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A Domain Decomposition Method for Simulating 2D External Viscous Flows

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ABSTRACT. Two-dimensional external incompressible viscous flows are simulated by means of a domain decomposition technique which combines a vortex method and a finite differences method. The vortex method is used in the flow region which is dominated by convection, whereas the finite differences method is used in the flow region where viscous diffusion matters.

1. Introduction

It is a wellknown fact in fluid mechanics that, for incompressible viscous flows, advection dominates viscous diffusion as the Reynolds number increases and, for external flows, vorticity tends to concentrate in wakes. These conditions are favourable for modelling such flows by means of particle methods. Furthermore since this class of methods is grid-free, it is suitable for tackling problems with moving boundaries. However, particle methods are less accurate as viscous effects are of the same order as or larger than that of advection; as a result, in boundary layers, *ie.* in the vicinity of physical boundaries, numerical approximations which are well suited to parabolic problems are needed. The remarks above led us to develop a domain decomposition method that combines both methods [6] [8].

2. Formulation of the problem

Consider p moving solids $(S_i)_{i=1,\dots,p}$ in a Galilean frame of reference $(O,\boldsymbol{i},\boldsymbol{j})$ of $\mathbb{R}^2$. Let $\boldsymbol{k}=\boldsymbol{i}\times\boldsymbol{j}$, define $(\boldsymbol{O}_i)_{i=1,\dots,p}$ origins of reference for each solid, and let $\boldsymbol{v}_i$ (resp. Ω_i) be the velocity of $\boldsymbol{O}_i$ (resp. the angular velocity of S_i). The solids are immersed in an incompressible Newtonian fluid which is at rest at infinity. The fluid domain, denoted by $\mathcal{D}$, is decomposed into $p+1$ open subdomains so that $\mathcal{D}=\mathcal{D}_0\cup_{i=1,\dots,p}\overline{\mathcal{D}_i}$, where the subdomains $\mathcal{D}_i$ are homeomorphic to a ring. It is hereafter assumed that the domain decomposition has been done so that convective effects are dominant in $\mathcal{D}_0$. Let B_i (resp. Γ_i) be the interface between $\mathcal{D}_i$ and S_i (resp. $\mathcal{D}_0$), and $\boldsymbol{n}_i$ be the outward normal to the boundary of $\mathcal{D}_i$ for $i=0,\dots,p$. The general strategy consists of replacing the original Navier–Stokes

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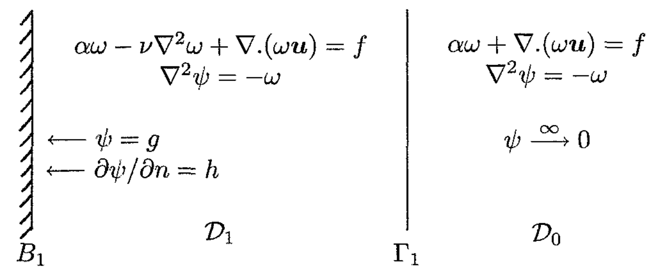
problem by subproblems in $\mathcal{D}_0, \ldots, \mathcal{D}_p$. In $\mathcal{D}_0$ the NS equations are formulated in terms of velocity and vorticity $(\boldsymbol{u}_0, \omega_0)$ and are approximated by means of a vortex method, whereas in each subdomain $\mathcal{D}_i$ they are formulated in terms of stream function and vorticity (ψ_i, ω_i) and are approximated by means of finite differences. Let $T > 0$, the NS problem is solved in $[0, T]$. Let $N \in \mathbb{N}$, $\delta t = T/N$, and $t_k = k\delta t$ so that $0 \le k \le N$; approximations of $(\boldsymbol{u}_0, \omega_0)$ and (ψ_i, ω_i) are sought in parallel in the time interval (t_k, t_{k+1}).

3. A model problem

Each subproblem is wellposed provided that some transmission conditions on the physical variables are imposed through interfaces Γ_i. Such conditions may be obtained by looking for an approximate parabolic decomposition of the advection–diffusion operator as in [3]. Another approach consists in taking into account the fact that in the vicinity of Γ_i viscous diffusion is dominated by advection. In order to illustrate this point, consider the model problem below:



Domain $\mathcal{D}_1$ is a vertical strip and $\mathcal{D}_0$ is a half plane. Parameters α and ν are positive constants and $\boldsymbol{u}$ is a constant velocity field. It is assumed here that ν is small and the solution is smooth enough so that $\nu\nabla^2\omega$ is negligible in the vicinity of Γ_1. This model amounts to a crude version of the χ-formulation of NS equations as advocated in [1] [2].

