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Parallel Iterative Solution of Symmetric Coupled FE/BE–Equations via Domain Decomposition

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ABSTRACT. The use of the FEM and BEM in different subdomains of a non–overlapping Domain Decomposition (DD) and their coupling over the coupling boundaries (interfaces) brings about several advantages in many practical applications. The resulting coupled FE–BE–DD equations can be reformulated as a system of linear algebraic equations with a symmetric, but indefinite system matrix. This paper provides a parallelization and a preconditioning of Bramble/Pasciak's CG (1988) applied to the symmetric, indefinite coupled FE–BE–DD equations. Both the parallelization and the preconditioning are essentially based on the domain decomposition approach. The parallelized algorithm is well suited for computations on MIMD computers with local memory and message passing principle.

1. Introduction

The Domain Decomposition (DD) approach offers many possibilities to marry the advantages of the Finite Element Method (FEM) to those of the Boundary Element Method (BEM) in many practical applications. For instance, in the magnetic field computation for electric motors, we can use the BEM in the air subdomains including the exterior of the motor more successfully than the FEM which is prefered in ferromagnetic materials where nonlinearities can occur in the partial differential equation (pde), or in subdomains where the right–hand side doesn't vanish [10]. The same is true for many problems in solid mechanics [11] and in other areas of mathematical physics. Various methods for coupling the FEM and the BEM can be found in [18]. A very straightforward and promising coupling technique was proposed by M. Costabel [3] and others. In the different

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subdomains of a non–overlapping domain decomposition, we use either the standard finite element (f.e.) Galerkin method or a mixed–type boundary element (b.e.) Galerkin method which are weakly coupled over the coupling boundaries (interfaces) Γ_C. The mixed b.e. Galerkin method makes use of the full Cauchy data representation on the b.e. subdomain boundaries via the Calderón projector. So, one drawback of the method is the necessity of calculating weakly singular, singular and hypersingular integrals. However, the progress that has been made in the combined analytical and numerical integration of such integrals recently removes this drawback more and more (see,e.g.,[**11**]). On the other hand, one can benefit from this mixed variational formulation in constructing highly efficient, parallel iterative solvers for the resulting f.e./b.e. DD equations. This topic has been considered much less extensive than the construction techniques, the numerical integration methods and the discretization error estimates (see [**12, 13, 16**] for b.e. equations).

In this paper we make use of the advantages of the mixed variational f.e./b.e. discretization and propose a algorithm for solving the coupled f.e./b.e. equations. First of all, the coupled f.e./b.e. equations can be reformulated in the symmetric and indefinite form

$$\left(\begin{array}{c|cc} K_\Lambda & -K_{\Lambda C} & \mathbb{O} \\ \hline -K_{C\Lambda} & -K_C & -K_{CI} \\ \mathbb{O} & -K_{IC} & -K_I \end{array}\right) \left(\begin{array}{c} \underline{u}_\Lambda \\ \underline{u}_C \\ \underline{u}_I \end{array}\right) = \left(\begin{array}{c} \underline{f}_\Lambda \\ \hline -\underline{f}_C \\ -\underline{f}_I \end{array}\right). \tag{1.1}$$

The coefficients of the unknown vectors $\underline{u}_C, \underline{u}_I$ and $\underline{u}_\Lambda$ approximate the unknown potential function (displacements in elasticity) at the nodes on the coupling boundary Γ_C, the unknown potential function (displacements) at the nodes in the f.e. subdomains and the normal derivatives (stresses) at the nodes on the boundary of the b.e. subdomains. We provide a preconditioning and a parallelization of Bramble/Pasciak's Conjugate Gradient (CG) method [**1**] applied to the symmetric and indefinite system (1.1). Both the parallelization and the preconditioning are essentially based on the DD approach. In order to derive the preconditioner, we use the Additive Schwarz Method (ASM) [**5, 8**]. The components of the preconditioner can be chosen such that the resulting algorithm is, at least, almost asymptotically optimal with respect to the operation count both in the 2D and 3D cases provided that an asymptotically optimal matrix–by–vector multiplication technique [**9**] is applied in the 3D case. Using a special data distribution, we parallelize the preconditioning equation and the remaining algorithm in such a way that the same amount of communication is needed as in the earlier introduced and well studied parallel PCG for solving symmetric and positive definite f.e. equations [**8**]. The parallelized algorithm presented in [**14**] is well suited for computations on MIMD computers with local memory and message passing principle (e.g. transputers). The hypercube seems to be the most suitable architecture for the implementation, at least, on a reasonable number of processors [**6**].

