Contemporary Mathematics
Volume 157, 1994

DOMAIN DECOMPOSITION for the STOKES EQUATIONS in STREAMFUNCTION FORMULATION

Wilhelm Heinrichs

Abstract. Multi domain pseudo spectral approximation of fourth order boundary value problems are considered. An iterative method with interface relaxation is investigated. Numerical results are presented for two and more subdomains.

1. Introduction

We provide a theoretical analysis of the so called patching collocation method for the Stokes equations in streamfunction formulation. This method was introduced by Orszag [5] for second order problems. Suppose that the initial domain is decomposed into rectangular regions inside of which Gauss-Lobatto points are considered. Then, the equation to be approximated is collocated at the internal nodes of each subdomain, while continuity of the piece-wise polynomial solution and its derivatives up to the order three are required at the nodes of the interfaces.

1991 Mathematics Subject Classification. Primary 65N35, 65L10.

The final version of this paper has been submitted for publication elsewhere.

© 1994 American Mathematical Society
0271-4132/94 $1.00 + $.25 per page

This is the first treatment of domain decomposition methods for fourth order problems. A theoretical analysis for the corresponding second order problems was given by Quarteroni et al. [3]. We extend these results to fourth order problems. We investigate an iterative procedure which corresponds to the Dirichlet-Neumann scheme for second order problems. For making the iteration convergent we have to introduce a relaxation parameter. We investigate different choices of relaxation parameters. It turns out that the best interface relaxation is given by a minimal residual strategy. We also present numerical results in the case of more than two subdomains. For an increasing number of subdomains the convergence of the interface relaxation becomes very slow and for more than about ten subdomains the relaxation procedure diverges. This is due to the fact that the interface matrix becomes indefinite. At the moment it is an open question how to construct an efficient preconditioner for the interface matrix.

2. Interface relaxation

We develop our analysis on a simple model problem. Given a function f, we have to determine the solution u of the following fourth order problem:

$$(1) \qquad \begin{cases} u^{IV} = f \quad in \ \ ]-a,b[, \\ \\ u(-a) = u'(-a) = u(b) = u'(b) = 0. \end{cases}$$

As already observed in [3] for second order problems, the simple iterative method which solves the fourth order problems on each subdomain alternatively does not in general converge. Certain restrictions on a and b such as $a > b$ are necessary (see [3, Prop. 2.1]). In [3] an interface relaxation procedure is proposed which makes the method convergent for all values of a and b. We also prefer this procedure for fourth order problems. Further, from the previous considerations we obtain a quite natural splitting of the boundary conditions. The relaxation procedure leads to the following algorithm. We look for two sequencies of functions: $v^{(n)}$ on $]-a,0[$, and $w^{(n)}$ on $]0,b[$, with $n \geq 1$, which satisfy:

$$(2)\qquad \begin{cases} v^{(n)}_{xxxx} = f \quad in\ ]-a,0[, \\ v^{(n)}(-a) = v^{(n)}_x(-a) = 0, \\ v^{(n)}(0) = \lambda_n, \\ v^{(n)}_{xx}(0) = \mu_n \end{cases}$$

and

$$(3)\qquad \begin{cases} w^{(n)}_{xxxx} = f \quad in\ ]0,b[, \\ w^{(n)}(b) = w^{(n)}_x(b) = 0, \\ w^{(n)}_x(0) = v^{(n)}_x(0), \\ w^{(n)}_{xxx}(0) = v^{(n)}_{xxx}(0). \end{cases}$$

In (2), λ_1, μ_1 are given real numbers, and the λ_n, μ_n $(n \geq 2)$ are recursively defined by the formula:

$$\begin{aligned} \lambda_{n+1} &= \theta^n_\lambda w^{(n)}(0) + (1-\theta^n_\lambda)\lambda_n, \\ \mu_{n+1} &= \theta^n_\mu w^{(n)}_{xx}(0) + (1-\theta^n_\mu)\mu_n, \quad n \geq 1. \end{aligned}$$

Here θ^n_λ, $\theta^n_\mu \in]0,1]$ are relaxation parameters. Let us first assume that in (2) θ^n_λ, θ^n_μ are fixed independent of n. We write θ_λ, θ_μ instead of θ^n_λ, θ^n_μ. In the spectral scheme we choose $v^{(n)}$, $w^{(n)} \in \mathbf{P_N}$ where $\mathbf{P_N}$ denotes the space of polynomials whose degree is less or equal to N. We use a collocation method where the fourth order differential equations are enforced at $N-3$ Chebyshev Gauss-Lobatto nodes. In $(-1,1)$ they are defined by $x_j = \cos\frac{j\pi}{N-2}$, $j = 1,\ldots,N-3$. The following theorem clarifies for which parameters the interface relaxation converges.

