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Overlapping Domain Decomposition Methods for the Obstacle Problem

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ABSTRACT. In this paper overlapping domain decomposition methods are applied to the numerical solution of nonlinear grid variational problems arising from the approximation of the obstacle problem by the piecewise linear finite element method. This method is important for nonlinear boundary value problems for two reasons: It provides the possibility of using parallel processing, and, what is perhaps more important, the means for isolating the neighbourhood of the free boundary for a special treatment. In the major part of the domain the problem is linear and traditional efficient solvers for linear problems can be applied. We give the sufficient conditions of the convergence of the method and formulate the convergence result. Moreover, we give some considerations about overlapping domain decomposition methods with monotone operators.

1. Introduction

In this paper we consider the numerical solution of two-dimensional obstacle-like free boundary problems by overlapping domain decomposition methods. Domain decomposition methods are a widely researched area for linear problems, but probably paper [8] was the first to use overlapping domain decomposition methods for the solving of variational inequalities. Thereafter, quite a few papers (for example, [1], [5], [6], [9]) have been issued on this topic. Our aim is to consider different solution algorithms based on the domain decomposition methodology.

We shall study the numerical solution of nonlinear grid variational problems arising from the FE/FD-approximations of the two-dimensional obstacle problem by domain decomposition methods. We formulate the method in the variational

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inequality form as well as give an equivalent operator formulation and the convergence result.

Finally, in the last section we shall consider overlapping domain decomposition methods with monotone operators. In this special case we can find properties which are very useful when constructing numerical solution algorithms: we can prove that with a suitable initial guess the algorithm is monotonically convergent, and, using this result, we can find two-sided approximations for the location of the free boundary and use this information to choose suitable domain decomposition procedures.

2. Formulation of the problem

2.1. Differential problem.

Let Ω be a bounded domain with the piecewise smooth boundary $\partial\Omega$, and let f be a given smooth function. Then the obstacle problem is formulated in the differential form: Find $u \in C^1(\overline{\Omega})$ (such that $\exists\ \mathcal{L}u$ a.e.):

$$\begin{aligned} \mathcal{L}u - f &\geq 0 \\ u &\geq \psi \quad \text{a.e. in } \Omega \\ (\mathcal{L}u - f)\cdot(u-\psi) &= 0 \\ u &= 0 \qquad \text{on } \partial\Omega\ , \end{aligned} \tag{1}$$

where

$$\mathcal{L} = -\nabla a \nabla + \bar{b}\cdot\nabla + c \tag{2}$$

is an elliptic operator, and the coefficients $\bar{b} = (b_1, b_2, ..., b_p)$, $p = 1, 2, 3$, a, b_1, b_2,...,b_p and c are given smooth functions. We assume further that $c \geq 0$ and $\inf_\Omega a > 0$. $\psi \in C^1(\Omega)$ is an obstacle function such that $\psi|_{\partial\Omega} \leq 0$.

Problem (1) can be presented in the variational inequality form: Let $a(\cdot,\cdot)$: $H_0^1(\Omega) \times H_0^1(\Omega) \mapsto \mathbb{R}$ be a bilinear form,

$$a(u,w) = \int_\Omega [a\nabla u\cdot\nabla w + (\bar{b}\cdot\nabla)uw + cuw]\ d\Omega,$$

and

$$(f,w) = \int_\Omega fw\ d\Omega.$$

We define a closed, convex subset K of $H_0^1(\Omega)$ by

$$K = \left\{v\ :\ v \in H_0^1(\Omega),\ \ v \geq \psi \quad \text{a.e. in } \Omega\right\}.$$

Now the obstacle problem reads: Find $u \in K$ such that

$$a(u, v-u) \geq (f, v-u) \quad \forall v \in K. \tag{3}$$

It should be noted that in the case $\bar{b} = 0$ the problem (3) is equivalent to the constrained minimization problem: Find $u \in K$ such that

$$(4) \qquad J(u) = \min_{u \in K} J(v), \quad J : K \mapsto \mathbb{R}, \quad J(v) = a(v,v) - 2(f,v).$$