The remainder of the paper is organized as follows: In Section 2, we recall the mixed DD coupled domain and boundary integral variational formulation (Section 2.1) and the corresponding coupled f.e./b.e. Galerkin discretization (Section 2.2). The third Section is devoted to the preconditioning of Bramble/Pasciak's CG. The parallelization of this algorithm is discussed in [**14**], where the reader can also find the first numerical results. Finally, we draw some conclusions (Section 4).

2. The Coupled FE/BE–DD–Discretization

2.1. Mixed Variational Formulation. Let us consider the model Dirichlet potential problem

$$\text{-div}\,(a\,\nabla u) = f \quad \text{in } \Omega \quad \text{and} \quad u = g \quad \text{on } \Gamma_D = \Gamma = \partial\Omega, \tag{2.1}$$

in a plane, bounded domain $\Omega \subset \mathbb{R}^2$ with a Lipschitz-continuous, piecewise smooth boundary Γ . Boundary value problems of the kind (2.1) typically arise, for instance, in the magnetic field computation for electric motors [**10**]. In this case, the coefficient function $a(.)$ usually varies in orders of magnitude in different subdomains of the domain Ω . So, we divide Ω into p non–overlapping, simply connected subdomains Ω_i $(i = 1, 2, \ldots, p)$ called also substructures or superelements such that $\overline{\Omega} = \overline{\Omega}_B \cup \overline{\Omega}_F$, with $\overline{\Omega}_B = \cup_{i=1}^{q} \overline{\Omega}_i$ and $\overline{\Omega}_F = \cup_{i=q+1}^{p} \overline{\Omega}_i$ and assume that

$$\begin{cases} f = 0 \text{ in } \Omega_B,\ f \in L_2(\Omega_F),\ a \in L_\infty(\Omega_F), \text{ with } a \geq \text{const} > 0 \text{ in } \Omega, \\ a(x) = a_i = \text{const} > 0 \text{ in } \Omega_i \text{ for } i = 1, 2, \ldots, q,\ g \in H^{1/2}(\Gamma). \end{cases} \tag{2.2}$$

Let us denote the coupling boundaries (interfaces) of $\overline{\Omega}$, $\overline{\Omega}_B$ and $\overline{\Omega}_F$ by

$$\begin{cases} \Gamma_C = \Gamma_{BC} \cup \Gamma_{FC},\ \overline{\Gamma}_C = \overline{\Gamma}_{BC} \cup \overline{\Gamma}_{FC} = \Gamma_C \cup \Gamma_D,\ \Gamma_{BC} = \\ \overline{\Gamma}_{BC} \backslash \Gamma_D, \overline{\Gamma}_{BC} = \bigcup_{i=1}^{q} \Gamma_i,\ \Gamma_{FC} = \overline{\Gamma}_{FC} \backslash \Gamma_D,\ \overline{\Gamma}_{FC} = \bigcup_{i=q+1}^{p} \Gamma_i \end{cases} \tag{2.3}$$

respectively, where $\Gamma_i = \partial\,\Omega_i$ $(i = 1, 2, \ldots, p)$. Further we assume that there exist two positive constants $\underline{r}_i$ and $\overline{r}_i$ such that

$$0 < \underline{r}_i \leq \text{diam}(\Omega_i) \leq \overline{r}_i \quad \text{and} \quad 0 < \underline{c}_1 \leq \overline{r}_i / \underline{r}_i \leq \overline{c}_2 \tag{2.4}$$

for all $i = 1, 2, \ldots, p$, with fixed (i-independent) constants $\underline{c}_1$ and $\overline{c}_2$. Denote $H = \max\{\overline{r}_i\}$ and suppose, for simplicity, that $H_B = \max\{\overline{r}_1 \ldots \overline{r}_q\} < 1$. The last condition ensures the positive definiteness of the single layer potential operator [**12, 16**].

