_\Gamma (B(\psi)[\eta] + [\psi]B(\eta))dy. \end{aligned}$$

The function $\phi_{2,H}$ is an approximate one-dimensional delta function, and $B$ is an approximation of the derivative with respect to $x$ at a point on $\Gamma$. The form $b(\cdot,\cdot)$ is a symmetric approximation of the part of (5) that is on $\Gamma$. The form $b(\cdot,\cdot)$ is based on the family of (nonsymmetric) forms used in [1]. For the analysis of the scheme presented here it is important that $b(\cdot,\cdot)$ is symmetric. The use of the symmetric version of the boundary form is reminiscent of the work of Joachim Nitsche in [3].

## 3. Analysis of the Domain Decomposition Method

First we need the basic properties of the form $b(\cdot,\cdot)$. We use $\|\cdot\|$ as the $L^2(\Omega)$ norm and for functions in $H^1(\Omega_1) \times H^1(\Omega_2)$ we define the norm

$$(7) \qquad |||\psi|||^2 = a(\psi,\psi) + \frac{1}{H}\int_\Gamma [\psi]^2 dy.$$

**Lemma 1.** *For $\psi \in H^1(\Omega_1) \times H^1(\Omega_2)$,*

$$a(\psi,\psi) + b(\psi,\psi) \geq \frac{1}{2}|||\psi|||^2.$$

*Proof.* Integration by parts, application of the Cauchy inequality, and use of $\alpha\beta \leq \frac{\epsilon}{2}\alpha^2 + \frac{1}{2\epsilon}\beta^2$ for $\epsilon > 0$, give the following relations:

$$\begin{aligned} b(\psi,\psi) &= 2\int_\Gamma B(\psi)[\psi]dy \\ &= 2\int_\Gamma \phi_{2,H}(\tfrac{1}{2})[\psi]^2 dy + 2\iint \phi_{2,H}(x)\psi_x(x,y)[\psi](y)dxdy \\ &= \frac{2}{H}\int_\Gamma [\psi]^2 dy + 2\iint \phi_{2,H}(x)\psi_x(x,y)[\psi](y)dxdy \\ &\geq \frac{2-\epsilon}{H}\int_\Gamma [\psi]^2 dy - \frac{2}{3\epsilon}\|\psi_x\|^2 \\ &\geq \frac{1}{2H}\int_\Gamma [\psi]^2 dy - \frac{4}{9}\|\psi_x\|^2. \end{aligned}$$

In the last step we used $\epsilon = 3/2$. The lemma follows easily. $\square$

Note that because of the symmetry of $b(\cdot,\cdot)$ we have

$$b(\psi^{s;\frac{1}{2}}, \delta\psi^s) = (b(\psi^{s+1}) - b(\psi^{s-1}))/(4\Delta t),$$

where we used $b(\psi)$ as an abbreviation for $b(\psi,\psi)$. We will use this convention on other bilinear forms as well.

Next we exhibit an "energy equality" for the discrete solution; for this purpose we take $f$ to vanish identically and $d = 0$. In (6) use $v = \delta U^{n+\frac{1}{2}}$, multiply by $2\,\Delta t$ and sum on n to get that

$$\|U^n\|_{\mathcal{E}}^2 = \|U^0\|_{\mathcal{E}}^2, \tag{8}$$

where the "energy norm" is

$$\|U^n\|_{\mathcal{E}}^2 = \|\partial U^n\|^2 + \frac{1}{2}((a+b)(U^{n+\frac{1}{2}}) + (a+b)(U^{n-\frac{1}{2}})) - \frac{\Delta t^2}{2} b(\partial U^n).$$

Since

$$\begin{aligned} b(\psi) &= 2\int_\Gamma B(\psi)[\psi]dy \\ &\leq 2\|[\psi]\|_{L^2(\Gamma)}\|B(\psi)\|_{L^2(\Gamma)} \\ &\leq 2\|[\psi]\|_{L^2(\Gamma)}\sqrt{\frac{2}{H^3}}\|\psi\|, \end{aligned}$$

