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Spectral multidomain methods for the simulation of wave propagation in heterogeneous media

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ABSTRACT. Domain decomposition is used to approximate elastic wave propagation in heterogeneous media. The spatial discretization is based on a Fourier-Legendre collocation method stemming from a variational formulation of the problem at hand. Time marching techniques are discussed, and some numerical tests are presented.

1. Introduction

The use of numerical simulation in engineering seismology is motivated by the need of (a) evaluating the influence of the different factors that affect earthquake ground motion at sites on the Earth surface, i.e. source process, propagating medium, and near-surface geological irregularities, and (b) assessing a priori the earthquake hazard at a given site in the assumption that the foregoing factors are known. While numerical simulation by itself cannot provide yet reliable estimates of the seismic motion at a given site, mainly because of the lack of basic physical information, it remains possibly the only tool available to obtain physical understanding of wave motion in the strongly heterogeneous media characterizing the upper Earth crust and to interpret instrumental observations. The attention will be focused here on the modifications introduced in upward propagating earthquake waves by the presence of surficial soil deposits, i.e. alluvial valleys of different shapes, and of topographic irregularities such as mountains and canyons.

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For viscoelastic behavior of the materials involved, the most efficient numerical methods used in recent years to analyze such seismic site effects include finite elements, wavenumber expansion extended to vertically inhomogeneous media (Bard and Gariel, 1986), boundary integral equation methods (Bouchon et al., 1989), and finite differences with irregular grids (Moczo, 1989).

The use of the Fourier pseudospectral algorithm with explicit time-marching, originally introduced in forward modeling analyses of exploration geophysics which involve at most a flat Earth surface (e.g. Kosloff et al., 1984), has been systematically explored and adapted for engineering seismology applications in 2D models with an irregular free surface by Paolucci, 1989, Tagliani, 1989 and Faccioli, 1991. Numerical tests have shown that the method is competitive with the previous ones as regards computational efficiency. However, the presence of a non-periodic boundary condition, i.e. the stress-free Earth surface, causes the loss of the spectral accuracy of the method, and requires an *ad hoc* treatment of the stress discontinuity.

Illustrated in this paper is an alternative pseudospectral approach for 2D problems in which Fourier collocation with regular grid spacing is retained in the horizontal direction, while a Legendre collocation algorithm is introduced in the vertical direction. The clustering of collocation points at the extremes of the domain appears well suited for handling the free surface boundary condition, while the ensuing severe restriction on the time step size is alleviated by the domain decomposition, which also provides a natural framework to account for different material parameters in different regions of the model.

The performance of the method is illustrated by 1D tests on a layered halfspace with strong impedance contrast, and by a 2D model of a semi-elliptical valley embedded in a homogeneous halfspace for which analytical solutions are available.

2. Mathematical formulation of the problem

If we refer to the idealized situation depicted in Fig.1, the one-dimensional elastic wave equation can be formulated as

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial z}\left(\beta^2 \frac{\partial u}{\partial z}\right), \quad 0 \le z \le L, \quad t \ge 0 \tag{2.1}$$

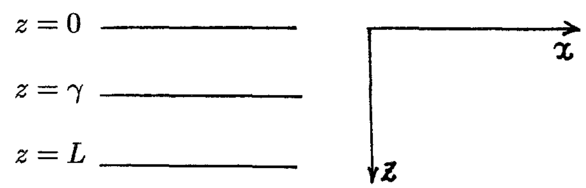


Fig. 1 -One–dimensional domain partition

where u is the horizontal displacement, and $\beta = \beta(z)$ is a given function (the velocity of propagation). Equation (2.1) needs to be supplemented by two initial conditions, say $u(z, t=0) = \varphi(z)$ and $\frac{\partial u}{\partial t}(z, t=0) = \psi(z)$, along with two boundary conditions. The one at $z = 0$ (the free surface) frequently takes the form of a zero stress, i.e.

$$\frac{\partial u}{\partial z} = 0 \;\; at \;\; z = 0, \;\; t \geq 0. \tag{2.2}$$