Following M.Costabel [**3**], G.C.Hsiao and W.L.Wendland [**12**], W.L.Wendland [**18**] and others, we can reformulate the b.v.p (2.1) as a mixed DD coupled domain and boundary integral variational problem: Find $(\lambda, u) \in \mathbb{V}_g = \Lambda \times \mathbb{U}_g$:

$$a(\lambda, u; \eta, v) = \langle F_{\Omega_F}, v \rangle \ \forall\, (\eta, v) \in \mathbb{V}_0 = \Lambda \times \mathbb{U}_0 \tag{2.5}$$

where

$$
(2.6)\quad \left\{\begin{array}{rcl}
a(\lambda,u;\eta,v) & := & a_B(\lambda,u;\eta,v)+a_F(u,v),\\
a_B(\lambda,u;\eta,v) & := & \sum\limits_{i=1}^{q} a_i\left\{\langle D_iu_i,v_i\rangle_{\Gamma_i}+\langle\eta_i,V_i\lambda_i\rangle_{\Gamma_i}+\tfrac{1}{2}\langle\lambda_i,v_i\rangle_{\Gamma_i}\right.\\
 & & \left.-\tfrac{1}{2}\langle\eta_i,u_i\rangle_{\Gamma_i}+\langle\lambda_i,K_iv_i\rangle_{\Gamma_i}-\langle\eta_i,K_iu_i\rangle_{\Gamma_i}\right\},\\
a_F(u,v) & := & \sum\limits_{i=q+1}^{p}\int\limits_{\Omega_i} a(x)\nabla^T u(x)\nabla v(x)\,dx,\\
\langle F_{\Omega_F},v\rangle & := & \sum\limits_{i=q+1}^{p}\int\limits_{\Omega_i} f(x)v(x)\,dx,\\
\langle\lambda_i,v_i\rangle_{\Gamma_i} & := & \int\limits_{\Gamma_i}\lambda_iv_i\,ds,\ \text{and } u_i=u|_{\Gamma_i}, v_i=v|_{\Gamma_i}\ \text{etc.},
\end{array}\right.
$$

with the boundary integral operators V_j, K_j, D_j defined by the relations

$$
(2.7)\quad \left\{\begin{array}{rcl}
V_i\lambda_i(x) & := & \int\limits_{\Gamma_i} E(x,y)\lambda_i(y)\,ds_y\\
K_iv_i(x) & := & \int\limits_{\Gamma_i}\partial_y E(x,y)v_i(y)\,ds_y\\
D_iu_i(x) & := & -\partial_x\int\limits_{\Gamma_i}\partial_y E(x,y)u_i(y)\,ds_y \;=\; -\partial_x K_iv_i(x)
\end{array}\right.
$$

and the fundamental solution $E(x,y) = -(log|x-y|)/2\pi$ of the Laplacian. The mapping properties of the boundary integral operators (2.7) on Sobolev spaces are now well known [**4**]. The manifold $\mathbb{U}_g$ and the spaces $\mathbb{U}_0$ and Λ are defined by the corresponding energy and trace spaces [**12, 14**].

The existence and uniqueness of the solution of the variational problem (2.5) can be easily shown [**12**].

2.2. Coupled BE/FE–Discretization in the Nodal Basis. Let us divide the last (p-q) subdomains $\overline{\Omega}_i$ $(i = q+1, q+2, \ldots, p)$ into finite elements δ_r such that this discretization process results in a conform triangulation of $\overline{\Omega}_F$, and let us continue the discretization process to $\overline{\Gamma}_{BC}$ in an analogous (one-dimensional) manner for linear elements (see, e.g.,[**11**] for higher–order elements).