In domain $\mathcal{D}_1$ we have to deal with a biharmonic problem, which requires two transmission conditions. Let G be the Green function of ∇^2 in $\mathbb{R}^2$, an updated boundary value of ψ_1^{k+1} on Γ_1 is given by the Green identity:

$$\psi_1^{k+1} = -\int_{\mathcal{D}_0 \cup \mathcal{D}_1} \omega^k G \mathrm{d}v + \int_{B_1} [g\frac{\partial G}{\partial n_1} - Gh]\mathrm{d}l, \tag{1}$$

The transmission condition on ω_1 is obtained by recalling that in the vicinity of the interface ω is almost solution to a hyperbolic equation of the first order. For this kind of equation, information flows along the characteristics; hence, an updated value of ω_1 can be obtained by integrating the hyperbolic equation on a small distance in the upwind direction. Formally this operation amounts to enforcing the Robin-like boundary condition: $\alpha\omega + \nabla.(\omega\boldsymbol{u}) = f$ on Γ_1.

In $\mathcal{D}_0$, the hyperbolic and elliptic equations are uncoupled, *ie.* the elliptic equation does not need to be solved in $\mathcal{D}_0$. Hence, a transmission condition is required only by the hyperbolic equation. Once again, this condition is obtained by formally imposing $\alpha\omega + \nabla.(\omega\boldsymbol{u}) = f$; this is done in practice by performing an upwind integration. See [7] for other elliptic–hyperbolic coupling techniques.

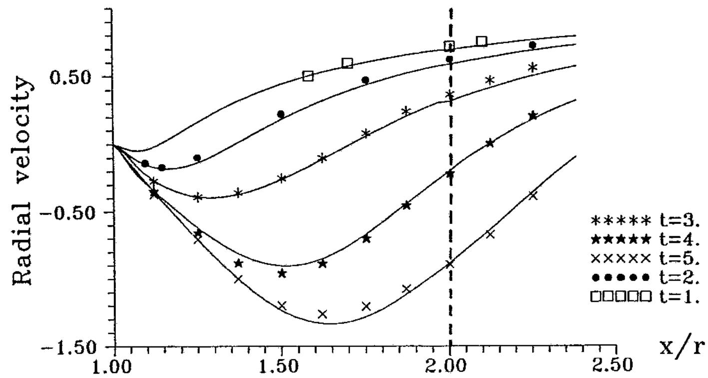


Figure 1: Comparison between numerical (solid lines) and experimental (symbols) velocity profils. Dashes at $x/r = 2$ represent the interface Γ.

4. Solution in $\mathcal{D}_0$

In $\mathcal{D}_0$, the fluid motion is studied in a frame of reference which moves with the mean velocity of the p solids $\boldsymbol{v}_\infty(t)$. The advection–diffusion equation of ω_0 is approximated by:

$$(2) \qquad \partial\omega_0^{k+1}/\partial t + \nabla.(\omega_0^{k+1}\boldsymbol{u}_0^k) = \nu\nabla^2\omega_0^{k+1},$$

$$\text{where,} \quad \boldsymbol{u}_0^k = -\boldsymbol{v}_\infty + \int_{\mathcal{D}} \omega^k \nabla G \times \boldsymbol{k} \mathrm{d}v + \sum_{j=1}^{p} \int_{B_j} [(\boldsymbol{n}_j \times \boldsymbol{v}_{ej}) \times \nabla G + (\boldsymbol{n}_j.\boldsymbol{v}_{ej})\nabla G]\mathrm{d}l,$$

and $\boldsymbol{v}_{ej} = \boldsymbol{v}_j + \Omega_j \times (\boldsymbol{y} - \boldsymbol{O}_j)$. Eq. (2) is locally considered as a hyperbolic equation whose right hand side, $\nu\nabla^2\omega_0^{k+1}$, is a perturbation. In the framework of §3, the required transmission condition is:

$$(3) \qquad j = 1, \ldots, p, \qquad \omega_0^{k+1}(\boldsymbol{x}) = \omega_j^k(\boldsymbol{x} - \boldsymbol{u}_j^k\delta t), \text{ if } \boldsymbol{u}_0^k(\boldsymbol{x}).\boldsymbol{n}_0(\boldsymbol{x}) < 0$$

Problem (2), supplemented by Dirichlet data (3), is approximated be means of a particle method (*cf.* [6] for details on this method).

