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# Some Recent Results on Schwarz Type Domain Decomposition Algorithms

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**Abstract.** Numerical experiments have shown that two-level Schwarz methods, for the solution of discrete elliptic problems, often perform very well even if the overlap between neighboring subregions is quite small. This is true to an even greater extent for a related algorithm, due to Barry Smith, where a Schwarz algorithm is applied to the reduced linear system of equations that remains after that the variables interior to the subregions have been eliminated. A supporting theory is outlined.

**1. Introduction** Over the last decade, a considerable interest has developed in Schwarz methods and other domain decomposition methods for elliptic partial differential equations. Among them are *two-level, additive Schwarz* methods first introduced in 1987; cf. Dryja and Widlund [16,13,17,18,29]. As shown in Dryja and Widlund [18], a number of other domain decomposition methods, in particular those of Bramble, Pasciak, and Schatz [3,4], can also be derived and analyzed using the same framework. Recent efforts by Bramble, Pasciak, Wang, and Xu [5], and Xu [30] have extended the general framework making a systematic study of multiplicative Schwarz methods possible. The multiplicative algorithms are direct generalizations of the original alternating method discovered more than 120 years ago by H.A. Schwarz [23]. For other current projects, which also use the Schwarz framework, see Dryja, Smith, and Widlund [15], Dryja and Widlund [19,21] and Widlund [29]. Proofs of most of the results of this paper can be found in Dryja and Widlund [20] or can be derived straightforwardly using the same technical tools. Here we will only discuss the additive algorithms.

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We begin our discussion by reexamining the block-Jacobi/conjugate gradient method applied to finite element approximations of linear, second order, elliptic problems. Upper and lower bounds on its rate of convergence are given. This analysis pinpoints the weakness of the method and suggests two means of improvement, namely the introduction of overlap between the diagonal blocks and the addition of a component of the preconditioner corresponding to a second, or more, levels of discretization. When a two-level method is used, the restrictions of the discrete elliptic problem to overlapping subregions, into which the given region has been decomposed, are solved exactly or approximately. In addition, in order to enhance the convergence rate, the preconditioner includes a global problem of relatively modest dimension.

The main result of our early study of two-level Schwarz methods shows that the condition number of the operator, which is relevant for the conjugate gradient iteration, is uniformly bounded if the overlap between neighboring subregions is sufficiently generous in proportion to the diameters of the subregions.

Our current work has been inspired very directly by several series of numerical experiments that indicate that the rate of convergence is quite satisfactory even for a small overlap and that the running time of the programs is often the smallest when the overlap is at a minimum. The number of conjugate gradient iterations is typically higher in such a case but this can be compensated for by the fact that the local problems are smaller and therefore cheaper to solve; cf. in particular Bjørstad, Moe, and Skogen [1], Bjørstad and Skogen [2], Cai [6,7], Cai, Gropp, and Keyes [8], and Skogen [24]. If the local problems are themselves solved by an iterative method, then a smaller overlap will give better conditioned local problems and therefore a higher rate of convergence; see Skogen [24] for a detailed discussion of this effect. All this work also shows that these algorithms are relatively easy to implement. Recent experiments by Gropp and Smith [22] for problems of linear elasticity provide strong evidence that these methods can be quite effective even for very large and ill-conditioned problems. We can show that the condition number of the preconditioned operator for the algorithm, introduced in 1987 by Dryja and Widlund [16], is bounded from above by $const.(1 + (H/\delta))$. Here $H$ measures the diameter of a subregion and $\delta$ the overlap between neighboring subregions; thus $H/\delta$ is a measure of the aspect ratio of the subregion common to two overlapping neighboring subregions.

Our main focus is a very interesting method, introduced in 1989 by Barry Smith [27,25]. It is known as the *vertex space* (or *Copper Mountain*) algorithm. Numerical experiments, for problems in the plane, have shown that this method converges quite rapidly even for problems, which were originally very ill-conditioned, even if the overlap is very modest; cf. Smith [25]. For additional work on variants of this method, see Chan and Mathew [9,10], Chan, Mathew, and Shao [11].