To avoid the homogenization of the Dirichlet boundary condition, we suppose that g=0 on Γ_D for simplicity. Define now the usual nodal b.e./f.e. basis

$$
(2.8)\qquad \Phi = [\varphi_1,\ldots,\varphi_{N_\Lambda};\varphi_{N_\Lambda+1},\ldots,\varphi_{N_\Lambda+N_C};\varphi_{N_\Lambda+N_C+1},\ldots,\varphi_N],
$$

where $N = N_\Lambda + N_C + N_I$, $N_\Lambda = N_{\Lambda,1}+\cdots+N_{\Lambda,q}$ and $N_I = N_{I,q+1}+\cdots+N_{I,p}$. Thus, the first N_Λ basis functions are used to approximate λ_1 on Γ_1 ($N_{\Lambda,1}$ basis functions), ... , λ_q on Γ_q ($N_{\Lambda,q}$ basis functions); the next N_C basis functions are used to approximate u on Γ_C; and, finally, the last N_I basis functions are defined in $\Omega_{q+1}(N_{I,q+1}$ basis functions), ... , in $\Omega_p(N_{I,p}$ basis functions) for approximating u in the subdomains $\Omega_{q+1},\ldots,\Omega_p$. Here and in the following the

indices "Λ", "C" and "I" mark quanties corresponding to the λ's on $\overline{\Gamma}_{BC}$, to the $u|_{\Gamma_C}$'s on the coupling boundary Γ_C and to the interior $\Omega_I = \bigcup_{i=q+1}^{p} \Omega_i$ of the f.e.-subdomains, respectively.

The f.e./b.e. subspace

$$\mathbb{V} \equiv \mathbb{V}_h = \Lambda_h \times \mathbb{U}_{0h} \subset \mathbb{V}_0 \tag{2.9}$$

is now defined by the finite dimensional spaces

$$\Lambda_{h,i} = \operatorname{span} \Phi_{\Lambda,i} \ (i = 1, \ldots, q) \ \text{ and } \ \mathbb{U}_{0h} = \operatorname{span} \Phi_U \tag{2.10}$$

with $\Lambda_h = \Lambda_{1,h} \times \cdots \times \Lambda_{q,h}$. The bases Φ_U and $\Phi_{\Lambda,i}$ are given by the relations

$$\Phi_U = [\varphi_{N_\Lambda+1}, \ldots, \varphi_{N_\Lambda+N_C}, \varphi_{N_\Lambda+N_C+1}, \ldots, \varphi_N] \quad \text{and} \tag{2.11}$$

$$\Phi_{\Lambda,i} = [\varphi_{\Sigma_i+1}, \ldots, \varphi_{\Sigma_i+N_{\Lambda,i}}] \quad i = 1, 2, \ldots, q\,, \tag{2.12}$$

with $\Sigma_1 = 0$ and $\Sigma_i = N_{\Lambda,1} + \cdots + N_{\Lambda,i-1}$ for i=2,3, ... ,q . Once the basis Φ for $\mathbb{V}$ is chosen, the Galerkin f.e./b.e. approximation to (2.5) results in the system

$$K\underline{u} = \underline{f} \tag{2.13}$$

of coupled linear f.e./b.e. equations. Because of the properties of the bilinear form a(.,.), the system matrix K is positive definit (p.d.), but non–symmetric. Note that asymptotic convergence estimates can be found in [**12**].

Taking into account the arrangement of the unknowns given in (2.8), we can rewrite system (2.13) in the block form

$$\begin{pmatrix} K_\Lambda & -K_{\Lambda C} & \mathbb{O} \\ K_{C\Lambda} & K_C & K_{CI} \\ \mathbb{O} & K_{IC} & K_I \end{pmatrix} \begin{pmatrix} \underline{u}_\Lambda \\ \underline{u}_C \\ \underline{u}_I \end{pmatrix} = \begin{pmatrix} \underline{f}_\Lambda \\ \underline{f}_C \\ \underline{f}_I \end{pmatrix}. \tag{2.14}$$

