Contemporary Mathematics
Volume 157, 1994

# A Domain Decomposition for The Transport Equation

LUCIA GASTALDI

Abstract. We apply the domain decomposition to a transport equation in two dimensions and to its discretization by finite elements. The decomposition leads to a family of problems coupled through interface equations (Steklov- Poincaré equations), which are resolved via the Richardson iterative procedure.

In the mathematical modelling of several problems in Physics and Engineering the following nonlinear hyperbolic equation arises

$$\frac{\partial u}{\partial t} + \operatorname{div} \mathbf{b}(u) = f \text{ in } \Omega\times]0,T[,$$

with initial and boundary conditions. Here $\Omega$ is a bounded domain in $\Re^2$ with Lipschitz, piecewise $\mathbf{C}^{1,1}$ boundary $\partial\Omega$ and $\mathbf{b}$ is a vector valued function depending on $x,t$ and on the unknown $u$.

Using an implicit scheme to advance in time, combined with some linearization technique we are led to consider the linear transport problem:

$$\text{(1)} \qquad \begin{cases} Lu \equiv \operatorname{div}(\mathbf{b}u) + b_0 u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega^{in}. \end{cases}$$

Here $\mathbf{b}$, $b_0$, $f$ and $g$ are known functions (which may depend on the values of $u$ at the preceding time step). In the boundary condition we have used the following notation: for $\Sigma \subset \partial\Omega$ ($\mathbf{n}$ outward normal) we set

$$\begin{aligned} \Sigma^{in} &= \{x \in \tilde{\Sigma} : \ (\mathbf{b}{\cdot}\mathbf{n})(x) < 0\}, \\ \Sigma^{out} &= \{x \in \tilde{\Sigma} : \ (\mathbf{b}{\cdot}\mathbf{n})(x) > 0\}, \\ \Sigma^{0} &= \Sigma \backslash (\overline{\Sigma^{in}}^{0} \cup \overline{\Sigma^{out}}^{0}), \end{aligned}$$

where $\tilde{\Sigma}$ is the largest subset of $\Sigma$ where $\mathbf{n}$ is defined and Lipschitz continuous.

---

1991 *Mathematics Subject Classification.* 65N30
Work partially supported by IAN/CNR (Pavia,Italy) and MURST 40% Analisi Numerica e Matematica computazionale

The final version of this paper will be submitted for publication elsewhere.

We will assume the following hypotheses on the data:

$$\mathbf{b} \in \mathbf{W}^{1,\infty}(\Omega), \; b_0 \in \mathbf{L}^{\infty}(\Omega), \; f \in \mathbf{L}^2(\Omega); \tag{2}$$

$$\frac{1}{2}\operatorname{div}\mathbf{b} + b_0 \geq \beta > 0, \text{ in } \Omega; \tag{3}$$

$$g \in \mathbf{L}^2_{\mathbf{b}}(\partial\Omega^{in}), \tag{4}$$

where for $\Sigma \subset \partial\Omega$

$$\mathbf{L}^2_{\mathbf{b}}(\Sigma) = \{v : \sqrt{|\mathbf{b}{\cdot}\mathbf{n}|}v \in \mathbf{L}^2(\Sigma)\}.$$

Under these assumptions, the solution $u$ to (1) exists, is unique and satisfies:

$$u_{|\partial\Omega^{out}} \in \mathbf{L}^2_{\mathbf{b}}(\partial\Omega^{out}). \tag{5}$$

In order to solve problem (1) we adopt a domain decomposition method associated with the discretization by finite elements.

More precisely, let us partition $\Omega$ into two nonoverlapping subdomains $\Omega_1$ and $\Omega_2$ and denote by $\Gamma$ their common boundary, that is

$$\Gamma = \partial\Omega_1 \cap \Omega = \partial\Omega_2 \cap \Omega.$$

In addition, $\Gamma_i^{in}$ and $\Gamma_i^{out}$ stand respectively for the inflow and the outflow part of $\Gamma$ with respect to $\Omega_i$.