Theorem. *The interface relaxation procedure* (2), (3) *converges for any value a and b provided* $\theta_\lambda \in]0, \frac{2}{\alpha}[$, $\theta_\mu \in]0, \frac{2}{\beta}[$ *and*

$$(4)\qquad 0 < \theta_\mu < \frac{2-\alpha\theta_\lambda}{\beta - \frac{\kappa}{2}\theta_\lambda}.$$

Here κ is a positive constant, defined by

$$\kappa = \alpha\beta - \gamma\delta > 0,$$

where α, β, γ, δ are given by

$$\begin{cases} \alpha = \frac{b^3+3a^2b+4a^3}{4a^3}, \ \beta = \frac{3b^2+a^2+4ab}{4ab}, \\ \gamma = \frac{b(b^2-a^2)}{8a}, \ \delta = \frac{3(b^2-a^2)}{2a^3b}. \end{cases}$$

In particular, in the case $a = b$ one obtains convergence for all $(\theta_\lambda, \theta_\mu)$ $\in]0,1[^2$.

Proof. One easily verifies the result by studying the interface matrix. $\oplus$

If we want to find optimal relaxation parameters we have to determine θ_λ, θ_μ in such a way that the spectral radius of A becomes minimal. A short computation shows that the optimal parameters are given by:

$$\theta_{\lambda,opt} = \frac{1}{\alpha}, \ \theta_{\mu,opt} = \frac{1}{\beta}. \tag{5}$$

The corresponding spectral radius of A is: $\rho_{opt}(A) = \sqrt{1 - \kappa/\alpha\beta} < 1$. Hence, if $a = b$ one obtains $\theta_{\lambda,opt} = \theta_{\mu,opt} = 1/2$ and convergence after two steps. If $a \neq b$ we do not have convergence after a fixed number of steps independently of a, b. This result is really different from the case of second order problems where the exact solution is always obtained after two steps. A dynamical parameter choice is obtained by applying a minimal residual relaxation (MRR). Here the parameter is chosen such that the discrete L^2-norm of the difference between two iterates becomes minimal. Another attractive choice is defined by the conjugate gradient (CG) method. The CG method is applicable since the 2×2 matrix B is symmetric with respect to a modified inner product. In table 1 we present the number of interface relaxations for different a, b and parameter choices. It becomes obvious that for $b >> a$ the methods converge slowly. For all a, b the MRR and CG methods yield the best interface relaxations.

Interval lenghts		Relaxation parameters		
a	b	(5)	MRR	CG
2	2	2	3	3
4	2	16	11	4
2	4	25	13	4
8	2	19	18	4
2	8	79	17	5

Table 1. Number of interface relaxations.

Furthermore we were interested in the convergence behavior of the interface relaxation in the case of more than two subdomains. Here we consider the case of n_d, $n_d \in \mathbf{N}$ subdomains. The interval $[-a, b]$ is subdivided in n_d subdomains $[a_{i-1}, a_i]$, , $i = 1, \ldots, n_d$ where

$$a_i \;=\; -a + ih,\; h = \frac{a+b}{n_d}$$

for $i = 0, \ldots, n_d$.
Then we look for solutions u_i, $i = 1, \ldots, n_d$ defined in the subdomains $[a_{i-1}, a_i]$. Now the interface relaxation is formulated for odd n_d, i.e. $n_d \in 2\mathbf{N} + 1$. Let λ_i^0, μ_i^0, ρ_i^0, σ_i^0, $i = 1, \ldots, n_d$ be given real numbers. Then the interface relaxation for n_d subdomains proceeds as follows: We look for n_d sequencies u_i^n, $i = 1, \ldots, n_d$, $n \geq 1$, which satisfy:
i odd :