we see that

$$\begin{aligned} \frac{\Delta t^2}{2} b(\partial\psi^n) &\leq \Delta t^2 \|[\partial\psi^n]\|_{L^2(\Gamma)}\sqrt{\frac{2}{H^3}}\|\partial\psi^n\| \\ &\leq \|\partial\psi^n\|^2 + \frac{\Delta t^2}{H^2}(|||\psi^{n+\frac{1}{2}}|||^2 + |||\psi^{n-\frac{1}{2}}|||^2) \\ &\leq \|\partial\psi^n\|^2 + \frac{1}{4}(|||\psi^{n+\frac{1}{2}}|||^2 + |||\psi^{n-\frac{1}{2}}|||^2) \end{aligned}$$

provided $\Delta t \leq H/2$. This gives that the "energy" is nonnegative under this time step constraint. Similarly, if $\Delta t \leq H/4$,

$$\|\psi^n\|_{\mathcal{E}}^2 \geq \frac{1}{2}\|\partial\psi^n\|^2 + \frac{1}{8}(|||\psi^{n+\frac{1}{2}}|||^2 + |||\psi^{n-\frac{1}{2}}|||^2). \tag{9}$$

For each $t$ take $W(t)$ in $\mathcal{M}$ to be the projection of $u$ into $\mathcal{M}$ with respect to the form $a(\cdot,\cdot)$; i.e., for all $v \in \mathcal{M}$,

$$a(u - W, v) = 0.$$

If $k = 0$ define the projection with $k$ replaced by 1. Let

$$\eta = u - W. \tag{10}$$

**Theorem 2.** *Suppose that the solution $u$ is sufficiently smooth and that $U^{\pm\frac{1}{2}}$ are defined by $W^{\pm\frac{1}{2}}$, respectively. Then there is a $C$ such that*

$$\begin{aligned} &\max_n \|\partial(u-U)^n\| \\ &\leq C\left(\Delta t^2 + H^{2.5} + \|\eta_{tt}\|_{L^1(0,T;L^2(\Omega))} + H^{-\frac{1}{2}}(\|\eta_t\|_{L^\infty(Q)} + \|\eta\|_{L^\infty(Q)})\right), \end{aligned}$$

*provided* $\triangle t \leq H/4$.

*Proof.* With $\rho^{n+\frac{1}{2}} = \partial^2 u^{n+\frac{1}{2}}$, $\mu^{n+\frac{1}{2}} = B(u^{n+\frac{1}{2}}) - u_x^{n+\frac{1}{2};\frac{1}{2}}$, and $\vartheta = W - U$, we get that

$$\begin{aligned}(\partial^2\vartheta^{n+\frac{1}{2}}, v) + a(\vartheta^{n+\frac{1}{2};\frac{1}{2}}, v) + b(\vartheta^{n+\frac{1}{2}}, v) &= (\rho^{n+\frac{1}{2}} + \partial^2\eta^{n+\frac{1}{2}}, v) \\ &\quad +(\mu^{n+\frac{1}{2}}, [v])_\Gamma + b(\eta^{n+\frac{1}{2}}, v).\end{aligned} \tag{11}$$

Express $b(\vartheta^{n+\frac{1}{2}}, v)$ as $b(\vartheta^{n+\frac{1}{2};\frac{1}{2}}, v) - \frac{1}{2}\triangle t^2 b(\partial^2\vartheta^{n+\frac{1}{2}}, v)$. Use $v = \delta\vartheta^{n+\frac{1}{2}}$ to get that

$$\begin{aligned}\frac{1}{2\triangle t}\left[\|\vartheta^{n+1}\|_{\mathcal{E}}^2 - \|\vartheta^n\|_{\mathcal{E}}^2\right] &= (\rho^{n+\frac{1}{2}} + \partial^2\eta^{n+\frac{1}{2}}, \delta\vartheta^{n+\frac{1}{2}}) + (\mu^{n+\frac{1}{2}}, [\delta\vartheta^{n+\frac{1}{2}}])_\Gamma \\ &\quad +b(\eta^{n+\frac{1}{2}}, \delta\vartheta^{n+\frac{1}{2}}).\end{aligned} \tag{12}$$

To get a bound for $\vartheta$ from this relation we sum on $n$ and then sum the last two terms by parts in time (since the energy norm gives no control of the time differences on $\Gamma$). The boundary terms that come from summing by parts in time are $(\mu^{n+\frac{1}{2}}, [\vartheta^{n+\frac{1}{2}}])$ and $b(\eta^{n+\frac{1}{2}}, \vartheta^{n+\frac{1}{2}})$. The first of these is bounded by a small multiple of $\|\vartheta^n\|_{\mathcal{E}}^2$ plus a $O(H^5)$-term; this gives the $H^{2.5}$-term in the final result. The second term is similarly treated. The discrete Gronwall inequality then gives the bound on $\vartheta$. The triangle inequality is applied to finish the proof. □

