Contemporary Mathematics
Volume 157, 1994

A DOMAIN DECOMPOSITION METHOD USING SPARSE GRIDS

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ABSTRACT. We present preconditioners for linear systems arising from sparse and full grid discretizations of PDE's and discuss their application to the domain decomposition method.

1. FULL AND SPARSE GRIDS, FULL AND SPARSE GRID PROBLEMS

We consider a partial differential equation

$$Lu = f$$

in the unit square $\bar{\Omega} = [0,1]^2 \subset \mathbb{R}^2$ with a linear, elliptic operator L of second order and appropriate boundary conditions. For reasons of simplicity, we restrict ourselves to the case of homogeneous boundary conditions.

For the discretization of the problem, we use a grid $\Omega_{i,j}$ with grid sizes $h_x = 2^{-i}$ and $h_y = 2^{-j}$ in x- and y-direction which is associated to the space $V_{i,j}$ of piecewise bilinear functions vanishing on the boundary. The dimension of $V_{i,j}$ is $(2^i - 1)(2^j - 1) = O(2^{i+j})$. The Galerkin approach now leads to the linear system $L_{i,j}u_{i,j} = f_{i,j}$ that can be solved effciently by a multigrid method, for example. Usually, an equidistant grid $\Omega_{k,k}$ associated to the space $V_{k,k}$ with dimension of $O(2^{2k})$ is used. Note that this space can be decomposed as

$$V_{k,k} = \sum_{i=1}^{k} \sum_{j=1}^{k} V_{i,j}.$$

Alternatively, we consider the so-called sparse grid space

$$V_{k,k}^S = \sum_{i=1}^{k} \sum_{j=1}^{k+1-i} V_{i,j}$$

with dimension $n_{k,k}^S = (k-1) \cdot 2^k - 1 = O(k \cdot 2^k)$. $V_{k,k}^S$ is associated to the sparse grid $\Omega_{k,k}^S$ (see Figure 1). Note that a product type hierarchical basis is advisable for

1991 *Mathematics Subject Classification.* 65F10, 65N20, 65N30.
Key words and phrases. sparse grids, combination method, preconditioning.
partially supported by DFG-Sonderforschungsbereich SFB 0342
This paper is in final form and no version of it will be submitted for publication elsewhere.

0271-4132/94 $1.00 + $.25 per page

the representation of a function of $V^S_{k,k}$. For details, see [1], [10], [11]. The Galerkin approach now leads to the associated sparse grid problem

$$L^S_{k,k} u^S_{k,k} = f^S_{k,k}. \tag{1.1}$$

Note that $L^S_{k,k}$ is more densely populated than $L_{k,k}$. However, based on the hierarchical basis mentioned above, there exists a product type representation of $L^S_{k,k}$ allowing a matrix vector multiplication in $O(n^S_{k,k})$ operations.

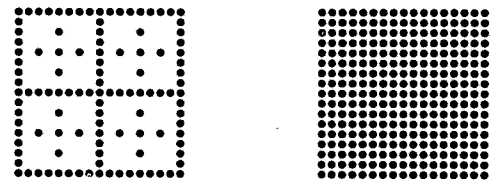

FIGURE 1. The sparse grid $\Omega^S_{4,4}$ and the associated full grid $\Omega_{4,4}$.

While the full grid problem possesses $O(4^k)$ unknowns, the sparse grid problem (1.1) only possesses $O(k \cdot 2^k)$ unknowns. The accuracy of the sparse grid solution $u^S_{k,k}$, however, is practically the same as for the full grid solution $u_{k,k}$. In [1] it is shown that both $|u - u_{k,k}|_E$ and $|u - u^S_{k,k}|_E$ are of the order $O(2^{-k})$, provided that the solution u is sufficiently smooth to satisfy

$$\left|\frac{\partial u^4}{\partial x^2 \partial y^2}\right| \leq C. \tag{1.2}$$

Thus, it is often cheaper to solve the sparse grid problem instead of the full grid problem. The solution $u^S_{k,k}$ of (1.1) can be obtained efficiently by special multigrid methods ([1], [6], [3]), that involve in general quite complicated data structures and algorithms. In the following, we present a simple preconditioner for (1.1).

2. THE COMBINATION METHOD AS A SPARSE GRID PRECONDITIONER

Consider now the so-called combination solution

$$u^C_{k,k} = \sum_{i=1}^{k} u_{i,k+1-i} - \sum_{i=1}^{k-1} u_{i,k-i} \tag{2.3}$$

that has been introduced in [4]. To obtain $u^C_{k,k}$, we have to solve k different problems $L_{i,j} u_{i,j} = f_{i,j}, i = 1,..,k, j = k+1-i$, each with about 2^k unknowns, and $k-1$ different problems $L_{i,j} u_{i,j} = f_{i,j}, i = 1,..,k-1, j = k-i$, each with about 2^{k-1} unknowns, and to combine their bilinearly interpolated solutions. This leads to an approximate solution of the sparse grid problem defined on $\Omega^S_{k,k}$. The method is illustrated in Figure 2.