When Smith's algorithm is used, the given large linear system of algebraic equations, resulting from a finite element discretization of an elliptic problem, is first reduced in size by eliminating all variables associated with the interiors of the non-overlapping *substructures* $\Omega_i$ into which the region has been subdivided. The reduced problem is known as the *Schur complement* system and the remaining degrees of freedom are associated with the set $\{\partial\Omega_i\}$ of substructure boundaries, which form the interface $\Gamma$ between the substructures. The preconditioner of this

domain decomposition method, which can be classified as a *Schwarz method on the interface,* is constructed from a coarse mesh problem, with the substructures serving as elements, and a potentially large number of local problems. The latter correspond to an overlapping covering of $\Gamma$, with each subset corresponding to a set of adjacent interface variables.

Smith's main theoretical result, given in [27,25], is quite similar to that for the original two-level Schwarz method; the condition number of this domain decomposition algorithm is uniformly bounded for a class of second order elliptic problems provided that there is a relatively generous overlap between neighboring subregions that define the subdivision of the domain decomposition method. We can now show that the condition number of the iteration operator grows only in proportion to $(1+\log(H/\delta))^2$. Even for a minimal overlap of just one mesh width $h$, which corresponds to a block Jacobi method enhanced by a coarse space solver, this bound is as strong as those for the well known iterative substructuring methods considered by Bramble, Pasciak, and Schatz [3,4], Dryja [12], Dryja, Proskurowski, and Widlund [14], Smith [26], and Widlund [28]; cf. also Dryja, Smith and Widlund [15]. We also note that the successful iterative substructuring methods for problems in three dimensions, require the use of more complicated coarse subspace and that therefore Smith's method seems to offer an advantage.

## 2. The Finite Element Problem and Block-Jacobi Methods

We write our continuous and finite element elliptic problems as: Find $u \in V$, such that

$$a(u,v) = f(v), : \forall\, v \in\ V: ,$$

and, find $u_h \in V^h$, such that

$$a(u_h, v_h) = f(v_h), : \forall\, v_h \in\ V^h: , \tag{1}$$

respectively. We assume that the bilinear form $a(u,v)$ is selfadjoint and elliptic and that it is bounded in $V \times V$. In the case of Poisson's equation, the bilinear form is defined by

$$a_\Omega(u,v) = \int_\Omega \nabla u \cdot \nabla v : dx: . \tag{2}$$

The bilinear form $a(u,v)$ is directly related to the Sobolev space $H^1(\Omega)$ that is defined by the semi-norm and norm

$$|u|^2_{H^1(\Omega)} = a_\Omega(u,u) \text{ and } \|u\|^2_{H^1(\Omega)} = |u|^2_{H^1(\Omega)} + \|u\|^2_{L^2(\Omega)},$$

respectively. To avoid unnecessary complications, we confine our discussion to Poisson's equation, to homogeneous Dirichlet conditions, to continuous, piecewise linear finite elements and to a polygonal region $\Omega$ in two or three dimensions. It is well known that the resulting space $V^h \subset V = H^1_0(\Omega)$, i.e. it is conforming.

The finite element problem (1) can be written as a linear system of algebraic equations

$$Kx = b, \tag{3}$$

where the elements $k_{i,j} = a(\varphi_i, \varphi_j)$ of the *stiffness matrix* $K$ are given in terms of the finite element basis functions $\varphi_i$. The right hand side, $b$, has components $b_i = f(\varphi_i)$. In addition, we assume that the fine triangulation, associated with $V^h$ is obtained through the refinement of large elements, the substructures, $\Omega_i, i = 1, \cdots, N$. We call the corresponding finite element space $V^H$. We note that $V^H \subset V^h$ and we assume that the two triangulations are shape regular.

