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Overlapping Domain Decomposition Methods for Parabolic Problems

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ABSTRACT. In this paper overlapping domain decomposition methods are applied to the numerical solution of FE-systems with singularly perturbed elliptic operators arising from implicit approximations of parabolic problems. The algebraic representation of overlapping domain decomposition preconditioners as well as convergence estimates of the solution algorithms are given.

1. Introduction

Recently, a number of methods based on domain decomposition ideas [7], [8] has been proposed for the numerical solution of parabolic problems and algebraic systems arising from discretization of these problems via implicit schemes [1], [3], [4], [6], [9]. In this paper we continue the presentation of overlapping domain decomposition methods proposed originally in [11] and developed in [12], [13] for solving algebraic systems arising from mesh discretizations of unsteady convection-diffusion problems via implicit schemes. For the sake of simplicity, we reduce the presentation to a model problem.

Let Ω be an open bounded polygon in $\mathbb{R}^2$ with the boundary $\partial\Omega$, and $\Gamma_0 \subset \partial\Omega$ is a union of closed segments. Consider the unsteady convection-diffusion problem: find $u = u(x,t)$ such that

$$(1) \qquad \begin{aligned} \frac{\partial u}{\partial t} + \mathcal{L}u &= f && \text{in } \Omega \times (0; \Delta t] \\ u &= 0 && \text{on } \Gamma_0 \times (0; \Delta t], \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \Gamma_1 \times (0; \Delta t] \\ u(x,0) &= u^0 && \text{in } \Omega \end{aligned}$$

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where $\Delta t > 0$ is given, u_0 is a given smooth function satisfying the homogeneous boundary conditions from (1), Γ_1 is a union of the open segments, i.e. $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $\partial\Omega = \Gamma_0 \cup \overline{\Gamma}_1$, and an operator $\mathcal{L}$ is defined by

$$\mathcal{L}u = -\nu\Delta u + (\vec{b} \circ \nabla)u. \tag{2}$$

Here $\nu = \text{const} > 0$ is given and $\vec{b}$ is a given smooth vector-function. The problem (1)–(2) with sufficiently small Δt arises, for instance, from the approximation of the Navier-Stokes equations by the operator-splitting methods [**2**].

The weak formulation of (1) is: find $u(t) = u(x,t) \in V_0$, $t \in (0, \Delta t]$, such that

$$(\frac{du}{dt}, v) + a(u,v) = (f,v) \quad \forall v \in V_0, \tag{3}$$

where

$$V_0 = \{v \;:\; v \in H^1(\Omega),\; v = 0 \text{ on } \Gamma_0\} \tag{4}$$

and

$$a(u,v) = \int_\Omega [\nu \cdot \nabla u \circ \nabla v + v \cdot (\vec{b} \circ \nabla)u]\, d\Omega \tag{5}$$

is the bilinear form generated by the operator $\mathcal{L}$. It is a well known fact that positive constants c_0, c_1 and c_2 exist such that the inequalities

$$\begin{aligned} c_0\|v\|_{H^1} - c_1\|v\|_{L_2} &\le a(v,v), \\ a(u,v) &\le c_2\|u\|_{H^1} \cdot \|v\|_{H^1} \end{aligned} \tag{6}$$

are valid, $\forall u, v \in V_0$. Here $\|\cdot\|_{H^1}$ and $\|\cdot\|_{L_2}$ are the classical $H^1(\Omega)$ and $L_2(\Omega)$ norms.

Let Ω_h be a regular triangular covering [**5**] of Ω such that $\Gamma_0 \cap \overline{\Gamma}_1$ belongs to the set of mesh nodes, and V_h is the corresponding piecewise linear finite element subspace of V_0. Under the assumptions made the standard FE-method coupled with the Crank-Nicholson scheme: find $u_h \in V_h$ such that

$$(\frac{u_h - u_0}{\Delta t}, v) + a(\frac{u_h + u_0}{2}, v) = (f,v) \quad \forall v \in V_0, \tag{7}$$

leads to the algebraic system

$$A_\omega v = g, \tag{8}$$

where

$$A_\omega = K + \omega^2 M. \tag{9}$$

Here K is the stiffness matrix generated by the bilinear form $a(\cdot,\cdot)$, M is the mass matrix, and $\omega^2 = \frac{2}{\Delta t}$. Note that the solution vector corresponds to the function $u_h - u_0$ and a right-hand side vector g is defined by the residual functional

$$\xi(v) = -a(u_0, v) + (f,v). \tag{10}$$

The matrix A_ω as well as the matrices K and M can be defined by the assembling procedure:

$$A_\omega = \{A_\omega^e\},\ K = \{K^e\},\ M = \{M^e\}, \tag{11}$$

where K^e and M^e are local stiffness and mass matrices and

$$A_\omega^e = K^e + \omega^2 M^e. \tag{12}$$

These representations are very convenient for the description of domain decomposition preconditioners, in particular those which will be defined in the next section.