2.2. Mesh problems.

Let $\mathcal{T}_h$ be a mesh partitioning of Ω into triangles e_i, $i = 1, \dots, k$, and let V_h be the piecewise linear finite-element subspace of $H_0^1(\Omega)$. We define a closed, convex subset K_h by

$$K_h = \{v \ : \ v \in V_h, \ \ v \geq \psi_h \text{ in } \Omega\},$$

where ψ_h is a FE-approximation of the obstacle function ψ. The finite-element problem is: find $u_h \in K_h$ such that

$$(5) \qquad a(u_h, v - u_h) \geq (f, v - u_h) \quad \forall v \in K_h.$$

Using the standard finite-element discretization procedure this leads to the algebraic problem: find $u \in \widehat{K}$ such that

$$(6) \qquad (Au, v - u) \geq (\hat{f}, v - u) \quad \forall v \in \widehat{K},$$

where

$$\widehat{K} = \{v \ : \ v \in \mathbb{R}^N, \ v \geq \hat{\psi}\},$$

and $\hat{f}$, $\hat{\psi} \in \mathbb{R}^N$: $\hat{\psi}_i = \psi_h(x_i)$, $x_i \in \Omega_h$. The algebraic problem (6) has the following equivalent formulation:

$$(7) \qquad \begin{aligned} (Au)_i &\geq \hat{f}_i \\ u_i &\geq \hat{\psi}_i, \quad i = 1, ..., N \quad (x_i \in \Omega_h). \\ (Au - \hat{f})_i \cdot (u - \hat{\psi})_i &= 0 \end{aligned}$$

We consider only two practically important particular cases. We assume that A is either symmetric and positive definite or an M-matrix. The first case we obtain immediately if the bilinear form $a(\cdot,\cdot)$ is symmetric and positive definite. Under such assumptions problem (7) is equivalent to the minimization problem: find $u \in \widehat{K}$ such that

$$(8) \qquad J(u) = \min_{v \in \widehat{K}} J(v),$$

where

$$(9) \qquad J(v) = (Av, v) - 2(v, \hat{f})$$

is the energy functional. To satisfy the above conditions we have to assume that $\bar{b} = 0$. This case is investigated in details in [**3**], [**4**], [**8**].

To satisfy the condition that A is an M-matrix is a much more complicated task. Within the finite element approximation it can be reached under the conditions $\bar{b} = 0$, $\bar{c} = 0$, and angles of triangles e_j are sufficiently regular[**2**]. To obtain the M-property in more general situation we have to combine the FE-approach with other approximation procedures like mass lumping, upwind schemes and finite difference/finite volume schemes. In the latter case the mesh problem should be formulated in terms of mesh functions: find a mesh function u_h such that

$$\begin{aligned} \mathcal{L}_h u_h &\geq f_h \\ u_h &\geq \psi_h \qquad \text{in } \Omega_h \\ (\mathcal{L}_h u_h - f_h) \cdot (u_h - \psi_h) &= 0 \\ u_h &= 0 \qquad \text{on } \partial\Omega_h, \end{aligned} \tag{10}$$

which leads to the corresponding algebraic problem (7). Here $\mathcal{L}_h$ is a monotone mesh operator, f_h and ψ_h are mesh approximations of f and ψ, and Ω_h and $\partial\Omega_h$ mesh partitionings of Ω and $\partial\Omega$, respectively.

3. An overlapping domain decomposition method for the obstacle problem

In this section we shall formulate the overlapping domain decomposition method for the finite-dimensional obstacle problem presented in the previous section.

We decompose the original grid domain Ω_h into m overlapping grid subdomains $\Omega_h^{(i)}$, i.e.

$$\Omega_h = \bigcup_{i=1}^{m} \Omega_h^{(i)},$$

such that every mesh node x belongs to the interior of at least one subdomain $\Omega_h^{(i)}$. We denote $\Omega_i = \Omega_h^{(i)}$ to simplify the notations.

For a given grid function $w_h \in K_h$ and a grid subdomain Ω_i we define a subset $K_h(\Omega_i; w_h)$ of K_h by

$$K_h(\Omega_i; w_h) = \{v \,:\, v \in K_h,\ v = w_h \text{ in } \Omega \setminus \Omega_i\}.$$

Now we can formulate the iterative procedure - the overlapping domain decomposition method - for the obstacle problem:

ALGORITHM 1. *Let* $u_h^{k+\frac{i}{m}} \in K_h$ *be given,* $k \geq 0$, $i \geq 0$. *Solve successively for* $k = 0, 1, \ldots,$ *and* $i = 1, \ldots, m$, *the subproblems: find* $u_h^{k+\frac{i+1}{m}} \in K_h(\Omega_i; u_h^{k+\frac{i}{m}})$ *such that*

$$a\left(u_h^{k+\frac{i+1}{m}}, v - u_h^{k+\frac{i+1}{m}}\right) \geq \left(f, v - u_h^{k+\frac{i+1}{m}}\right) \quad \forall v \in K_h\left(\Omega_i; u_h^{k+\frac{i}{m}}\right). \tag{11}$$

Algorithm 1 has also an operator formulation: by rewriting (11) in the terms of the operator $T_j = T(\Omega_j)$, we get: $u_h^n \in K_h$,

$$u_h^{n+1} = T u_h^n \equiv T_m(T_{m-1}(\ldots T_1(u_h^n)\ldots)), \quad n = 0, 1, \ldots.$$

REMARK 1. *This method can be easily formulated in terms of mesh functions and finite-dimensional vector spaces as well.*

THEOREM 1. *Under the assumptions made the overlapping domain decomposition method converges for any* $u_h^0 \in K_h$.