The standard block-Jacobi method corresponds to a block diagonal preconditioner $K_J$ which is the direct sum of diagonal blocks of $K$. Each block corresponds to an index set $\mathcal{N}_i$. Without limiting the generality, we here assume that $\mathcal{N}_i$ is associated with the nodes of $\Omega_i$ and some of those of its boundary $\partial\Omega_i$. There is no overlap between the index sets, but we note that any two sets

$$\Omega_{i,h} = \cup_{j \in \mathcal{N}_i} supp(\varphi_j)$$

that correspond to neighboring substructures have an overlap on the order of $h$. This follows from a well known property of the support, $supp(\varphi_j)$, of the basis functions. We also note that a subspace is associated to each index set. Additive Schwarz methods, of which this block Jacobi method is a simple example, are often described in terms of subspaces $V_i$ and projections onto these subspaces. Here we will primarily use matrix language.

The rate of convergence of this preconditioned conjugate gradient method is estimated in terms of the condition number

$$\kappa(K_J^{-1}K) = \lambda_{max}(K_J^{-1}K)/\lambda_{min}(K_J^{-1}K).$$

It is easy to show that the eigenvalues of $K^{-1}K_J$ are given by the Rayleigh quotient

$$\frac{\sum_{i=1}^N a(u_i, u_i)}{a(u,u)}, \quad u = \sum_{i=1}^N u_i, \quad u_i \in V_i. \tag{4}$$

In the block Jacobi case, and indeed for all methods considered in this paper, it is easy to derive an upper bound for the eigenvalues of $K_J^{-1}K$, which is independent of $h$ and $H$. A sharp lower bound can be derived quite directly from an explicit formula for the $u_i$. The crucial bound, obtained by using calculus, is

$$a(u_i, u_i) \leq (1 + C\frac{H}{h}) a_{\Omega_{i,h}}(u,u) + C\frac{1}{Hh}\|u\|^2_{L_2(\Omega_{i,h})}. \tag{5}$$

¿From this bound, which cannot be improved, we see that $\lambda_{min}(K_J^{-1}K)$, disappointingly, is on the order of $Hh$. We note that it was shown in Widlund [28] that any preconditioner built only from solvers on local subregions of diameter on the order of $H$, necessarily must satisfy $\kappa(P) \geq C/H^2$.

**3. Remedies** There are two simple devices that can help improve the performance of methods of this kind.

The first involves the introduction of a set $\{\Omega_{i,\delta}\}$ of overlapping subregions constructed from the substructures $\Omega_i$ by including all degrees of freedom of all nodes within the distance $\delta$ of the substructure. It is shown in Dryja and Widlund [20]

that this results in an estimate such as (5) with $h$ replaced by $\delta$. In the proof of this result the following representation is used,

$$a(P^{-1}u, u) = inf_{u=\sum u_i} a(u_i, u_i);$$

cf. formula (4). Here $P = \sum_{i=1}^{N} P_i$ is the operator relevant to the iterative method being considered, with the $P_i$ the $a$-orthogonal projection onto the subspace $V_i$. The subspaces $V_i$ no longer form a direct sum and the freedom of choice in the representation of $u$ can be turned to an advantage.

The second remedy is the introduction of a component corresponding to the coarse space $V^H$. Algorithmically, this is accomplished by adding a term $R^T(K^H)^{-1}R$ to the preconditioner based on the local problems alone. The matrix $R$, which represents a *restriction operator,* is rectangular, with one row for each interior vertex of the coarse triangulation, and it expresses the standard basis functions in $V^H$ in terms of the standard basis of $V^h$.

The following result is established in Dryja and Widlund [20].