Then problem (1) is equivalent to the two subproblems:

$$\begin{cases} Lu_1 = f \text{ in } \Omega_1, \\ u_1 = g \text{ on } \partial\Omega^{in} \cap \partial\Omega_1^{in}, \\ u_1 = u_2 \text{ on } \Gamma_1^{in}, \end{cases} \tag{6}$$

$$\begin{cases} Lu_2 = f \text{ in } \Omega_2, \\ u_2 = g \text{ on } \partial\Omega^{in} \cap \partial\Omega_2^{in}, \\ u_2 = u_1 \text{ on } \Gamma_2^{in}. \end{cases} \tag{7}$$

Notice that the last boundary conditions in (6) and (7) express the continuity of the solution along the interface, where $\Gamma$ is not tangential to the transport field $\mathbf{b}$. Hence the two subproblems turn out to be coupled unless either $\Gamma_1^{in}$ or $\Gamma_2^{in}$ are empty. In these cases we have a simplification of the problem: the two subproblems can be solved sequentially by a single sweep; even more, when both $\Gamma_1^{in}$ and $\Gamma_2^{in}$ are empty, the two subproblems are independent from each other. But when the streamlines cross several times the interface or there is a circulating transport, the subproblems are strictly coupled and some strategies are needed to resolve them in practice.

There are two possibilities: the first one is to introduce an iteration-by-subdomain procedure, that is:

($a$) assign $\psi_1^0$ in $\mathbf{L}^2_{\mathbf{b}}(\Gamma_1^{in})$;
($b$) solve (6), with $u_1 = \psi_1^0$ on $\Gamma_1^{in}$;
($c$) set $\psi_2^0 =_{def} u_{1|\Gamma_1^{out}}$,
($d$) solve (7), with $u_2 = \psi_2^0$ on $\Gamma_2^{in}$;
($e$) set $\psi_1^1 =_{def} u_{2|\Gamma_2^{out}}$;
($f$) go to ($b$) and iterate.

Otherwise, we can reduce the coupled problems (6) and (7) to an equation involving only the value along the interface. To this aim let us consider, for every $\psi_i \in \mathbf{L}^2_{\mathbf{b}}(\Gamma_i^{in})$, $i = 1, 2$, the solution $u_{\psi i}$ of the problem:

$$\begin{cases} Lu_{\psi i} = f \text{ in } \Omega_i, \\ u_{\psi i} = g \text{ on } \partial\Omega^{in} \cap \partial\Omega_i^{in}, \\ u_{\psi i} = \psi_i \text{ on } \Gamma_i^{in}, \end{cases} \tag{8}$$

Then we define the operators:

$$\Sigma_i : \mathbf{L}^2_{\mathbf{b}}(\Gamma_i^{in}) \mapsto \mathbf{L}^2_{\mathbf{b}}(\Gamma_i^{out}) \quad i = 1, 2 \tag{9}$$

given by

$$\Sigma_i \ \psi_i \ = \ u_{\psi i|\Gamma_i^{out}}. \tag{10}$$

Then the Steklov-Poincaré equation is:

$$S \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \tag{11}$$

where

$$S = \begin{pmatrix} \mathbf{b}\cdot\mathbf{n}_1 & 0 \\ 0 & \mathbf{b}\cdot\mathbf{n}_2 \end{pmatrix} \begin{pmatrix} \mathbf{I} & -\Sigma_2 \\ -\Sigma_1 & \mathbf{I} \end{pmatrix}. \tag{12}$$

Once this equation is solved, we can reconstruct the solution in the whole domain by solving the problems (8) for $i = 1, 2$.

To calculate a solution of the Steklov-Poincaré equation (11) we apply the Richardson iterative algorithm:

$$\begin{pmatrix} \psi_1^{n+1} \\ \psi_2^{n+1} \end{pmatrix} = (\mathbf{I} + QS) \begin{pmatrix} \psi_1^n \\ \psi_2^n \end{pmatrix}, \tag{13}$$

where

$$Q = \begin{pmatrix} \mathbf{I} & \Sigma_2 \\ \Sigma_1 & \mathbf{I} \end{pmatrix} \begin{pmatrix} \frac{1}{\mathbf{b}\cdot\mathbf{n}_1} & 0 \\ 0 & \frac{1}{\mathbf{b}\cdot\mathbf{n}_2} \end{pmatrix} \tag{14}$$

is the preconditioning operator.

We remark that the sequence, generated by this iterative algorithm, is the same as the one produced by the iteration-by-subdomain procedure ($a$)-($f$). Moreover, the following convergence theorem holds:

**Theorem 1** *For every* $(\psi_1^0, \psi_2^0)$ *the sequence defined in* (13) *converges to the solution of the Steklov-Poincaré equation* (11).

This result (which holds under mild assumptions on the geometry of the interface) is based upon sharp estimates of the $\mathbf{L}^2_\mathbf{b}$-norm of the outflow value of $u_{\psi i}$ in terms of the $\mathbf{L}^2_\mathbf{b}$-norm of the inflow value $\psi_i$ (see [1] for the details).