The system matrix K satisfies the identity

$$(K\underline{u}, \underline{v}) = a(\lambda, u; \eta, v) \tag{2.15}$$

for all $\underline{u}, \underline{v} \in \mathbb{R}^N$ with $\underline{u} \leftrightarrow (\lambda, u) = \Phi\underline{u} \in \mathbb{V}_h$ and $\underline{v} \leftrightarrow (\eta, v) = \Phi\underline{v} \in \mathbb{V}_h$.
The latter f.e./b.e. isomorphisms $\underline{u} = \left(\underline{u}_\Lambda^T, \underline{u}_C^T, \underline{u}_I^T\right)^T \leftrightarrow (\lambda, u)$ means

$$\underline{u}_1 = \underline{u}_\Lambda = \left(\underline{u}_{\Lambda,1}^T, \ldots, \underline{u}_{\Lambda,q}^T\right)^T \leftrightarrow \lambda = \Phi_\Lambda \underline{u}_\Lambda := \left(\Phi_{\Lambda,1}\underline{u}_{\Lambda,1}; \ldots, \Phi_{\Lambda,q}\underline{u}_{\Lambda,q}\right) \in \Lambda_h$$

and

$$\underline{u}_2 = \left(\underline{u}_C^T; \underline{u}_I^T\right)^T \leftrightarrow u = \Phi_U \underline{u}_2 = \sum_{i=N_\Lambda+1}^{N} u^{(i)} \varphi_i \in \mathbb{U}_{0h}.$$

The same is true for the correspondence $\underline{v} \leftrightarrow (\eta, v)$.

Now we are interested in highly parallelizable and efficient iterative methods for solving the system (2.14) of the coupled f.e./b.e. equations. A promising

approach to the construction of such methods consists in the reformulation of system (2.14) as the symmetric, but indefinite system

$$\begin{pmatrix} K_1 & K_{12} \\ K_{21} & -K_2 \end{pmatrix} \begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \end{pmatrix} = \begin{pmatrix} \underline{f}_1 \\ \underline{f}_2 \end{pmatrix} \tag{2.16}$$

by multiplying the last two block equations in (2.14) by (-1), where we use the notations $K_1 = K_\Lambda, \underline{f}_1 = \underline{f}_\Lambda$ and

$$K_{21} = K_{12}^T = \begin{pmatrix} -K_{\Lambda C} \\ \mathbb{O} \end{pmatrix}, K_2 = \begin{pmatrix} K_C & K_{IC} \\ K_{CI} & K_I \end{pmatrix}, \underline{f}_2 = \begin{pmatrix} -\underline{f}_C \\ -\underline{f}_I \end{pmatrix}.$$

In the next section, we apply Bramble/Pasciak's method of transforming system (2.16) into a equivalent system the system matrix of which is positive definite and self–adjoint with respect to a specially chosen inner product [1].

3. Parallelization and Preconditioning of Bramble/Pasciak's CG via Domain Decomposition

3.1. Bramble/Pasciak's CG for Symmetric and Indefinite Systems.

Let us suppose that there is a symmetric and positive definite matrix

$$C_1 \left(= C_\Lambda = \operatorname{diag}\left[C_{\Lambda,i}\right]_{i=1,\ldots,q}\right)$$

satisfying the spectral equivalence inequalities

$$\underline{\gamma}_1 C_1 \leq K_1 \leq \overline{\gamma}_1 C_1 \tag{3.1}$$

with positive spectral equivalence constants $\underline{\gamma}_1$ and $\overline{\gamma}_1$. Without restriction of generality, we assume that $\underline{\gamma}_1 > 1$. The transformed system

$$M\underline{u} = \underline{g} \tag{3.2}$$

with the system matrix

$$M = \begin{pmatrix} C_1^{-1}K_1 & C_1^{-1}K_{12} \\ K_{21}C_1^{-1}(K_1 - C_1) & K_2 + K_{21}C_1^{-1}K_{12} \end{pmatrix} \tag{3.3}$$