Note that the solutions of the different problems arising in the combination method can be computed independently. Thus, the parallelization of the method is straightforward. For results, see [2] or [8]. The solution of each problem is computed on a standard grid. Therefore, the complicated data structures and

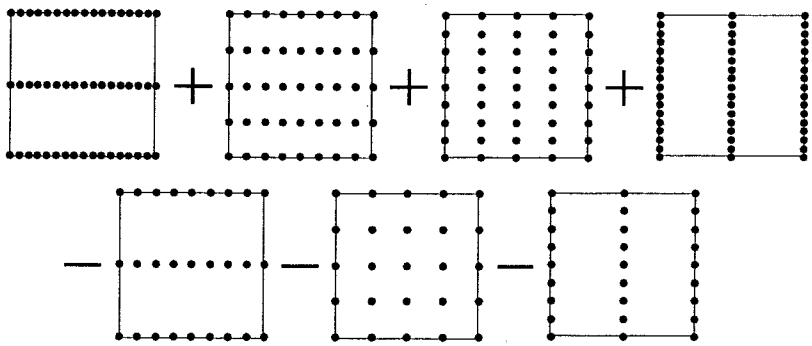

FIGURE 2. The linear combination of grids $\Omega_{i,j}$ and solutions $u_{i,j}$ with $i = 1, ..k, j = k + 1 - i$ and $i = 1, .., k - 1, j = k - i, k = 4$.

algorithms necessary for the sparse grid approach are avoided, and we can use standard codes. A multigrid method with semi-refinement to cope with the grid distorsion is recommended.

The combination method leads to the preconditioner

$$(B^S_{k,k}) = \sum_{i=1}^{k} P^{k,k,S}_{i,k+1-i} L^{-1}_{i,k+1-i} R^{i,k+1-i}_{k,k,S} - \sum_{i=1}^{k} P^{k,k,S}_{i,k-i} L^{-1}_{i,k-i} R^{i,k-i}_{k,k,S}. \tag{2.4}$$

for the sparse grid problem (1.1). Here, $P^{k,k,S}_{i,j} : V_{i,j} \to V^S_{k,k}$ denotes the interpolation operator and $R^{i,j}_{k,k,S} : V^S_{k,k} \to V_{i,j}$ denotes the restriction operator defined as the adjoint of $P^{k,k,S}_{i,j}$, $R^{i,j}_{k,k,S} = (P^{k,k,S}_{i,j})^T$.

It can be shown that $u^C_{k,k} = (B^S_{k,k})^{-1} f^S_{k,k}$. Note that the combination solution $u^C_{k,k}$ is not equal to u^S_k, but a very good approximation to it. It can be shown that

$$\kappa((B^S_{k,k}) L^S_{k,k}) = O(1), \tag{2.5}$$

and by using the combination method as a preconditioner within an appropriate iterative method like a conjugate gradient or a defect correction iteration, we obtain a simple algorithm that needs a number of iterations independent of k to solve the sparse grid problem to a prescribed accuracy.

However, since the discrete solution u^S_k differs from u by the discretization error anyway, it makes no sense to solve the sparse grid problem to machine accuracy. It is sufficient to perform only as many iterations as are necessary to reach the discretization error. To demonstrate this and the property (2.5) we consider the model problem $\Delta u = f$ with its exact solution $x \cdot (1-x) \cdot cos(x \cdot \pi/2) \cdot y \cdot (1-y) \cdot cos(y \cdot \pi/2)$ and Dirichlet boundary conditions. We perform conjugate gradient iterations for (1.1), preconditioned by (2.4). The convergence history of the iteration for the cases $k = 3, 4, .., 8$ is shown in Figure 3 (left). We clearly see that the convergence rate and thus the condition number of $(B^S_{k,k}) L^S_{k,k}$ are independent of k.

In the right part of Figure 3, we show a zoom of the convergence history for the first two iteration steps. Additionally, we indicate where the discretization error (horizontal lines) is reached. We see that the discretization error accuracy is already gained after only one iteration step, where just the combination method is