It follows from (6) that for any sufficiently small Δt ($\omega^2 \gg 1$) the matrix A_ω is positive definite.

2. Overlapping domain decomposition preconditioners

Ω_h is a union of elementary cells (triangles) e_h. Partition Ω_h into nonoverlapping subdomains $\Omega_h^{(k)}$, $k = \overline{1,m}$. Every $\Omega_h^{(k)}$ is also a union of e_h. Define the matrix $A_\omega^{(k)}$ related to $\Omega_h^{(k)}$ by

$$A_\omega^{(k)} = \{A_\omega^e\} \tag{13}$$

taking into account those cells e, which belong to $\Omega_h^{(k)}$. It follows that

$$A_\omega = \{A_\omega^{(k)}\} = \sum_{k=1}^{m} P_k \begin{bmatrix} A_\omega^{(k)} & 0 \\ 0 & 0 \end{bmatrix} P_k^T \tag{14}$$

with suitable permutation matrices P_k.

To define the required overlapping domain decomposition preconditioners we embed every subdomain $\Omega_h^{(k)}$ into a larger subdomain $\widehat{\Omega}_h^{(k)}$, assume that every interior mesh node of Ω_h belongs to the interior of at least one subdomain $\widehat{\Omega}_h^{(k)}$ and denote by $\widehat{\Gamma}_h^{(k)}$ the closure of $\partial\widehat{\Omega}_h^{(k)} \setminus \partial\Omega_h$.

For every subdomain $\widehat{\Omega}_h^{(k)}$, we divide the set of mesh nodes into two groups: the first one collects all nodes belonging to the interior of $\widehat{\Omega}_h^{(k)}$ and to $\partial\Omega_h \setminus \widehat{\Gamma}_h^{(k)}$, while the second one comprises those nodes which belong to $\widehat{\Gamma}_h^{(k)}$. After that, the matrix $\widehat{A}_\omega^{(k)}$ can be represented in the following 2×2 block form:

$$\widehat{A}_\omega^{(k)} = \begin{bmatrix} \widehat{A}_{11}^{(k)} & \widehat{A}_{1\Gamma}^{(k)} \\ \widehat{A}_{\Gamma 1}^{(k)} & \widehat{A}_{\Gamma\Gamma}^{(k)} \end{bmatrix}. \tag{15}$$

We also introduce the matrices

$$\widehat{A}_k = P_k \begin{bmatrix} \widehat{A}_{11}^{(k)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P_k^T \tag{16}$$

and

$$H_k \equiv \widehat{A}_k^+ = P_k \begin{bmatrix} [\widehat{A}_{11}^{(k)}]^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P_k^T. \tag{17}$$

In this section, we first describe an overlapping domain decomposition (ODD) preconditioner based on the superposition idea [**11**]. For the sake of simplicity, we assume hereafter in this section that $\partial\Omega = \Gamma_0$, i.e. $\Gamma_1 = \emptyset$.

For every subdomain we define $\Omega_h^{(k)}$ a nonnegative diagonal matrix Q_k with diagonal elements equal to one for every node belonging to the interior of $\Omega_h^{(k)}$ and equal to zero for every node belonging to the interior of $\Omega_h \setminus \Omega_h^{(k)}$. Thus, diagonal elements of Q_k corresponding to nodes of $\Gamma_h^{(k)} = \partial\Omega_h^{(k)} \setminus \partial\Omega_h$ are not defined exactly. We assume that

$$\sum_{k=1}^{m} Q_k = E, \tag{18}$$

where E is the identity matrix.

We define the required ODD-preconditioner by

$$H = \sum_{k=1}^{m} H_k Q_k, \tag{19}$$

where H_k are taken from (17). The motivation for such a preconditioner is that in the case $\Omega_h^{(k)} = \Omega_h$, $k = \overline{1,m}$, we have

$$H = A^{-1}, \tag{20}$$

i.e. we get the (theoretically) optimal preconditioner. The reality is quite far from (20).