and the right–hand side

$$\underline{g} = \left[\underline{g}_1^T, \underline{g}_2^T\right]^T \quad , \underline{g}_1 = C_1^{-1}\underline{f}_1, \quad \underline{g}_2 = K_{21}C_1^{-1}\underline{f}_1 - \underline{f}_2, \tag{3.4}$$

is obviously equivalent to system (2.16). The system matrix M is self–adjoint and positive definite with respect to the new inner product [.,.] defined by

$$\left[\begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \end{pmatrix}, \begin{pmatrix} \underline{v}_1 \\ \underline{v}_2 \end{pmatrix}\right] := ((K_1 - C_1)\underline{u}_1, \underline{v}_1) + (\underline{u}_2, \underline{v}_2) \tag{3.5}$$

for all $\underline{u} = \left(\underline{u}_1^T, \underline{u}_2^T\right)^T$ and $\underline{v} = \left(\underline{v}_1^T, \underline{v}_2^T\right)^T \in \mathbb{R}^N$. Because of assumption $\underline{\gamma}_1 > 1$, the operator $K_1 - C_1$ is positive definite and inequalities

$$\underline{\alpha}_1 \left(K_1 \underline{u}_1, \underline{u}_1\right) \leq \left((K_1 - C_1)\underline{u}_1, \underline{u}_1\right) \leq \overline{\alpha}_1 \left(K_1 \underline{u}_1, \underline{u}_1\right) \quad \forall \underline{u}_1 \in \mathbb{R}^{N_1} \tag{3.6}$$

hold, with $\underline{\alpha}_1 = 1 - \left(1/\underline{\gamma}_1\right)$ and $\overline{\alpha}_1 = \alpha = 1 - (1/\overline{\gamma}_1)$. Moreover, J.H.Bramble and J.F.Pasciak proved the spectral equivalence inequalities (Theorem 1 in [1])

$$\underline{\lambda}\,[R\,\underline{u}, \underline{u}] \leq [M\,\underline{u}, \underline{u}] \leq \overline{\lambda}\,[\underline{R}\,\underline{u}, \underline{u}] \quad \forall \underline{u} \in \mathbb{R}^N \tag{3.7}$$

with the spectral equivalence constants

$$\underline{\lambda} = \left(1 + \frac{\alpha}{2} + \sqrt{\alpha + \frac{\alpha^2}{4}}\right)^{-1} \quad \text{and} \quad \overline{\lambda} = \frac{1 + \sqrt{\alpha}}{1 - \alpha} \tag{3.8}$$

and with the regularisator

$$R = \begin{pmatrix} I_1 & \mathbb{O} \\ \mathbb{O} & K_2 + K_{21} K_1^{-1} K_{12} \end{pmatrix} \tag{3.9}$$

from which we can easily derive a preconditioner C for M provided that a preconditioner C_2 for $K_2 + K_{21}K_1^{-1}K_{12}$ is known. In the next subsection we use the ASM in order to construct the preconditioner C_2.

3.2. The ASM–DD Block Preconditioner C_2. Let us construct a symmetric and positive definite block preconditioner C_2 for the matrix

$$K_2 + K_{21}K_1^{-1}K_{12} = \begin{pmatrix} K_C + K_{C\Lambda}K_\Lambda^{-1}K_{\Lambda C} & K_{CI} \\ K_{IC} & K_I \end{pmatrix} \tag{3.10}$$

by means of the Additive Schwarz Method (ASM). Beside the nodal basis

$$\Phi_2 = \Phi_U = \left(\underline{\Phi}_{N_\Lambda+1}, \ldots, \underline{\Phi}_{N_\Lambda+N_C}, \ldots, \underline{\Phi}_N\right), \tag{3.11}$$

we consider the approximate discrete harmonic basis [8]

$$\tilde{\Phi}_2 = \Phi_2 \tilde{V}_2 \tag{3.12}$$

obtained by the basis transformation matrix

$$\tilde{V}_2 = \begin{pmatrix} I_C & \mathbb{O} \\ -D_I^{-1}K_{IC} & I_I \end{pmatrix} = \left(\tilde{V}_{2,C}\ \tilde{V}_{2,I}\right), \tag{3.13}$$

where D_I is an appropriately chosen non–singular matrix. We suppose that the matrix $D_I = \operatorname{diag}\left(D_{I,i}\right)_{i=q+1,\ldots,p}$ has the same block diagonal structure as K_I and that systems with the system matrix $D_{I,i}$ can be solved much faster than systems with the original matrix $K_{I,i}$ (see [8] for example). The symmetry and the positive definitness of D_I are not required.