5. Solution in $\mathcal{D}_i$

In $\mathcal{D}_i$, the fluid motion is studied in a non-inertial frame of reference that is linked to S_i, and NS equations are formulated in terms of relative stream function ψ_i and absolute vorticity ω_i. Hence, the PDE's to be solved are:

$$(4) \qquad \partial\omega_i^{k+1}/\partial t + \nabla.(\omega_i^{k+1}\nabla \times (\psi_i^{k+1}\boldsymbol{k})) = \nu\nabla^2\omega_i^{k+1}$$

$$(5) \qquad \nabla^2\psi_i^{k+1} = 2\Omega_i - \omega_i^{k+1}$$

Note that this system has a form which is identical to that it would have if it was written in an inertial frame of reference. This invariance with respect to the frame of reference characterizes formulations of NS equations which are based on vorticity. The system above is supplemented by boundary conditions on B_i:

$$(6) \; \psi_i^{k+1} = \psi_{B_i}^{k+1}, \quad \frac{\partial\psi_i^{k+1}}{\partial n} = 0, \text{ and } \int_{B_i} \frac{\partial\omega_i^{k+1}}{\partial n_i}\mathrm{d}l = \frac{\dot{\Omega}_i}{\nu}\int_{B_i} [\boldsymbol{k} \times (\boldsymbol{y} - \boldsymbol{O}_i)].\mathrm{d}\boldsymbol{l}$$

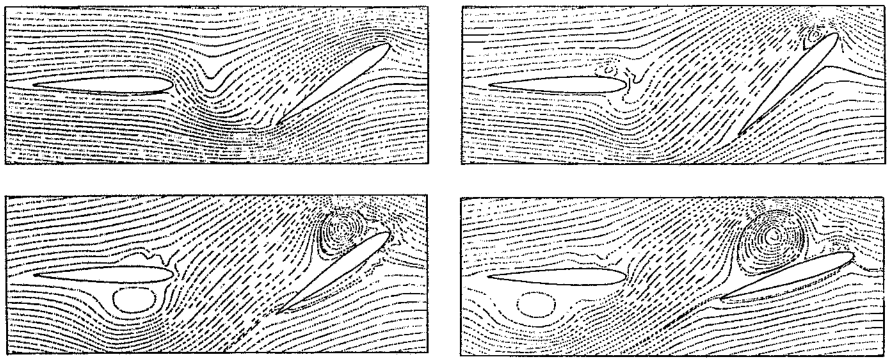


Figure 2: Streamline patterns about impulsively started tandem airfoils.

Furthermore, transmission conditions must be enforced across Γ_i as shown in §3.

The updated boundary value of ψ_i^{k+1} is provided by the Green identity based on (5) and (6):

$$(7)\quad \psi_i^{k+1}(\boldsymbol{x}) = -\psi_{ei}(\boldsymbol{x}) - \int_{\mathcal{D}_0 \cup_{j=1}^{p} \mathcal{D}_j} \omega^k G(\boldsymbol{y}-\boldsymbol{x})\mathrm{d}v + \sum_{j=1}^{p} \int_{B_j} (\psi_{ej} + \psi_{B_j}^k) \frac{\partial G(\boldsymbol{y}-\boldsymbol{x})}{\partial n_j} \mathrm{d}l - \int_{B_j} G(\boldsymbol{y}-\boldsymbol{x}) \frac{\partial \psi_{ej}}{\partial n_j} \mathrm{d}l,$$

where $\psi_{cj}(\boldsymbol{x}) = \boldsymbol{v}_j.[(\boldsymbol{x}-\boldsymbol{O}_j)\times \boldsymbol{k}] - \Omega_j|\boldsymbol{x}-\boldsymbol{O}_j|^2/2$ is the stream function of the entrainment velocity field of S_j. Note that condition (7) is global; *ie*, it transmits the whole spectrum of information to each subdomain at once, whereas classical Dirichlet-Neumann coupling conditions (*eg.* [7]) poorly transmit low frequencies.

By using the same arguments as that of §3, the transmission of information on vorticity is achieved on the subset of Γ_i where the flow enters $\mathcal{D}_i$ by:

$$(8)\qquad \omega_i^{k+1}(\boldsymbol{x}) = \omega_0^k(\boldsymbol{x} - \boldsymbol{u}_0^k \delta t), \text{ if } \boldsymbol{u}_i^k(\boldsymbol{x}).\boldsymbol{n}_i(\boldsymbol{x}) < 0.$$

As far as information transfer is concerned, this condition should be sufficient. Nevertheless, since (4) is approximated by means of a finite differences scheme which is adapted to parabolic problems, a boundary value of ω_i on Γ_i is required. Once again, the flow regime being almost hyperbolic, the piece of information that is missing is obtained by doing a Lagrangian integration of (4):

$$(9)\qquad \omega_i^{k+1}(\boldsymbol{x}) = \omega_i^k(\boldsymbol{x} - \boldsymbol{u}_i^k \delta t), \text{ if } \boldsymbol{u}_i^k(\boldsymbol{x}).\boldsymbol{n}_i(\boldsymbol{x}) \geq 0.$$

The (ψ_i, ω_i) problem as formulated above is linearized and solved by means of a finite differences method that has been developed in [4].