Define the distance between $\Gamma_h^{(k)}$ and $\widehat{\Gamma}_h^{(k)}$ by

$$d(\Gamma_h^{(k)}; \widehat{\Gamma}_h^{(k)}) = \inf_{\substack{x \in \Gamma_h^{(k)} \\ y \in \widehat{\Gamma}_h^{(k)}}} |x - y|. \tag{21}$$

Let us assume that

$$\max_{1 \le k \le m} d(\Gamma_h^{(k)}; \widehat{\Gamma}_h^{(k)}) \ge c_3 \frac{1}{\omega} \ln(c_4 \omega^\alpha) \tag{22}$$

with some positive constants c_3, c_4 and α. Then, according to the results from [**12**], [**13**] the following proposition can be proved: for given positive constants c_5 and β the constants c_3, c_4 and α in (22) can be chosen such that the inequality

$$\|E - HA\|_1 \le \frac{c_5}{\omega^\beta} \tag{23}$$

is valid for all sufficiently large values of ω. Here $\|\cdot\|_1$ is the finite-dimensional analogue of the $H^1(\Omega)$ norm. In our case, the norm $\|\cdot\|_1$ is generated by the matrix A_ω from (9) when $\vec{b}(x) \equiv 0$ in Ω and $\omega^2 = \nu \equiv 1$.

To analyze the above result we have to remind that $\omega^2 = \frac{2}{\Delta t}$ and Δt is assumed to be sufficiently small. It follows from (23) that the prescribed accuracy ε for the solution vector of (8) can be reached with preconditioner (19) even in one iteration step. In fact, it can even happen practically that for this goal we have to choose $\Omega_h^{(k)} = \Omega_h$, $k = \overline{1,m}$. At the same time, this result is very attractive at least asymptotically: for sufficiently small Δt the value of $d(\Gamma_h^{(k)}; \widehat{\Gamma}_h^{(k)})$ can be chosen like $\mathcal{O}(\sqrt{\Delta t} \ln \frac{1}{\Delta t})$.

More detailed analysis of the correspondence between constants in (22) and (23) was done in **[12]** for the model convection-diffusion equation and in **[10]** for the model heat equation. Numerical results on parallel computers were analyzed in **[14]**.

We define the overlapping domain decomposition preconditioner via restrictions by

$$H = \sum_{k=1}^{m} Q_k H_k. \tag{24}$$

Again, in the case $\widehat{\Omega}_h^{(k)} = \Omega_h$, $k = \overline{1,m}$, we get equality (20).

For preconditioner (24) the proposition, which was done for preconditioner (19) is still valid. Moreover, under some special assumptions this proposition remains valid for quasi-linear parabolic equations like one-dimensional Burgers equation (such an extension was done by G. S. Abdoulaev, the paper will appear in the second issue of the East-West Journal of Numerical Mathematics). For quasi-linear problems the preconditioner H will not be a matrix any more. In this case H_k will be nonlinear subdomain solution operators, and Q_k will be the same matrices.

3. Applications and generalizations

Here we describe only one application and one generalization of the above domain decomposition procedures.

The application of preconditioner (24) is related to the separation of the global FE-problem (7) into several completely independent subdomain problems with interface boundary Γ_h.

We define Γ_h by

$$\Gamma_h = \bigcup_k \Gamma_h^{(k)}. \tag{25}$$

and embed Γ_h into a mesh subdomain G_h such that the inequality

$$d(\Gamma_h; \partial G_h) \geq \frac{c_6}{\omega} \ln(c_7 \omega^\alpha) \tag{26}$$

is valid with some positive constants c_6, c_7 and α. It follows from the above proposition and the regularity of Ω_h that for given positive constants c_8 and β

the constants c_6, c_7 and α exist such that the inequality

$$\|Q_\Gamma(A^{-1} - H_G)g\|_M \le \frac{c_8}{\omega}\|g\|_M \tag{27}$$

is valid for all sufficiently large values of ω. Here M is the mass matrix and

$$H_G = P_G \begin{bmatrix} A_G^{-1} & 0 \\ 0 & 0 \end{bmatrix} P_G^T \tag{28}$$

with a suitable permutation matrix P_G and Q_Γ is a nonnegative diagonal matrix, whose diagonal elements are equal to one for the nodes belonging to Γ_h and zero otherwise.

It is clear that components of the vector $Q_\Gamma H_G g$ can be used to divide the global system (7) into m independent approximate subproblems for subdomains $\Omega_h^{(k)}$.

Here we discuss briefly only one generalization of the above domain decomposition procedures. Instead of (17) we define matrices H_k by

$$H_k = P_k \begin{bmatrix} [A_\omega^{(k)}]^{-1} & 0 \\ 0 & 0 \end{bmatrix} P_k^T, \tag{29}$$

i.e. instead of the Dirichlet boundary conditions on $\widehat{\Gamma}_h^{(k)}$ we suggest to use the Neumann boundary conditions.

It can be shown that all above theoretical conclusions are still valid. At the same time, it is clear that the Neumann boundary conditions are much more convenient for the construction of efficient inner subdomain iterative procedures.

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