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Interface conditions for heterogeneous domain decomposition: Coupling of different hyperbolic systems

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ABSTRACT. We consider a domain decomposition method for coupling of different sets of hyperbolic systems of partial differential equations. The domain consists of several types of subdomains, each type is characterized by a hyperbolic system. We show how we can construct interface conditions between subdomains, and show that for a specific problem (ocean/bottom acoustics problem) that the coupled system is well-posed, and that the Chebyshev spectral collocation approximation to this problem is time-stable.

1. Introduction

We consider the solution of a physical problem in a domain containing several types of media, and assume that the physics in each medium is governed by a set of hyperbolic partial differential equations (PDEs). For the sake of simplicity consider two media and let one medium be a subdomain. This situation is illustrated in figure 1.

The PDE systems are formulated as first order systems and we allow the PDEs to be quasi-linear.

$$u_t^i + \sum_{j=1}^{d} A_j(u^i)\, u_{x_j}^i = f^i \qquad \text{in} \quad \Omega_i \times (0,T), \quad i = 1,2 \tag{1.1}$$

where Ω_i is an open bounded domain in $\mathbb{R}^m$, $m = 1,2$ or 3, u and f are vector functions: $u^i, f^i : (\Omega_i \times (0,T)) \to \mathbb{R}^n$, $n \geq 1$. $A_j(u^i)$ are $n \times n$ matrices possibly depending on u^i. The system (1.1) is supplemented by initial conditions of the form $u^i(x,0) = \phi^i(x)$, $x \in \Omega_i$, by boundary conditions on $\Gamma_i \times (0,T)$ and by

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interface conditions on $\Gamma_I \times (0, T)$. The boundary conditions and the interface conditions will be discussed below.

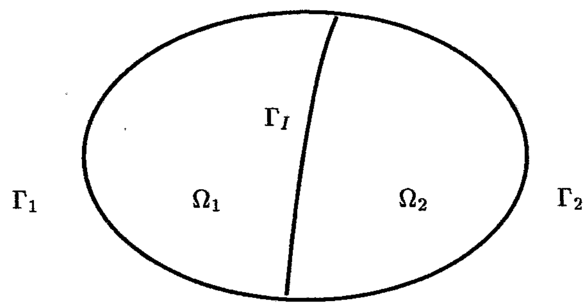

FIGURE 1. The two different media case

Since we discuss hyperbolic systems, the boundary conditions will have to be specified using the characteristic variables of these PDE systems, see e.g. [**5**], [**2**], [**1**], [**10**].

Physical examples with different media in the domain occurs frequently, e.g. in vibration problems. A typical example is the coupling between an acoustic field and an elastic body. Coupled vibration problems are studied in detail in e.g. [**4**] and [**7**], but without reference to domain decomposition and interface conditions.

Our goal is to construct efficient numerical methods for such coupling problems, and in [**10**] we have presented a model for ocean/bottom acoustics using the domain decomposition method. The purpose of this paper is to provide a better basis for the model presented in [**10**]. The rest of the paper is organized as follows: In section 2 we present the interface conditions in a general form together with a specific example. The well-posedness of the coupled problem and the time-stability of the discretized version of the example in section 2 is presented in section 3. The space discretization is by Chebyshev spectral collocation. For numerical results from the ocean/bottom acoustic model we refer to [**1**], [**10**] and [**9**].

2. Boundary and interface conditions

The interface conditions between the subdomains (of the same type or not) must be based on the physical boundary conditions of the problem. Examples of such physical boundary conditions in the case of continuum mechanics are: (The summation convention is understood)

a) Continuous normal velocity: $u_i^1 n_i = u_i^2 n_i$

b) Continuity in momentum: $\sigma_{ij}^1 n_j = \sigma_{ij}^2 n_j = \gamma n_i$.

c) Continuous heat flux: $\nu^1 n_i (\nabla T^1)_i = \nu^2 n_i (\nabla T^2)_i$.

where $\{n_i\}$ are the components of the normal vector at a point on the interface, σ_{ij}^k the components of the stress tensor in Ω_k, γ a quantity representing surface tension, ν^k the thermal conductivity in Ω_k and T^k the temperature in Ω_k.

The construction of the interface conditions is done as follows: Consider a point p of the interface and denote by $\{\psi_i^1\}$ and $\{\psi_i^2\}$ the locally one-dimensional (normal to the interface) characteristic variables in the two media. Take corresponding characteristic variables, i.e. those containing corresponding variables in the physical boundary conditions, say ψ_k^1 and ψ_l^2 and consider a pair of "ghost" characteristics $\hat{\psi}_k^1$ and $\hat{\psi}_l^2$ which have the following meaning: If we consider the situation seen from Ω_1 then some incoming characteristic, which is unknown since its impact comes from Ω_2, should be combined with the outgoing characteristic in order to get the corrected values of the physical variables at the interface point. The situation is illustrated in figure 2. The same reasoning applies to Ω_2.

