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Domain Decomposed Preconditioning For Faulted Geological Blocks

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ABSTRACT. The maturation of hydrocarbons in the earth is a complex physical process influenced by heat distribution, fluid flow, and geochemistry over a geological time scale. Time-dependent geometry imposed by geological faulting can further complicate the physics of the process. Due to the different scales of thermal diffusion and geological events, it is reasonable to assume that the temperature reaches equilibrium before further movement along a fault is introduced. Here, the steady-state thermal diffusion equation is solved using conjugate gradients (CG) with domain decomposed preconditioning on a domain that assumes geometries imposed by vertical faulting.

1. Domain Decomposition for Faulted Geological Blocks

1.1. Introduction. Numerical modeling of the maturation of hydrocarbons in the earth is progressively becoming a more important tool in the oil industry. Among the factors influencing the maturation process are heat distribution, fluid flow, geochemistry and geological faulting over a geological time scale.

In this paper one aspect of the overall model is considered. The steady-state heat equation on a domain that assumes geometries imposed by vertical faulting is solved using CG with domain decomposition preconditioning. In a physical sense, the steady-state assumption implies that temperatures have reached equilibrium before movement along the fault imposes a new geometry at some subsequent time. This is reasonable considering the time scales of geological events and thermal diffusion.

Geologists have observed that block movement in the earth can cause changes in the material properties of the fault zone, such as rock hardening. Since heat

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FIGURE 1. Geometry of moving blocks for two different configurations

flow across the fault is related to the material properties of the fault and boundary conditions, the temperature distribution depends on both the domain geometry and fault location.

1.2. Heat Flow on Faulted Geological Blocks. Geological faulting imposes time-dependent domain geometries as illustrated in Figure 1. The vertical fault suggests a natural partition of the entire domain Ω into the two subdomains Ω_1 and Ω_2 and the fault domain Ω_3, with Ω_3 by far the smallest of the three domains (exaggerated in Figure 1).

Portions of the subsurface can be exposed resulting from block movement. This converts interior nodes to boundary nodes. Thus, boundary conditions are dependent on the changing geometry. In addition, changes in the material properties of the fault zone correspond to temporal variations of the thermal conductivities in Ω_3.

Consider the numerical solution of a two-dimensional steady-state heat equation:

$$-\frac{\partial}{\partial x}\left(c_x\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(c_y\frac{\partial u}{\partial y}\right) = f, \qquad (x,y) \in \Omega,$$

subject to Dirichlet boundary conditions on $\partial\Omega$ and where $c_x, c_y \geq 0$. This self adjoint elliptic partial differential equation has simple anisotropy when $c_x \neq c_y$. The five-point finite difference discretization of the operator gives rise to a symmetric positive definite linear system. This linear system can be solved efficiently with the preconditioned conjugate gradient (PCG) method, provided an appropriate preconditioning is used [**2**].

1.3. Discretization. A staggered grid is used for discretization of the PDE, with thermal conductivities defined at locations midway between unknown temperatures. This generates a symmetric positive definite linear system which is solved via CG with domain decomposed preconditioning.

The ordering used within the three domains can be described as follows. In Ω_1 (the left block) the nodes are ordered from top to bottom and left to right. In Ω_2 (the right block) the nodes are ordered from top to bottom and right to left. The discretization for Ω_3 (the fault) consists of only two columns of nodes which are ordered top to bottom and left to right. This ordering induces a preferred flow of information in the incomplete factorization; information about the coefficients moves naturally toward the fault domain.