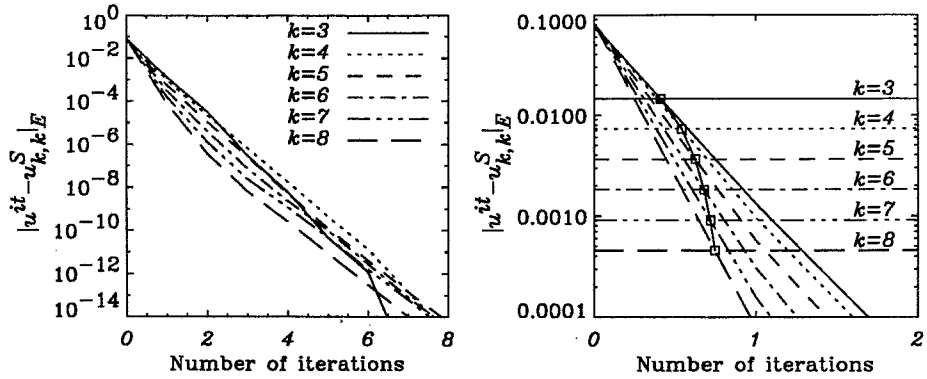

FIGURE 3. Convergence history of CG for preconditioner (2.4).

involved. Consequently, for any u that satisfies the requirement (1.2), the error of the combination solution is within the discretization error accuracy and only one iteration step is necessary.

3. THE COMBINATION METHOD USED IN A FULL GRID PRECONDITIONER

Now, we turn to the problem discretized on the full grid $\Omega_{k,k}$

$$L_{k,k}u_{k,k} = f_{k,k}. \tag{3.6}$$

Let $P^{k,k}_{i,j} : V_{i,j} \to V_{k,k}$ be the interpolation operator and let $R^{i,j}_{k,k} : V_{k,k} \to V_{i,j}$ be the restriction operator defined as the adjoint of $P^{k,k}_{i,j}$, $R^{i,j}_{k,k} = (P^{k,k}_{i,j})^T$. The preconditioner associated to the combination method can now be written as

$$B_{k,k} = \sum_{i=1}^{k} P^{k,k}_{i,k+1-i} L^{-1}_{i,k+1-i} R^{i,k+1-i}_{k,k} - \sum_{i=1}^{k} P^{k,k}_{i,k-i} L^{-1}_{i,k-i} R^{i,k-i}_{k,k}. \tag{3.7}$$

However, only functions of the sparse grid subspace $V^S_{k,k}$ are affected by $B_{k,k}$, and it can be shown that $B_{k,k}$ is of deficient rank. An iteration for the full grid problem preconditioned by $B_{k,k}$ only would converge to its fix point which is contained in the subspace $V^S_{k,k}$. However, the solution $u_{k,k} \in V_{k,k}$ is never reached. In the multigrid context, $B_{k,k}$ can be considered as a certain sort of multivariate additive coarse grid correction similar to [7].

Therefore, for the full grid problem we consider the following additive preconditioner

$$C_{k,k} = B_{k,k} + \omega \cdot diag(L_{k,k})^{-1} \tag{3.8}$$

that exhibits full rank. It incorporates $B^S_{k,k}$ on the subspace $V^S_{k,k}$ and a Jacobi type preconditioner on the full space $V_{k,k}$. Here, ω is a properly chosen damping parameter ($\omega = 1.0$, e.g.).

Let $\Phi^{GS}_{k,k}$ denote the Gauss-Seidel iteration operator on grid $\Omega_{k,k}$. Additionally, we consider the multiplicative iteration operator

$$M_{k,k} = \Phi^{GS}_{k,k} \cdot (I_{k,k} - B_{k,k}L_{k,k}) \tag{3.9}$$

that consists of the multivariate additive coarse grid correction induced by $B_{k,k}$ followed by one Gauss-Seidel smoothing step on the fine grid $\Omega_{k,k}$.

It turns out that the condition number is no longer optimal. For properly chosen ω,

$$\kappa(C_{k,k} \cdot L_{k,k}) = O(k^{\alpha}) \tag{3.10}$$

holds where α was determined experimentally to range between 3 and 4. Analogously, the asymptotic convergence rate $\rho(M_{k,k} \cdot L_{k,k})$ is dependent on k. This behavior can be explained by means of Fourier analysis. Compare also [9]. For our previous model problem, the convergence history of the CG-iteration with preconditioner (3.8) is shown in figure 4. For the MG-like iteration with operator (3.9), analogous results have been obtained.

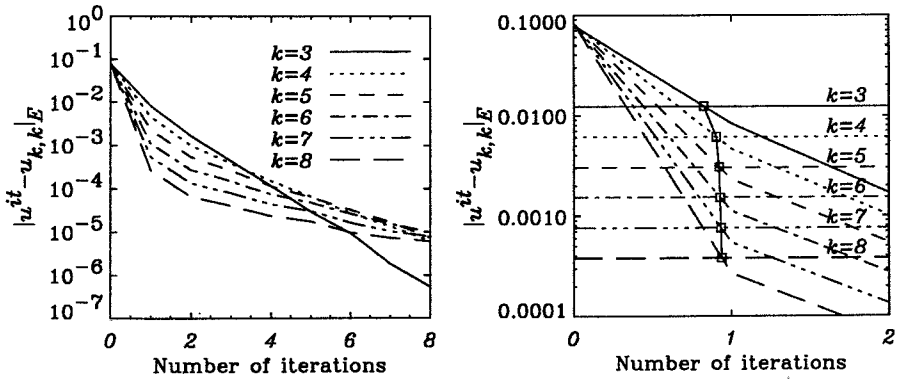

FIGURE 4. Convergence history of CG for preconditioner (3.8).