For what concerns the lower boundary, a convenient approach is to enforce the radiation condition

$$\frac{\partial u}{\partial z} = \frac{1}{\beta}\frac{\partial u}{\partial t} \;\; at \;\; z = L, \;\; t \geq 0 \tag{2.3}$$

A two-dimensional model is easily derived considering the domain $\Omega = \{(x, z) : -M \leq x \leq M, \; 0 \leq z \leq L\}$, and assuming M large enough in order for u to be considered periodic at $x = -M$ and $x = M$. In such a case, the model equation becomes

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial z}(\beta^2 \frac{\partial u}{\partial z}) + \frac{\partial}{\partial x}(\beta^2 \frac{\partial u}{\partial x}), \; 0 \leq z \leq L, \; -M \leq x \leq M, \; t \geq 0. \tag{2.4}$$

For the sake of simplicity, let us illustrate our multidomain formulation on the one-dimensional problem (2.1). To start with, we split the vertical interval $I = (0, L)$ into $I_1 \cup I_2$, where $I_1 = (0, \gamma]$ and $I_2 = [\gamma, L)$, for some $\gamma : \; 0 < \gamma < L$. Then for each $t \in (0, T)$, we look for $u_i(t)$, $i = 1, 2$ (the notation $u_i \equiv u \mid_{I_i}$ is understood)

$$u_1(t) = u_2(t) \;\; at \;\; z = \gamma \tag{2.5}$$

$$\sum_{i=1}^{2} \int_{I_i} \frac{\partial^2 u_i(t)}{\partial t^2} v dz = -\sum_{i=1}^{2} \int_{I_i} \beta^2 \frac{\partial u_i(t)}{\partial z} \frac{\partial v}{\partial z} dz + \beta \frac{\partial u_2(t)}{\partial t} v \mid_{z=L} \tag{2.6}$$

for any test-function v belonging to the space

$$H^1(0, L) = \{v \in L^2(0, L) : \frac{dv}{dz} \in L^2(0, L)\}$$

The *rationale* behind (2.5) is as follows: counterintegrating by parts on the right-hand side and playing suitably with the test functions v we find out that each u_i individually satisfies (2.1) in I_i. Besides, u_1 satisfies (2.2) and u_2 verifies (2.3). Further, since the test functions are continous across $z = \gamma$ we deduce from (2.6) that

$$\beta_1^2 \frac{\partial u_1(t)}{\partial z} = \beta_2^2 \frac{\partial u_2(t)}{\partial z} \;\; at \;\; z = \gamma \tag{2.7}$$

i.e., stresses are preserved across the interface point $z = \gamma$ (note that β may be different approaching $z = \gamma$ from different sides).

The extension to the two-dimensional case is straightforward provided we decompose Ω into $\Omega_1 = I_1 \times \{-M < x < M\}$ and $\Omega_2 = I_2 \times \{-M < x < M\}$. Denoting this time with u_i the restriction of u to Ω_i, $i = 1, 2$, instead of (2.6) we have

$$
\sum_{i=1}^{2} \int_{\Omega_i} \frac{\partial^2 u_i(t)}{\partial t^2} v\,dx dz = -\sum_{i=1}^{2} \int_{\Omega_i} \beta^2 \frac{\partial u_i(t)}{\partial z} \frac{\partial v}{\partial z} dx dz + \int_{-M}^{M} \frac{\partial u_2(t)}{\partial t} v \,|_{z=L}\, dx \tag{2.8}
$$

for all $t > 0$ and $v \in H^1_p(\Omega)$, where

$$
H^1_p(\Omega) = \{v \in L^2(\Omega) : \nabla v \in L^2(\Omega),\ v \text{ is periodic at } x = \pm M\}.
$$

In particular this still yields fulfillement of (2.4) upon Ω_1 and Ω_2, of the boundary conditions at $z = 0$ and $z = L$, along with the stress continuity equation (2.7) that holds now across the whole interface $\Gamma = \{-M < x < M,\ z = \gamma\}$.

The generalization to decomposition by more than two subdomains is also straightforward.