Now, the f.e./b.e. subspace $\mathbb{U}_{0h}$ can be represented in the form

$$\mathbb{U}_{0h} = \tilde{\mathbb{U}}_C + \tilde{\mathbb{U}}_I \tag{3.14}$$

of the direct sum of the subspaces

$$\tilde{\mathbb{U}}_C = span\,\Phi_2\tilde{V}_{2,C} \quad \text{and} \quad \tilde{\mathbb{U}}_I = span\,\Phi_2\tilde{V}_{2,I}. \tag{3.15}$$

Introduce on $\mathbb{U}_{0h}$, for all $u, v \in \mathbb{U}_{0h}$ the auxiliary bilinear form

$$\tilde{a}(u,v) = a_F(u,v) + \sum_{i=1}^{q} a_i \langle Du_i, v_i\rangle_{\Gamma_i} + \left(K_{C\Lambda}K_\Lambda^{-1}K_{\Lambda C}\underline{u}_C, \underline{v}_C\right) \tag{3.16}$$

with the f.e. part $a_F(.,.)$ defined in (2.6) and with the correspondence $u|_{\Gamma_C} \leftrightarrow \underline{u}_C, v|_{\Gamma_C} \leftrightarrow \underline{v}_C$. Now we can prove a strengthened Cauchy inequality for the decomposition (3.14) with respect to the symmetric and positiv bilinear form $\tilde{a}(.,.)$ defined on $\mathbb{U}_{0h} \times \mathbb{U}_{0h}$.

LEMMA 3.1. *For all $u \in \tilde{\mathbb{U}}_C$ and $v \in \tilde{\mathbb{U}}_I$, the strengthened Cauchy inequality*

$$|\tilde{a}(u,v)| \le \sqrt{\frac{\mu}{1+\mu}}\,(\tilde{a}(u,u))^{1/2}\,(\tilde{a}(v,v))^{1/2} \tag{3.17}$$

holds, where $\mu = \rho\left(\overset{\circ}{S}_C^{-1}\overset{\circ}{T}_C\right)$ is the spectral radius of $\overset{\circ}{S}_C^{-1}\overset{\circ}{T}_C$. The matrix $\overset{\circ}{S}_C = \overset{\circ}{K}_C - \overset{\circ}{K}_{CI}\,K_I^{-1}\,\overset{\circ}{K}_{IC}$ denotes the f.e. Schur complement and the operator $\overset{\circ}{T}_C$ is defined by the relation

$$\overset{\circ}{T}_C = \overset{\circ}{K}_{CI}\left(K_I^{-1} - D_I^{-T}\right)K_I\left(K_I^{-1} - D_I^{-1}\right)\overset{\circ}{K}_{IC}\,. \tag{3.18}$$

The constant in (3.17) is sharp, i.e. $\gamma = \cos \sphericalangle\left(\tilde{\mathbb{U}}_C, \tilde{\mathbb{U}}_I\right) = (\mu/(1+\mu))^{0.5} < 1$.