## 4. Numerical Experiments

We first present numerical results on the rate of convergence of the algorithm described and analyzed above. We consider the following test problem:

$$u_{tt} - \Delta u = f, \quad \text{on } \Omega \times [0, T], \tag{13}$$

with $\Omega = (0,1) \times (0,1)$. We choose $f$ and the initial and boundary data so that $u(x,y,t) = t^2\cos(\pi x)\cos(\pi y)$. We compare the errors for two domain decomposition scenarios and a fully implicit Galerkin procedure, by computing $e_h \equiv \|\partial(u-U)(\cdot,t)\|$ at time $t = 1$ for 10 by 10, 20 by 20, and 40 by 40 uniform meshes. In these runs $\triangle t = h$ and, in the domain decomposition cases, $H = 2h$. We note that the errors are of comparable size for the three cases considered, and approach zero like $h^2 + \triangle t^2$. Similar convergence rates were observed for parabolic problems using the same type of domain decomposition procedure [1], indicating that the error estimate derived above may not be sharp. In fact, as noted in [1], a better estimate than that demonstrated here can be derived for tensor-product rectangular meshes.

As mentioned in the introduction, our intent in this paper is to study damped second-order hyperbolic equations. Next we consider the damped equation

$$u_{tt} + u_t - \Delta u = 0, \quad \text{on } (0,1)^2 \times [0, T], \tag{14}$$

with

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \times [0, T], \tag{15}$$

and

$$u(x,y,0) = \begin{cases} (1 - 2(x+y))^2, & x + y \leq .5, \\ 0, & \text{otherwise,} \end{cases} \tag{16}$$

| | Implicit | | 2x1 DD | | 2x2 DD | |
|---|---|---|---|---|---|---|
| $h^{-1}$, $H$ | $e_h * 10^4$ | Rate | $e_h * 10^3$ | Rate | $e_h * 10^3$ | Rate |
| 10, .2 | 16.9 | – | 21.2 | – | 27.8 | – |
| 20, .1 | 4.33 | – | 5.26 | – | 6.30 | – |
| 40, .05 | 1.08 | 1.98 | 1.28 | 2.02 | 1.51 | 2.10 |

TABLE 1. Convergence in $h$: $u(x,t) = u_1(x,y,t)$

$$u_t(x,y,0) = 0. \tag{17}$$

In Figures 1 and 2, we compare the fully implicit Galerkin solution (no domain decomposition) at time $t = .5$ with a 2 by 2 domain decomposition solution. In these runs an 80 by 80 rectangular mesh was used, with $\Delta t = .0125$. In the domain decomposition runs, $H = 4h$. The figures indicate that the solutions are virtually identical.

The computer simulations described above were run on the Intel iPSC/860 Hypercube located at the National Science Foundation Center for Research on Parallel Computation at Rice University.

## 5. Conclusions

A noniterative, conservative, Galerkin domain decomposition procedure has been presented and analyzed for second order hyperbolic equations. The method uses nonoverlapping domain decomposition and the calculation of boundary information is very inexpensive compared to the cost of solving subdomain problems, at least for medium and coarse-grain decompositions. Because boundary information is calculated explicitly, the method does require that the time-step and the interface discretization parameter $H$ satisfy a stability inequality. On two test problems, the domain decomposition procedure gave numerical results comparable in quality to that of a fully implicit Galerkin procedure.

## References

1. C. N. Dawson and T. F. Dupont, *Explicit/implicit, conservative, Galerkin domain decomposition procedures for parabolic problems*, Math. Comp. **58** (1992), 21-34.
2. C. N. Dawson, Q. Du, and T. F. Dupont, *A Finite difference domain decomposition algorithm for the numerical solution of the heat equation*, Math. Comp. **57** (1991), 63-67.
3. J. Nitsche, *Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind*, Abh. Math. Sem. Univ. Hamburg, **36** (1971), 9-15.