3. Spectral approximation by the Legendre collocation method

Let I=(-1,1) denote a reference interval, N a positive integer (the polynomial degree), and $\overline{z}_j$, $0 \le j \le N$, the roots of $(1 - z_j^2)L'_N(z)$, L_N being the Legendre polynomial of degree N. We order $\overline{z}_j$ from left to right in a way that $\overline{z}_0 = -1$, $\overline{z}_N = 1$; the other points are simmetrically distributed around $z = 0$. Let $\overline{\omega}_j$ denote the weights associated with $\overline{z}_j$ in the Gauss-Lobatto formula. We remind that

$$
\sum_{j=0}^{N} \varphi(\overline{z}_j)\overline{\omega}_j = \int_{-1}^{1} \varphi(z) dz \tag{3.1}
$$

for all $\varphi \in \mathbb{P}_{2N-1}$ (hereby, $\mathbb{P}_n$ denotes the vector space of algebraic polynomials of degree $\le n$). For details see Davis and Rabinowitz, 1985 or Canuto et al., 1988.

The we map $\overline{z}_j$ into $z_j^{(i)}$ according to the transformation $I \to I_i$, i=1,2 and set $\omega_j^{(i)} = \overline{\omega}_j(meas(I_i)/2)$. Moreover we define

$$
(\varphi, \psi)_N^i = \sum_{j=0}^{N} \varphi(z_j^{(i)})\psi(z_j^{(i)})\omega_j^{(i)}, \quad i = 1, 2 \tag{3.2}
$$

and

$$V_N = \{\varphi \in C^\circ[0,L] : \varphi\,|_{I_i} \in \mathbb{P}_N,\ \ i = 1,2\}.$$

The spectral approximation to the multidomain problem (2.5), (2.6) is defined as follows.

For each $t > 0$ we look for $u_N^{(i)} \in \mathbb{P}_N$, i=1,2, such that

$$u_N^{(1)}(t) = u_N^{(2)}(t)\ \ at\ \ z = \gamma \tag{3.3}$$

(3.4)

$$\sum_{i=1}^{2}\Big(\frac{\partial^2 u_N^{(i)}}{\partial t^2}, v_N\Big)_N^i = -\sum_{i=1}^{2}\Big(\beta^2\frac{\partial u_N^{(i)}(t)}{\partial z}, \frac{\partial v_N}{\partial z}\Big)_N^i + \beta\frac{\partial u_N^{(2)}(t)}{\partial z} v_N\,|_{z=L}$$

for all test functions $v_N \in V_N$.

A differential interpretation can be provided owing to the property of exactness (3.1), which in turns implies:

$$(\varphi,\psi)_N^i = \int_{I_i}\varphi\psi dz\ \ if\ \ \varphi\psi \in \mathbb{P}_{2N-1} \tag{3.5}$$

As a matter of fact, using a standard technique from (3.4) we obtain

$$L_N^{(i)}u_N^{(i)}(t) \equiv \frac{\partial^2 u_N^{(i)}(t)}{\partial t^2} - \frac{\partial}{\partial z}\Big[J_N^i\Big(\beta^2\frac{\partial u_N^{(i)}}{\partial z}\Big)\Big] = 0 \tag{3.6}$$

$$at\ \ z_j^{(i)},\ \ 1 \le j \le N-1,\ \ i = 1,2.$$

For any function $\varphi \in C^\circ(I_i)$, $J_N^i\varphi \in \mathbb{P}_N$ denotes the polynomial interpolating φ at the Legendre points $z_j^{(i)}$, $j = 0,....,N$. For each j, (3.6) is obtained counterintegrating by parts in (3.4) and taking as v_N the Lagrangian polynomial $\psi_j^{(i)}(z)$ such that $\psi_j^{(i)}(z_k^{(i)}) = \delta_{kj}$ for $0 \le k \le N$. The boundary conditions stemming from (3.4) are obtained similarly, setting now $v_N = \psi_0^{(1)}$ or $v_N = \psi_N^{(2)}$, and taking the following "weak" form:

$$\frac{\partial u_N^{(1)}(t)}{\partial z} = -\omega_0^{(1)}L_N^{(1)}u_N^{(1)}(t)\ \ at\ \ z = 0, \tag{3.7}$$

$$\frac{\partial u_N^{(2)}}{\partial z} - \frac{1}{\beta}\frac{\partial u_N^{(2)}}{\partial t} = \omega_N^{(2)}L_N^{(2)}u_N^{(2)}(t)\ \ at\ \ z = L. \tag{3.8}$$