4. Overlapping domain decomposition methods with monotone operators

The final part of the paper considers overlapping domain decomposition methods with monotone operators. Previously, some aspects for this topic were given in [1], [7], [8], [9]. In this section we do not separate the finite and infinite-dimensional cases: the results given below are valid in both cases.

4.1. The convergence result.

Let us consider a subdomain problem defined in Algorithm 1 in the previous section, i.e. for a given subdomain $S \subset \Omega$ and for a given function $w \in K$, we define

$$K(S;w) = \{v \ : \ v \in K, \ \ v = w \text{ in } \Omega \setminus S\}, \tag{12}$$

and, find $u \in K(S;w)$ such that

$$a(u, v-u) \geq (f, v-u) \ \ \forall v \in K(S;w). \tag{13}$$

Correspondingly, we give an operator formulation with the operator R_S:

$$u = R_S(f;w) \ . \tag{14}$$

We make the following (monotonicity) assumptions for the operator R_S:

1. $f \leq f_1,\, w \leq w_1 \implies R_S(f;w) \leq R_S(f_1;w_1)$,
2. R_S is continuous.

Let us consider an iterative procedure, similar to Algorithm 1 in Section 3. Partition Ω into overlapping subdomains, $\Omega = \bigcup\limits_{i=1}^{n} \Omega_i$, and let $u^{k+\frac{i}{n}} \in K$ be given, $k \geq 0$, $i \geq 0$: Find $u^{k+\frac{i+1}{n}} \in K(\Omega_i; u^{k+\frac{i}{n}})$ such that $(i = 0, \dots, n-1$, $k = 0, 1, \dots)$

$$a\left(u^{k+\frac{i+1}{n}}, v - u^{k+\frac{i+1}{n}}\right) \geq \left(f, v - u^{k+\frac{i+1}{n}}\right) \ \ \forall v \in K\left(\Omega_i; u^{k+\frac{i}{n}}\right). \tag{15}$$

LEMMA 1. *If* $u^0 \in K$ *and* $a(u^0, v) \geq (f, v)$ $\forall v \in H_0^1(\Omega)$, $v \geq 0$, *then* $\forall\ k \geq 0$ *we have*

(i) $u^k \geq u^{k+1} \geq \psi$ *in* Ω*;*
(ii) $u^k = u^{k+1} \iff u^k$ *is the solution function.*

REMARK 2. *The latter assumptions can be established very easily for the bilinear form generated by operator* $\mathcal{L}$ *from (2) as well as for the corresponding mesh problem if* A *is an* M*-matrix.*

Lemma 1 has a useful consequence when we consider the obstacle problem as a free boundary problem, i.e. as a problem to find the boundary ∂G of the contact region G, where the solution function coincides with the given obstacle function,

$$G = \{x : x \in \Omega;\ u(x) = \psi(x)\}. \tag{16}$$

We define a subdomain $G_k \subset \Omega$ by

$$G_k = \{x : x \in \Omega;\ u^k(x) = \psi(x)\}. \tag{17}$$

COROLLARY 1.

$$G_k \subset G_{k+1}. \tag{18}$$

Using Lemma 1 we can prove

THEOREM 2. *Under the assumptions made the overlapping domain decomposition method is monotonically convergent.*

4.2. Two-sided approximations for free boundaries.

Theorem 2 gives us a sequence of "free boundaries" ∂G_k, which monotonically approaches the free boundary ∂G. In fact, by using the sequence $\{G_k\}_k$, it is possible to find two-sided approximations for the free boundary. Namely, let $u^k \in K$ and $G_k \subset \Omega$ be given. If we solve a linear problem: find $\hat{u}^k \in H_0^1(\Omega)$, $\hat{u}^k = \psi$ in G_k, such that

$$a(\hat{u}^k, v) = (f, v) \quad \forall v \in H_0^1(\Omega \setminus \overline{G}_k),$$

and define a new set by

$$\widehat{G}_k = \{x : x \in \Omega;\ \hat{u}^k(x) \le \psi(x)\},$$

we obtain

THEOREM 3.

$$G_{k-1} \subseteq G_k \subseteq G \subseteq \widehat{G}_k \subseteq \widehat{G}_{k-1}.$$

This result is naturally an important theoretical observation, but it has advantages in the practical computations, too. Namely, it gives two-sided approximations for the free boundary, which permit us to decompose the original domain into subdomains with linear and nonlinear subproblems on every iteration step.

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