**Theorem 1.** *For the method which uses the coarse space $V^H$ and an overlap of order $\delta$ between the subregions, the condition number of the additive Schwarz method satisfies*

$$\kappa(P) \leq C(1 + H/\delta).$$

*The constant is independent of the parameters $H$, $h$ and $\delta$.*

**4. Smith's Algorithm** This method has previously been described in Smith [27,25]. In the first step of this, and many other domain decomposition methods, the unknowns of the linear system of equations $Kx = b$ that correspond to the the interiors of the substructures are eliminated. We proceed as follows:

Let $K^{(i)}$ be the stiffness matrix corresponding to the bilinear form $a_{\Omega_i}(u_h, v_h)$ which represents the contribution of the substructure $\Omega_i$ to $a_\Omega(u_h, v_h)$. Let $x$ and $y$ be the vectors of nodal values that correspond to the finite element functions $u_h$ and $v_h$, respectively. The stiffness matrix $K$ of the entire problem can then be obtained by using the *method of subassembly* defined by the formula

$$x^T K y = \sum_i x^{(i)^T} K^{(i)} y^{(i)}.$$

Here $x^{(i)}$ is the subvector of nodal parameters associated with the closure of $\Omega_i$. The matrix $K^{(i)}$ is defined by

$$\begin{pmatrix} K_{II}^{(i)} & K_{IB}^{(i)} \\ K_{IB}^{(i)T} & K_{BB}^{(i)} \end{pmatrix}$$

and the subvector $x^{(i)}$ is divided into two, $x_I^{(i)}$ and $x_B^{(i)}$, corresponding to the variables which are interior to the substructure and those which are shared with other substructures, i.e. those associated with the nodal points of $\partial\Omega_i$. Since the interior variables of $\Omega_i$ are coupled only to other variables of the same substructure, they

can be eliminated locally and in parallel. The resulting reduced matrix is a Schur complement and is of the form

$$S^{(i)} = K_{BB}^{(i)} - K_{IB}^{(i)T} K_{II}^{(i)-1} K_{IB}^{(i)}. \tag{6}$$

¿From this follows that the Schur complement, corresponding to the global stiffness matrix $K$, is given by $S$ where

$$x_B^T S y_B = \sum_i x_B^{(i)T} S^{(i)} y_B^{(i)}. \tag{7}$$

Thus, if the local problems are solved exactly, what remains is to find a sufficiently accurate approximation of the solution of the linear system

$$S x_B = b_B. \tag{8}$$

Problem (8) is solved by an iterative method of additive Schwarz type. The most important difference between this algorithm and that of the previous subsection is that we are now working on $\Gamma = \bigcup \partial\Omega_i \setminus \partial\Omega$.

Smith's algorithm can now be described in terms of subspaces or alternatively in terms of index sets. We use the same coarse space as in the previous subsection, i.e. $V^H$, but we restrict its values to $\Gamma$. In the case when the original problem is two dimensional, we introduce one subspace for each interior edge and vertex of the substructures. An edge space is defined by setting all nodal values, except in the interior of the edge in question, to zero. Similarly, a vertex space is obtained by setting to zero all values at the nodes on $\Gamma$ that are at a distance greater than $\delta$. For many more details and a discussion of implementation details, see Smith [27,25].

In the case when the original problem is three dimensional, we introduce one subspace for each interior face, edge, and vertex. The elements of a face subspace vanish at all nodes on $\Gamma$ that do not belong to the interior of the face. Similarly, an edge space is supported in the strips of width $\delta$, which belong to the faces which have this edge in common. Finally, a vertex space is defined in terms of the nodes on $\Gamma$ that are within a distance $\delta$ of the vertex.

The following result is established in Dryja and Widlund [20]. The bound is stronger than in Theorem 1. Technically, this is related to the fact that we now work in the trace space $H^{1/2}(\Gamma)$ instead of the orginal space $H^1(\Omega)$.

**Theorem 2.** *The condition number of the vertex space method satisfies*

$$\kappa(P) \leq C(1 + \log(H/\delta))^2.$$

*The constant is independent of the parameters $H$, $h$ and $\delta$.*

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