Since both $\omega_0^{(1)}$ and $\omega_N^{(2)}$ behave like $1/N^2$ as N goes to infinity, it follows that (3.7) and (3.8) are the "relaxed" form of (2.2) and (2.3), respectively.

Similarly, from (3.4) we can obtain the following interface transmission relationship

$$\beta_1^2 \frac{\partial u_N^{(1)}(t)}{\partial z} - \beta_2^2 \frac{\partial u_N^{(2)}(t)}{\partial z} = \omega_N^{(1)} L_N^{(1)} u_N^{(1)}(t) - \omega_0^{(2)} L_N^{(2)} u_N^{(2)}(t) \tag{3.9}$$

Collecting (3.3) and (3.6)-(3.9), the problem is now closed, and provides a system of differential algebraic equations that should be discretized in time. At this stage, at least three different approaches can be pursued.

The first one is based on a fully explicit, second order leap-frog scheme. The results that are presented in the next section are obtained by this method.

A second approach entails advancing by a fully implicit Newmark method.

A latter possibility consists of advancing the differential equation (3.4) by the explicit leap-frog scheme at the interface solely. This would provide the updated value of $u_N^{(i)}$ at $z = \gamma$ to be used as boundary data for either I_1 and I_2, where the Newmark method can therefore be applied yielding two completely independent subproblems. This approach is based on an idea of Dawson and Dupont.

4. Numerical results

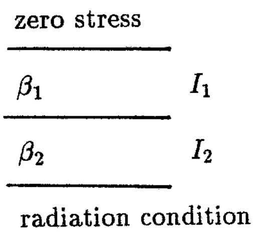
In our calculation, the time derivative in (2.1) and (2.4) is discretized by an explicit second order leap-frog scheme, taking as time-step $\Delta t = \min_i(\Delta t_i)$ where each Δt_i is obtained through a Von Neumann analysis, and reads

$$\Delta t_i = 2\sqrt{2}\frac{L_i}{max(\beta_i)} \cdot \frac{1}{N_i^2} \tag{4.1}$$

Here L_i= length of the i-th subdomain, while N_i= degree of the spectral solution in the i-th subdomain.

Because of (4.1) the number of discretization nodes should be kept small, and this condition can be satisfied by the multidomain technique.

4.1 1-D analysis. We first consider a model consisting of two parallel layers I_1 and I_2 (whose depth is 320 m), with velocity ratio $\frac{\beta_1}{\beta_2} = \frac{1}{4}$ (β_1=3000 m/s), 33 nodes of discretization in each subdomain and Δt=0.0002 sec, with boundary conditions illustrated in Fig. 2a. The upper boundary is assumed to be stress-free, to represent the Earth surface. Excitation is taken as a double impulse of assigned bandwith impinging at the base. The synthetic seismogram calculated at the surface is illustrated in Fig. 2b. In Fig. 2c the spectral ratio (Fourier spectrum of surface response/ input spectrum) for the computed seismogram is compared with the exact solution available for this simple model.