LEMMA 3.2. *If the symmetric and positive definite block preconditioners C_C and $C_I = diag(C_{I,i})_{i=\overline{1,q}}$ satisfy the spectral equivalence inequalities*

$$\underline{\gamma}_C C_C \le S_C + T_C + B_C + K_{C\Lambda}K_\Lambda^{-1}K_{\Lambda C} \le \overline{\gamma}_C C_C \quad \textit{and} \tag{3.19}$$

$$\underline{\gamma}_I C_I \le K_I \le \overline{\gamma}_I C_I \tag{3.20}$$

with positive spectral equivalence constants $\underline{\gamma}_C, \overline{\gamma}_C, \underline{\gamma}_I, \overline{\gamma}_I$, then the ASM preconditioner

$$C_2 = \begin{pmatrix} I_C & K_{CI}D_I^{-T} \\ \mathbb{O} & I_I \end{pmatrix}\begin{pmatrix} C_C & \mathbb{O} \\ \mathbb{O} & C_I \end{pmatrix}\begin{pmatrix} I_C & \mathbb{O} \\ D_I^{-1}K_{IC} & I_C \end{pmatrix} \tag{3.21}$$

is symmetric, positive definite and satisfies the spectral equivalence inequalities

$$\underline{\gamma}_2 C_2 \le K_2 + K_{21}K_1^{-1}K_{12} \le \overline{\gamma}_2 C_2 \tag{3.22}$$

with the spectral equivalence constants

$$\underline{\gamma}_2 = \min\left\{\underline{\gamma}_C, \underline{\gamma}_I\right\}(1-\gamma) \quad \textit{and} \quad \overline{\gamma}_2 = \max\left\{\overline{\gamma}_C, \overline{\gamma}_I\right\}(1+\gamma) \tag{3.23}$$

The proofs of Lemmas 3.1 and 3.2 are given in [**14**]. Combining Bramble/Pasciak's results (see Section 3.1) and the results of this section, we arrive at the following spectral equivalence theorem.

THEOREM 3.1. *Let us suppose that the conditions imposed above on* $C_\Lambda = C_1, C_C, C_I$ *and* D_I, *especially, the spectral equivalence inequalities* (3.1) *and* (3.19) *are satisfied. Then the f.e./b.e. DD preconditioner*

$$C = diag(I_1, C_2) \tag{3.24}$$

is self–adjoint and positive definite with respect to the inner product [.,.] and satisfies the spectral equivalence inqualities

$$\underline{\gamma}\,[C\underline{u}, \underline{u}] \leq [M\underline{u}, \underline{u}] \leq \overline{\gamma}\,[C\underline{u}, \underline{u}] \quad \forall \underline{v} \in \mathbb{R}^N, \tag{3.25}$$

with the spectral equivalence constants

$$\underline{\gamma} = \underline{\lambda} \min\left\{1, \underline{\gamma}_2\right\} \quad \textit{and} \quad \overline{\gamma} = \overline{\lambda} \max\left\{1, \overline{\gamma}_2\right\}, \tag{3.26}$$

where $\underline{\lambda}, \overline{\lambda}$ *and* $\underline{\gamma}_2, \overline{\gamma}_2$ *are defined by* (3.8) *and* (3.23), *respectively.*

In [**14**], the data distribution to the processors of a MIMD–computer with distributed memory and the parallelization of the ASM–DD–preconditioned Bramble/Pasciak CG is discussed in detail.

4. Conclusions

It follows from the results of Theorem 3.1 and from the papers mentioned above that there are such basis transformations D_I and such preconditioners C_Λ, C_C and C_I that the ASM–DD–preconditioned Bramble/Pasciak CG is asymptotically optimal or, at least, almost optimal with respect to the operation count and well suited for computation on MIMD computers with local memory and message passing principle [**7**]. The components D_I, C_Λ, C_C, and C_I of the preconditioner can be adapted to the case of strongly varying coefficients and to other specialities of the boundary value problem under consideration (see,e.g., [**2, 6, 8, 13, 15, 16**]). In the 3D case, these components are also avialable [**13, 16, 17**].

Our algorithm works in the case of coupled f.e./b.e. equations ($1 \leq q < p$) as well as for "pure" b.e. equations (q=p) and for "pure" f.e. equations (q=0). In the latter case, the algorithm coincides with the parallelized PCG for solving symmetric and positive definite f.e. equations presented earlier in [**6, 8**]. In this connection, we can expect that our algorithm processes similar excellent properties as the well studied and well tested PCG [**6, 7, 8**].

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