Let us consider a regular and quasi-uniform triangulation $\mathcal{T}_h(\Omega)$ of $\Omega$ compatible with the partition, that is

$$
\begin{aligned}
&\mathcal{T}_h(\Omega) = \cup_{i=1}^2 \mathcal{T}_h(\Omega_i),\\
&\text{if } \ell \text{ is a side of } T \in \mathcal{T}_h \text{ such that } T \cap \Gamma = \ell,\\
&\text{then either } \ell \in \Gamma^0, \text{ or } \ell \in \Gamma_1^{in}, \text{ or } \ell \in \Gamma_1^{out}.
\end{aligned}
\tag{15}
$$

Then let $\Sigma_{ih}$ be the discrete counterparts of $\Sigma_i$, obtained applying a finite element method to approximate the solutions of problems (6) and (7). Hence given $(\psi_{1h}, \psi_{2h})$, piecewise polynomial functions along $\Gamma_1^{in} \times \Gamma_2^{in}$, we have

$$
\begin{aligned}
\Sigma_{1h} &: \psi_{1h} \mapsto u_{h\psi 1|\Gamma_1^{out}},\\
\Sigma_{2h} &: \psi_{2h} \mapsto u_{h\psi 2|\Gamma_2^{out}},
\end{aligned}
\tag{16}
$$

where $u_{h\psi 1}$ and $u_{h\psi 2}$ are the approximate solutions of (6) and (7).

Then the discrete Steklov-Poincaré equation is:

$$
S_h \begin{pmatrix} \psi_{1h} \\ \psi_{2h} \end{pmatrix} = 0,
\tag{17}
$$

where

$$
S_h = \begin{pmatrix} \mathbf{b}\cdot\mathbf{n}_1 & 0 \\ 0 & \mathbf{b}\cdot\mathbf{n}_2 \end{pmatrix} \begin{pmatrix} \mathbf{I} & -\Sigma_{2h} \\ -\Sigma_{1h} & \mathbf{I} \end{pmatrix}.
\tag{18}
$$

Analogously, we can write also the discrete iterative procedure:

$$
\begin{pmatrix} \psi_{1h}^{n+1} \\ \psi_{2h}^{n+1} \end{pmatrix} = (\mathbf{I} + Q_h S_h) \begin{pmatrix} \psi_{1h}^{n} \\ \psi_{2h}^{n} \end{pmatrix},
\tag{19}
$$

where

$$
Q_h = \begin{pmatrix} \mathbf{I} & \Sigma_{2h} \\ \Sigma_{1h} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \frac{1}{\mathbf{b}\cdot\mathbf{n}_1} & 0 \\ 0 & \frac{1}{\mathbf{b}\cdot\mathbf{n}_2} \end{pmatrix}.
\tag{20}
$$

In order to analyze the convergence of the discrete sequence defined in (19), we must specify the finite element method we work with. Among the available methods to discretize the transport equation, we have chosen the *discontinuous Galerkin* and the *streamline diffusion* methods (see [2], and the references therein). These methods are both consistent and enjoy the same accuracy property. We describe briefly their application to the model transport equation (1); the extension to the related subproblems is straightforward.

The discontinuous Galerkin method consists in taking the subspace of $\mathbf{L}^2$ made by piecewise polynomial functions of degree $k$, $P_k$:

$$V_h(\Omega) = \{v \in \mathbf{L}^2(\Omega) : \; v_{|T} \in \mathbf{P}_k(T), \; \forall T \in \mathcal{T}_h\}.$$

Then the discretization of (1) reads: *find $u_h \in V_h(\Omega)$ such that $\forall v \in V_h(\Omega)$*

(21)
$$\sum_{T\in\mathcal{T}_h} \int_T (Lu_h - f)\, v\, dx = \int_{\partial\Omega^{in}} \mathbf{b}{\cdot}\mathbf{n}(u_h - g) v\, ds + \sum_{T\in\mathcal{T}_h} \int_{\partial T^{in}\cap\Omega} \mathbf{b}{\cdot}\mathbf{n}_T [u_h]_T (v)_T^i\, ds,$$

where we have used the following notation ($\mathbf{n}_T$ unit outward normal to the boundary of $T$):

$$\begin{aligned}
(v)_T^e &= \lim_{t\to 0^+} v(x + t\mathbf{n}_T),\\
(v)_T^i &= \lim_{t\to 0^-} v(x + t\mathbf{n}_T),\\
[v]_T &= (v)_T^i - (v)_T^e.
\end{aligned}$$