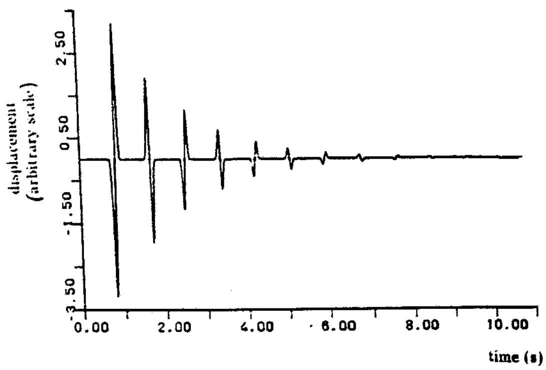


Fig.2a -Data for one–dimensional simulation

Fig.2b -Synthetic seismograms at the surface for the one–dimensional simulation

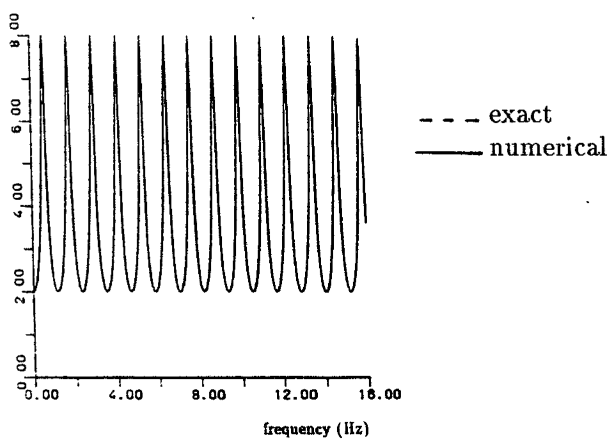


Fig. 2c -Spectral ratio for the computed seismogram

4.2 2-D analysis. A 2-D analysis was also performed, limited to two subdomains. The model is illustrated in Fig. 3a and represents an elliptical alluvial valley with 1:2 axis ratio, completely contained within the layer Ω_1. The ratio of the propagation velocity in alluvium, β_1 (3000 m/s) to that of the surrounding material, β_2, is 1/2. Each subdomain is vertically discretized with 20 nodes, while a constant Δx=20 m is used in the horizontal direction, and Δt=0.0002 sec. The excitation consists of a Ricker wavelet, with f_p=7.5 Hz, vertically impinging at the base of the model. The synthetic seismograms obtained at evenly spaced receivers on the surface are illustrated in Fig. 3b. A comparison between the surface spectral ratios calculated from synthetic seismograms at a frequency of

8.4 Hz and those obtained by a closed form solution (Wong et al., 1974) is shown in Fig. 3c. The small observed discrepancies in the spectral ratios are mainly due to the coarse grid used in the horizontal direction.

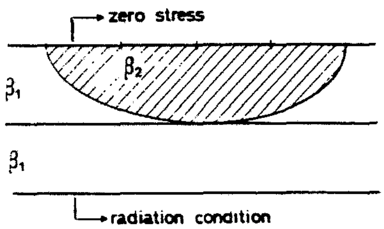


Fig.3a -Two–dimensional problem description

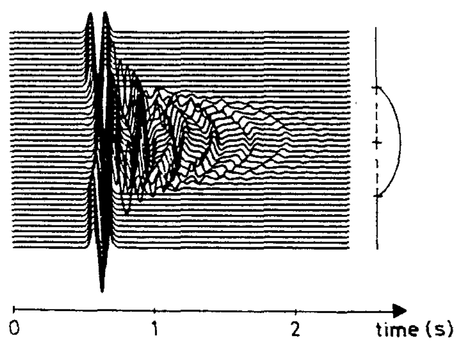


Fig.3b -Synthetic seismograms at the surface for the two-dimensional problem.

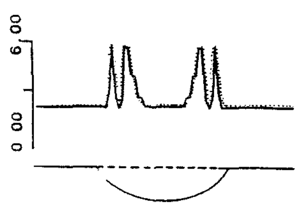


Fig.3c -Spectral ratio for the computed seismograms

5. Conclusions

The proposed method appears to provide accurate numerical solutions to 1D and 2D wave propagation problems encountered in the analysis of seismic site effects. The multidomain description allows an effective reduction of the number of vertical discretization points, and thus alleviate the heavy restriction on the time step size imposed by the explicit integration scheme and by the clustering of points near to the boundary. However, further work appears desirable to clarify the choice of the subdomains and of the number of discretization points when more complex heterogeneities are present.

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