6. Numerical results

Consider an impulsively started cylinder of radius r. In figure 1, experimental [5] and numerical velocity profils downstream the cylinder on the symmetry axis

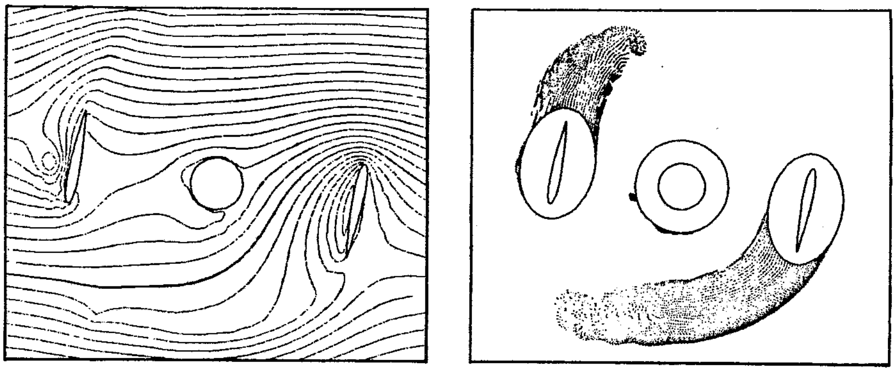


Figure 3: Shed particles and streamlines about a Darrieus windmill at $t = 2$.

are compared at time steps $t = 1, 2, 3, 4$, and 5. The finite differences subdomain, $\mathcal{D}_1$, is a ring the external radius of which is $2r$. The Reynolds number, $v_\infty r/\nu$, is set to 3000. Note the smooth matching of numerical results across the interface.

In figure 2 we present streamline patterns about impulsively started tandem airfoils at times $t = 1.5, 2.5, 3.5$ and 4.5. The leading airfoil oscillates in pitch and the rear one is fixed. Shown here is the interaction between the rear airfoil and the starting vortex of the leading airfoil. The fluid domain is decomposed into three subdomains. The Reynolds number $v_\infty C/2\nu$ is set to 3000, the reduced frequency of the oscillating airfoil $fC/2v_\infty$ is equal to 0.2 and $\alpha_{\max} = 45^o$. Note that the tandem airfoil problem or other problems of this kind would be difficult to treat by means of classical global approaches, for the flow domain would have to be either regridded or deformed at each time step.

The last example has been designed to illustrate versatility of the present method. In figure 3 is shown shed particles and streamline patterns about a Darrieus-like windmill at $t = 2.2$ after an impulsive start. There are four subdomains: one for each airfoil, one for the hub, and their complement $\mathcal{D}_0$. The windmill rotates in the anti-clockwise direction and the fluid moves from right to left with velocity v_∞. Let R be the windmill radius, the advance parameter $\Omega R/v_\infty$ is set to 2.16 and the Reynolds number $v_\infty r/\nu$ is equal to 3000.

REFERENCES

1. Y. Achdou, O. Pironneau, *C. R. Acad. Sci. Paris*, 312, Série I, 1991, 1005-1011.
2. F. Brezzi, C. Canuto, A. Russo, *Comp. Meth. in Appl. Mech. and Eng.*, 73, 1989, 317-330.
3. Ph. Charton, F. Nataf, F. Rogier, *C. R. Acad. Sci. Paris*, 313, Série I, 1991, 623-626.
4. O. Daube, T. P. Loc, *Journal de Mécanique*, 17, 5, 1978, 651-678.
5. T.P. Loc, R. Bouard, *J. Fluid Mech.*, 160, 1985, 93-117.
6. J.-L. Guermond, S. Huberson, W. Z. Shen, *C. R. Acad. Sci. Paris*, 314, Série II, 1992, 557-562, and *LIMSI report 91-9*, 1991.
7. A. Quarteroni, Domain decomposition methods for numerical solution of partial differential equations, *Univ. of Minnesota Supercomputer Inst., research report UMSI 90/246*, 1990.
8. W.-Z. Shen, Calcul d'écoulements tourbillonnaires visqueux incompressibles par une méthode de couplage, thèse de doctorat, Univ. Paris XI, February 1993.

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