The integral along $\partial\Omega^{in}$ enforces, in a weak way, the inflow boundary condition of (1), while the sum in the right hand side expresses, still in a weak way, the continuity of the discrete solution across the interelement boundaries, where $\mathbf{b}\cdot\mathbf{n}_T \neq 0$. For this reason, when we apply (21) to the subproblems (6) and (7), we obtain a scheme which is equivalent to the discretization of (1). Moreover, the following convergence result for the discrete iterative algorithm to solve the Steklov-Poincaré equation holds:

**Theorem 2** *Assume that the mesh is uniform close to the points of $\Gamma$ where $\Gamma_1^{in}$ meets $\Gamma_1^{out}$, then the sequence (19) is convergent.*

In the streamline diffusion method we consider

$$C_h(\Omega) = V_h(\Omega) \cap \mathbf{C}^0(\Omega),$$

then the discrete version of (1) reads: *find $u_h \in C_h(\Omega)$ such that $\forall v \in C_h(\Omega)$*

$$\sum_{T\in\mathcal{T}_h} \int_T (Lu_h - f)\,(v + \delta Lv)\, dx = \int_{\partial\Omega^{in}} \mathbf{b}{\cdot}\mathbf{n}(u_h - g) v\, ds, \tag{22}$$

where $\delta = Ch$, for a suitable positive constant $C$.

Also in this case the inflow boundary condition is imposed in a weak sense, by the integral in the right hand side, hence the solution does not take the exact value at the boundary. Therefore the discretization of the split problem (6) and (7) is no longer equivalent to the discrete single domain problem (22). However the two domain solution has the same accuracy as the single domain one.

The numerical results obtained confirm the theory presented. As an example let us consider the following case:

$$\begin{aligned}
&\Omega = ]-1,1[\times]-1,1[\\
&\mathbf{b} = (-y, x), \quad b_0 = 1, \quad \text{for } (x,y) \in \Omega_0,\\
&f(x,y) = \begin{cases} 0, & \text{if } x^2 + y^2 > 1 \text{ and } x > 0,\; y > 0,\\ 1, & \text{otherwise}, \end{cases}\\
&g(1,y) = 0, \quad y \in ]0,1[,\\
&g(x,y) = 1, \quad \text{elsewhere along } \partial\Omega_0^{in}.
\end{aligned}$$

The domain $\Omega$ is subdivided into $\Omega_1 = ]-1,1[\times]-1,0[$ and $\Omega_2 = ]-1,1[\times]0,1[$. The mesh is uniform, generated by partitioning each side of $\Omega_0$ into $2N$ equal subintervals. The finite element spaces are made by piecewise linear polynomials, and $\delta = h$ in (22).

The related large, sparse, nonsymmetric algebraic systems are solved by means of the Generalized Minimal RESidual algorithm.

| | Streamline Diffusion | | | Discontinuous Galerkin | | |
|---|---|---|---|---|---|---|
| | One dom. | Two subdom. | | One dom. | Two subdom. | |
| $2N$ | Error | Error | NIT | Error | Error | NIT |
| 8 | .21441 | .19760 | 6 | .22587 | .22587 | 5 |
| 16 | .15161 | .14620 | 6 | .14385 | .14385 | 5 |
| 32 | .11342 | .11183 | 6 | .09138 | .09138 | 5 |

Table: $\mathbf{L}^2$-error and number of iterations (NIT) for different values of $2N$.

The Table shows a comparison between the $\mathbf{L}^2$-norm of the relative approximation error in the case of a single domain and of two subdomains depending on $2N$. The number of iterations of (19) is also given. An overall comparison is made between the two possible choices of finite element spaces (21) and (22).

Notice that the error is relatively large, due to the jump discontinuity of the solution. However this error is "concentrated" around the discontinuity line. This feature is present both in the single domain and in the two domain solutions. Actually, the two domain solution is as good as the single domain one: in fact for $2N = 32$ we have that the $\mathbf{L}^2$-norm of the difference beteewn the two solutions is $.5411E-8$ in the case of discontinuous Galerkin method (21) and $.1985E-1$ in the case of streamline diffusion method (22). The result provided by the discontinuous Galerkin method is much better than the other one, because the two domain problem is equivalent to the single domain problem in the former case.

The Table suggests also that the number of iterations of (19) is independent of the mesh size.

## REFERENCES

1. F. Gastaldi and L. Gastaldi, *On a domain decomposition for the transport equation: theory and finite element approximation*, IMA J. of Num. Anal. (to appear).
2. C. Johnson, "Numerical solutions of partial differential equations by the finite element method," Cambridge Univ. Press, Cambridge, 1987.

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PAVIA, 27100 PAVIA, ITALY