$$\begin{aligned}
u^n_{i,xxxx} &= f \quad \text{in } (a_{i-1}, a_i), \\
i = 1:\; u_i^n(-a) &= u^n_{i,x}(-a) = 0, \\
i \geq 3:\; u^n_{i,x}(a_{i-1}) &= \lambda_i^{n-1},\; u^n_{i,xxx}(a_{i-1}) = \mu_i^{n-1} \\
i = n_d:\; u_i^n(b) &= u^n_{i,x}(b) = 0, \\
i \leq n_d - 2:\; u_i^n(a_i) &= \rho_i^{n-1},\; u^n_{i,xx}(a_i) = \sigma_i^{n-1}
\end{aligned}$$

i even :

$$\begin{aligned}
u^n_{i,xxxx} &= f \quad \text{in } (a_{i-1}, a_i), \\
u^n_{i,x}(a_{i-1}) &= u^n_{i-1,x}(a_{i-1}), \\
u^n_{i,xxx}(a_{i-1}) &= u^n_{i-1,xxx}(a_{i-1}), \\
u_i^n(a_i) &= u^n_{i+1}(a_i), \\
u^n_{i,xx}(a_i) &= u^n_{i+1,xx}(a_i).
\end{aligned}$$

Then set:

$$\begin{aligned}
\lambda_i^n &= \theta u_{i-1,x}^{n-1}(a_{i-1}) + (1-\theta)\lambda_i^{n-1}, \\
\mu_i^n &= \theta u_{i-1,xxx}^{n-1}(a_{i-1}) + (1-\theta)\mu_i^{n-1}, \text{ for } i = 3,\ldots,n_d,\ i \text{ odd} \\
\rho_i^n &= \theta u_{i+1}^{n-1}(a_i) + (1-\theta)\rho_i^{n-1}, \\
\sigma_i^n &= \theta u_{i+1,xx}^{n-1}(a_i) + (1-\theta)\sigma_i^{n-1}, \text{ for } i = 1,\ldots,n_d-2,\ i \text{ odd},
\end{aligned}$$

where θ denotes a positive relaxation parameter. The *degree of parallelism* of this algorithm is $\frac{n_d}{2}$.
We calculated the number of iterations IT which were necessary to reach an error of 10^{-9} for the difference of two interface iterates. The interface relaxation was started with the zero approximation. Furthermore we calculated the medium error reduction factor $\overline{\rho}$. The corresponding numerical results are presented in table 2. It was also observed that there are eigenvalues with negative real parts. Hence the interface matrix B becomes indefinite and the interface relaxation procedure will never converge. By applying GMRES to our problem we obtained convergence in 4, 10, 16 and 28 iterations for 3, 7, 11 and 15 subdomains. The number of iterations seems to be linear dependent on the number of subdomains. For an increasing number of subdomains GMRES also becomes very slow and for more than 30 subdomains it also diverges. Furthermore we obtained similar results for other splittings of boundary conditions.

n_d	IT	$\overline{\rho}$
2	2	—
3	27	0.184
4	72	0.772
5	182	0.927
6	462	0.968
7	1330	0.987
8	5476	0.997

Table 2. Numerical results for n_d subdomains.

Similar results are available for the two–dimensional interface relaxation applied to the biharmonic operator. Here we also obtained good results by using the MRR relaxation. The convergence once more becomes slow for $b >> a$. For an increasing number of subdomains ($n_d \geq 20$) the interface relaxation is no more efficient. For $n_d \geq 35$ it always diverges. Hence the same problem as in the one–dimensional case occurs but somewhat later.

References

1. Canuto, C., Hussaini, M.Y., Quarteroni, A., and Zang, T.A.: *Spectral methods in fluid dynamics.* Series of Computational Physics, Springer, Heidelberg–Berlin–New York 1989

2. Funaro, D.: Domain decomposition method for pseudospectral approximations, Part one: Second order equations in one dimension. Numer. Math. **52**, 329-344(1988)

3. Funaro, D., Quarteroni, A., and Zanolli, P.: An iterative procedure with interface relaxation for domain decomposition methods. SIAM J. Num. Anal. **25**, 1213-1236(1988)

4. Heinrichs, W.: Domain decomposition for fourth order problems. SIAM J. Numer. Anal. **30**, No. 2, April 1993

5. Orszag, S.A.: Spectral methods for problems in complex geometries. J. Comput. Phys. **37**, 70-92(1980)

Current address:
Mathematisches Institut der Heinrich-Heine-Universität Düsseldorf,
Universitätsstraße 1, W-4000 Düsseldorf 1 (GERMANY).
E–mail address: WHEINRIC@ze8.rz.uni-duesseldorf.de