1.4. Matrix Structure. The domain ordering produces the following symmetric positive definite block linear system for the nth configuration of the process:

$$A^{(n)}u = \begin{bmatrix} A_{11} & & A_{13}^{(n)} \\ & A_{22} & A_{23}^{(n)} \\ A_{31}^{(n)} & A_{32}^{(n)} & A_{33}^{(n)} \end{bmatrix} \begin{bmatrix} u_1^{(n)} \\ u_2^{(n)} \\ u_3^{(n)} \end{bmatrix} = \begin{bmatrix} b_1^{(n)} \\ b_2^{(n)} \\ b_3^{(n)} \end{bmatrix}.$$

Movement of geological blocks induces changes on the linear system. Block movement converts certain interior nodes to boundary nodes. The number of nodes in the fault domain Ω_3 decreases with movement of the blocks. The order of the submatrices $A_{i3}^{(n)} = \left(A_{3i}^{(n)}\right)^T$, $i = 1, 2$, and $A_{33}^{(n)}$, also decreases. Similarly the solution $u_3^{(n)}$ and the right hand side $b_3^{(n)}$ also change in order. Changes in the material properties of Ω_3 can also cause variations in the coefficients of $A_{33}^{(n)}$. The submatrices A_{11} and A_{22} remain fixed with block movement.

The matrix $A^{(n)}$ admits the following block LU factorization:

$$A^{(n)} = \begin{bmatrix} L_1 & & \\ & L_2 & \\ A_{31}^{(n)}L_1^{-T} & A_{32}^{(n)}L_2^{-T} & S^{(n)} \end{bmatrix} \begin{bmatrix} L_1^T & & L_1^{-1}A_{13}^{(n)} \\ & L_2^T & L_2^{-1}A_{23}^{(n)} \\ & & I \end{bmatrix},$$

where L_i is the Choleski factorizations of the ith, $i = 1, 2$, diagonal blocks of $A^{(n)}$. And, $S^{(n)} = A_{33}^{(n)} - A_{31}^{(n)}A_{11}^{-1}A_{13}^{(n)} - A_{32}^{(n)}A_{22}^{-1}A_{23}^{(n)}$ is the Schur complement of A_{11} and A_{22} on the matrix $A^{(n)}$.

1.5. Preconditioning. A preconditioning consists of an approximate factorization of $A^{(n)}$. Let $\tilde{L}_i$, $i = 1, 2$, and $\tilde{L}_3^{(n)}$ be some incomplete Choleski factors of the diagonal blocks of $A^{(n)}$. Then, $\tilde{M}^{(n)}$ is an approximate factorization of $A^{(n)}$ and is defined as

$$\tilde{M}^{(n)} = \begin{bmatrix} \tilde{L}_1 & & \\ & \tilde{L}_2 & \\ A_{31}^{(n)}\tilde{L}_1^{-T} & A_{32}^{(n)}\tilde{L}_2^{-T} & M_s^{(n)} \end{bmatrix} \begin{bmatrix} \tilde{L}_1^T & & \tilde{L}_1^{-1}A_{13}^{(n)} \\ & \tilde{L}_2^T & \tilde{L}_2^{-1}A_{23}^{(n)} \\ & & I \end{bmatrix}.$$

When $M_S^{(n)}$ is an appropriate preconditioner for the Schur complement $S^{(n)}$, the matrix $\tilde{M}^{(n)}$ is symmetric positive definite (Goovaerts [3]). The solution of $\tilde{M}^{(n)}z = r$ for a given vector r, is obtained by solving the following systems:

$$\tilde{L}_1 y_1 = r_1, \quad \tilde{L}_2 y_2 = r_2, \quad M_S^{(n)} y_3 = r_3 - A_{31}^{(n)}\tilde{L}_1^{-T} y_1 - A_{32}^{(n)}\tilde{L}_2^{-T} y_2$$

and

$$z_3 = y_3, \quad \tilde{L}_1^T z_1 = y_1 - \tilde{L}_1^{-1}A_{13}^{(n)} z_3, \quad \tilde{L}_2^T z_2 = y_2 - \tilde{L}_2^{-1}A_{23}^{(n)} z_3.$$

The solution of these systems requires two approximate solves in Ω_1 and Ω_2. We do not use $\tilde{M}^{(n)}$. Instead we use $M^{(n)}$ which is defined next and requires only a single solve per subdomain.