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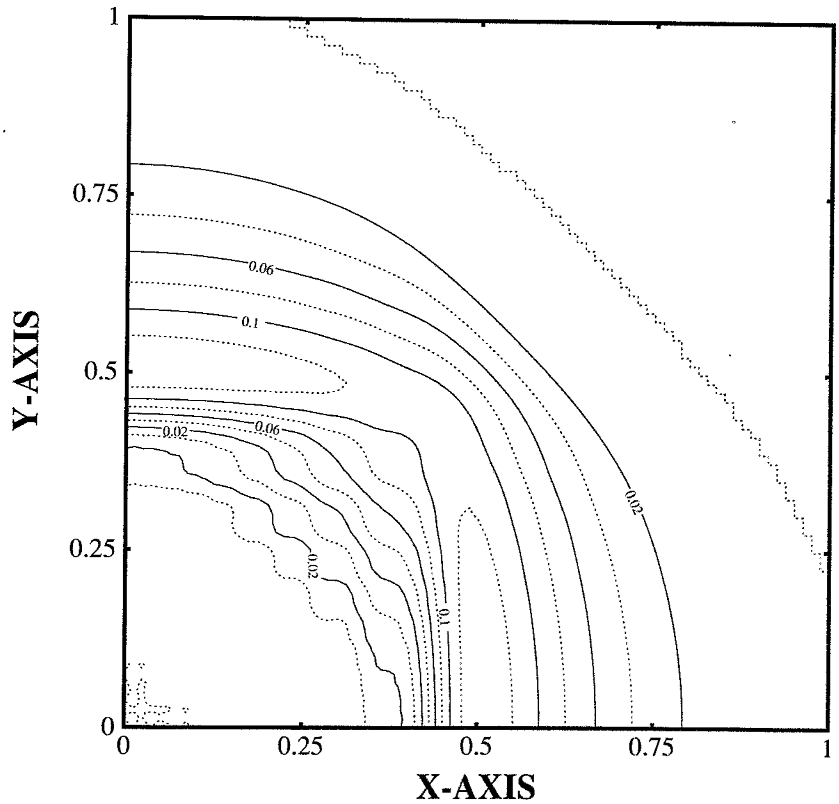


FIGURE 1. Implicit Galerkin solution for damped case

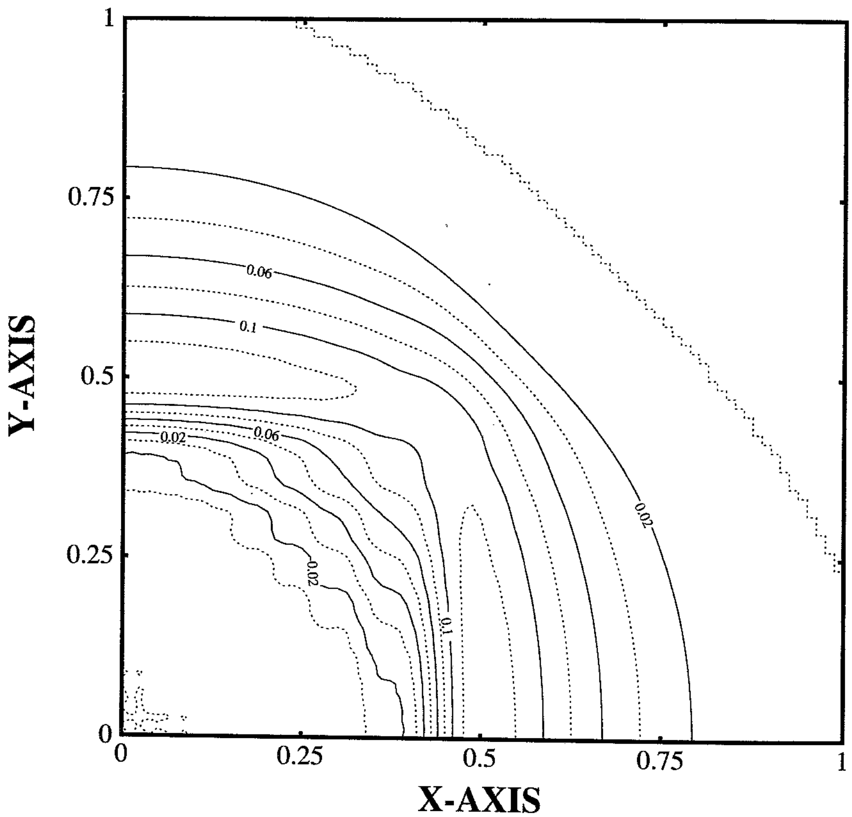


FIGURE 2. Domain decomposition-Galerkin solution for damped case