Note however, that the number of iteration steps necessary to reach discretization error accuracy is independent of k. We see in Figure 4 (right) that just one iteration step is sufficient for our purpose. With respect to the L_2-norm, about two or three iteration steps are necessary.

Thus, there is no use looking only for asymptotically optimal $O(1)$ preconditioners, if just discretization error accuracy is needed. We have seen that there exist preconditioners with suboptimal condition number, but with associated iterative methods where the number of iterations necessary to reach discretization error accuracy is independent of the mesh size.

4. THE COMBINATION METHOD AND THE DOMAIN DECOMPOSITION METHOD

To combine the combination method preconditioner and the domain decomposition method, basically two approaches exist.

The first one is straightforward. We apply the domain decomposition method (outer loop) with quadrilateral domains and use (3.8) or (3.9) for the solution of each subdomain problem (inner loop). To gain the advantages of the sparse grid approach within the domain decomposition method, we can solve only the sparse grid problem on each subdomain instead of the usual full grid problem by CG and (2.4), e.g. In this way, we use just a certain sparse grid for the overall problem.

Alternatively, the combination method can be applied globally (outer loop). See also figure 5 for a simple example. Then, each arising subproblem can be solved by the domain decomposition technique (inner loop).

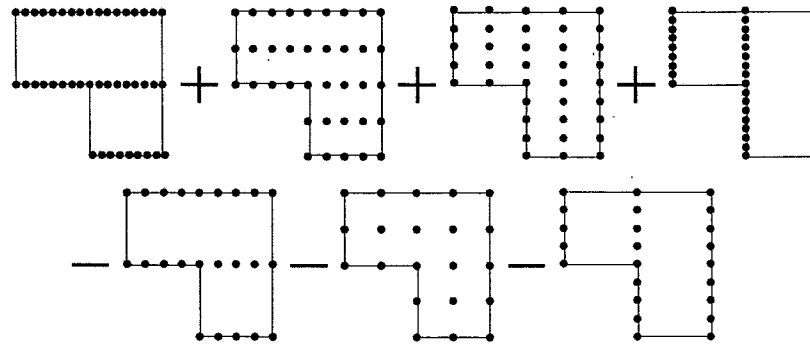

FIGURE 5. The linear combination of grids $\Omega_{i,j}$ and solutions $u_{i,j}$ with $i = 1, ..k, j = k + 1 - i$ and $i = 1, .., k - 1, j = k - i, k = 4$ for the L-shaped domain.

In both cases, the parallelization possibilities both of the domain decomposition method and of the combination method can be used simultaneously.

To show at last that the second approach involving the combination technique in the outer loop works well also for more general problems, we study the Navier-Stokes equations with complicated boundaries. We consider the example of a laminar flow over the skyline of Munich (2D idealization) with Reynolds number $Re = 500$. The discretization involves 33 quadrilateral subdomains with curved boundaries and graded meshes to resolve the effects caused by singularities due to reentrant corners. For each block $k = 6$ is chosen.

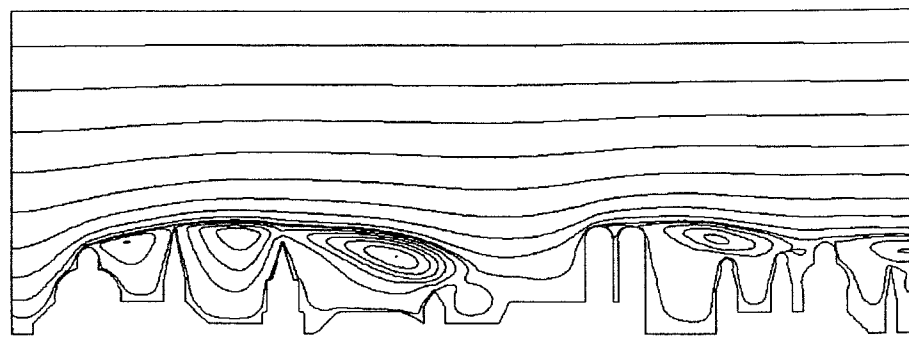

FIGURE 6. Laminar flow over the skyline of Munich, $Re = 500$.

Figure 6 shows the contour lines of the stream function computed for the sparse grid discretization by the combination method. Accuracy experiments with successive finer grids showed that the same order of accuracy was achieved with this method as for the solution obtained for the full grid approach. Further experiments and results can be found in [5].

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