The domain decomposition variation of the above factorization approximates $A_{3i}^{(n)}\tilde{L}_i^{-1}$ with $\tilde{A}_{3i}^{(n)}$, $(i = 1, 2)$, which are equivalent to the stencils of the incomplete Choleski factors for Ω_3 which communicate with Ω_1 and Ω_2. The factored preconditioning matrix $M^{(n)}$ defined by this is

$$M^{(n)} = \begin{bmatrix} \tilde{L}_1 & & \\ & \tilde{L}_2 & \\ \tilde{A}_{31}^{(n)} & \tilde{A}_{32}^{(n)} & \tilde{L}_3^{(n)} \end{bmatrix} \begin{bmatrix} \tilde{L}_1^T & & \tilde{A}_{13}^{(n)} \\ & \tilde{L}_2^T & \tilde{A}_{23}^{(n)} \\ & & \left(\tilde{L}_3^{(n)}\right)^T \end{bmatrix}.$$

The dependence on the block configuration (n) is restricted to Ω_3 and the connections it has with Ω_1 and Ω_2. Thus when the configuration of the blocks changes, the incomplete Choleski factors only have to be updated for Ω_3. For a large number of configurations, the computational cost can be reduced.

The preconditioning application $M^{(n)}z = r$ for a given vector r is performed by solving the following systems:

$$\tilde{L}_1 y_1 = r_1, \quad \tilde{L}_2 y_2 = r_2, \quad \tilde{L}_3^{(n)} y_3 = r_3 - \tilde{A}_{31}^{(n)} y_1 - \tilde{A}_{32}^{(n)} y_2$$
and
$$\left(\tilde{L}_3^{(n)}\right)^T z_3 = y_3, \quad \tilde{L}_1^T z_1 = y_1 - \tilde{A}_{13}^{(n)} z_3, \quad \tilde{L}_2^T z_2 = y_2 - \tilde{A}_{23}^{(n)} z_3.$$

Furthermore, parallelism is natural over Ω_1 and Ω_2.

Movement of the blocks requires an updating of the incomplete Choleski factors. However, the only incomplete Choleski factors required to be updated are for the submatrices associated with the fault itself.

The domain decomposed preconditioning $M^{(n)}$ is equivalent to an incomplete Choleski multiplicative Schwarz preconditioning with minimum overlap on the subdomains. Minimal overlap implies that no two domains share any interior nodes. A multiplicative Schwarz preconditioning with a larger overlap would require updating a larger number of nodes in the overlap region when the coefficients in the fault change. A relationship between overlapping and nonoverlapping domain decomposition is given in Chan and Goovaerts [**1**].

1.6. Convergence and Example. In this section convergence properties are examined; see Figure 2. An example of a sequence of block configurations is also presented. The ICCG(0) incomplete factorization of Meijerink and van der Vorst [4] is used on each subdomain with subdomain dependent ordering. Convergence rates of the domain decomposition preconditioning are compared for a single domain preconditioning and two different node orderings for the three domain case without block movement. The orderings in Figure 2 are: column ordering on a "single" domain where the columns are ordered left to right; "inward" ordering with the columns ordered left to right in Ω_1 and right to left in Ω_2 (node order increases toward the fault); and "outward" ordering with the columns ordered right to left in Ω_1 and left to right in Ω_2 (node order increases away from the fault). In each case the nodes within each column are

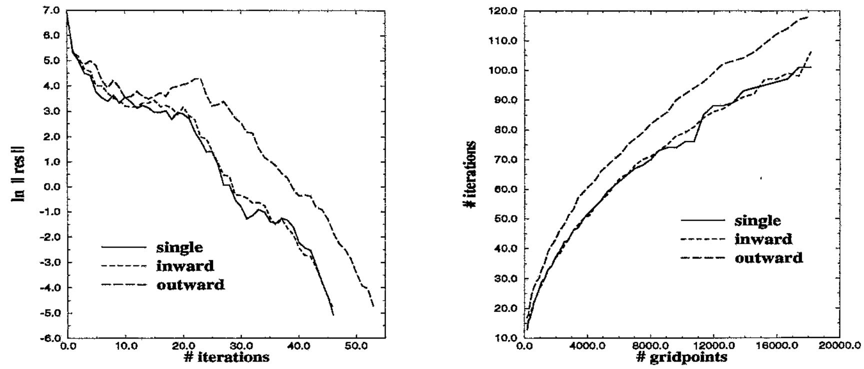

FIGURE 2. Convergence properties

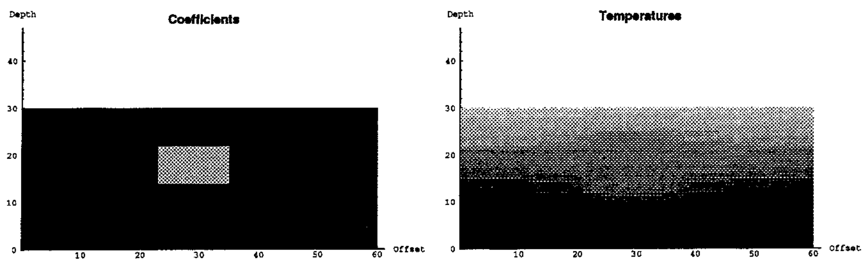

FIGURE 3. Coefficients and temperatures for initial configuration

ordered top to bottom. On the left of Figure 2 ln $\| r^k \|_\infty$ is plotted for an 81 x 41 grid, while on the right the number of iterations for convergence is displayed as a function of grid spacing. The coefficients for these convergence rates are described by the model on the left in Figure 3, where the contrast is 10 to 1, with the light shading denoting a coefficient of 10.

The computational example includes block movement with a single fault placed in the middle of a rectangular region. The coefficients and temperatures are displayed for three configurations in Figures 3, 4, and 5, where light and dark shades correspond to high and low temperatures respectively. The region of high conductivities is being sheared by the block movement.

Computations were completed on a Sun 4. The stopping criterion used requires that $\| r^k \|_\infty \leq 10^{-6} \| r^0 \|_\infty$. The convergence rates indicate that the domain decomposed preconditioning performed as well as a single domain preconditioning. The convergence rates did not deteriorate as a result of the additional parallelism.

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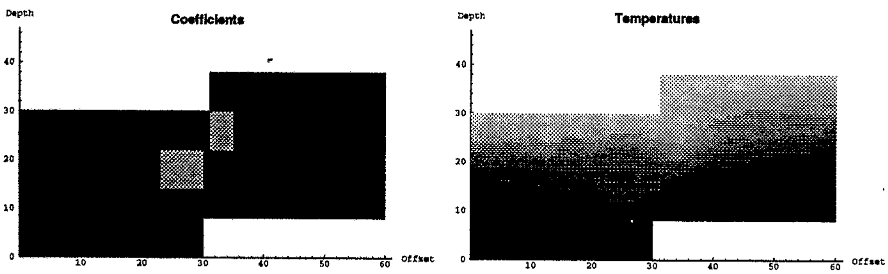

FIGURE 4. Coefficients and temperatures for intermediate configuration

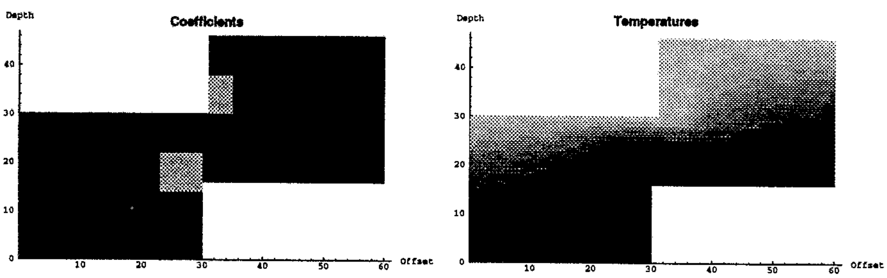

FIGURE 5. Coefficients and temperatures for final configuration

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