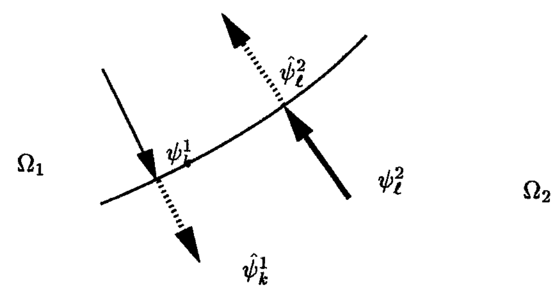

FIGURE 2. Real and "ghost" characteristics

Now perform the correction of the physical variables from ψ_k^1 and $\hat{\psi}_k^1$ in Ω_1 and from ψ_l^2 and $\hat{\psi}_l^2$ in Ω_2, and use the physical boundary conditions to find $\hat{\psi}_k^1$, $\hat{\psi}_l^2$ and the corrected values at the interface.

A differential type of interface conditions for hyperbolic systems is constructed in [**3**], and this method and the correction method are closely related. The main advantage of method in [**3**] is that the problem including boundary and/or interface conditions can be formulated as a single ODE set. In addition the differential approach allow for implicit time integrators, for a discussion of the problems in using implicit integrators, see [**5**].

Let us consider an example of the correction procedure. In [**10**] we studied the coupling of the equations for adiabatic wave motion in water:

$$\rho_t^1 + \nabla \cdot (\rho^1 u^1) = 0 \tag{2.1}$$

$$u_t^1 + (u \cdot \nabla) u^1 + \frac{1}{\rho} \nabla p = -g \tag{2.2}$$

$$p_t + (u^1 \cdot \nabla) p + \rho^1 C^2 \nabla \cdot u^1 = 0 \tag{2.3}$$

and the equations of linear elasticity in sea bottom:

$$\rho^2 u_t^2 = Div\,\sigma \tag{2.4}$$

$$\sigma_t = \lambda I \nabla \cdot u^2 + \mu(\nabla u^2 + (\nabla u^2)^T) \tag{2.5}$$

where ρ^i are the densities, u^i the velocity vectors, C the sound speed, p the pressure, σ the stress tensor and λ, μ the Lamé coefficients.

For simplicity we consider a horizontal interface (Ω_1 above Ω_2). In Ω_1 the "fast" outgoing characteristic is $\psi_1^1 = p - \bar{\rho}^1 \bar{C} v^1$ and the corresponding "ghost" characteristic is $\hat{\psi}_1^1 = \hat{p} + \bar{\rho}^1 \bar{C} \hat{v}^1$. The characteristic in Ω_2 corresponding to ψ_1^1 is $\psi_1^2 = \sigma_{22} - \rho^2 c_P v^2$ with its accompanying "ghost" characteristic. The correction procedure for the vertical velocity in the two media and the physical boundary condition gives the relation:

$$\frac{1}{2\bar{\rho}^1 \bar{C}}(\hat{\psi}_1^1 - \psi_1^1) = \frac{1}{2\rho^2 c_P}(\hat{\psi}_1^2 - \psi_1^2) \tag{2.6}$$

Similarly, the continuity of the vertical stress gives the relation:

$$\frac{1}{2}(\hat{\psi}_1^1 + \psi_1^1) = \frac{1}{2}(\hat{\psi}_1^2 + \psi_1^2) \tag{2.7}$$

Hence we can find $\hat{\psi}_1^1$ and $\hat{\psi}_1^2$ and we can compute the corrected values of p, v^1, σ_{22} and v^2 at the interface. For more details, see **[10]**. This procedure is also used if the interface has an arbitrary form. The details are in **[9]**, but the procedure is briefly as follows: If the interface at a point p forms an angle $\theta(p)$ with the horizontal, we rotate the coordinate system this angle so the horizontal becomes tangent to the interface at p. The components of u^1, u^2 and σ in this coordinate system is then computed and the correction procedure described above is applied to these components. Finally, the inverse coordinate transform is performed, and the corrected values are computed in the original coordinate system.

Remark 2.1. Note that the interface procedure can be put in the framework of the bicharacteristics method of Kopriva **[8]**. He uses this method for homogeneous domain decomposition. For details on both homogeneous and heterogeneous domain decomposition for the two PDE sets (2.1)- (2.3) and (2.4)-(2.5), see **[11]**.

Remark 2.2. For rectangular domains the corners are treated by rotating the coordinate system so that one axis bisects the angle of the corner. The correction procedure is then applied based on the components of the physical variables along this axis.

We have developed a model for acoustic propagation in water and bottom by using Chebyshev collocation on the equations (2.1)-(2.3) and (2.4)-(2.5), open boundary conditions (see **[1]**) and interface conditions as described above. The numerical experiments with this model shows that the interface procedure gives a time-stable domain decomposition method. Some results from the model are reported in [**9**] and [**1**].

3. Stability of the coupled problem

We would like to show that the PDE systems (2.1)-(2.3) and (2.4)-(2.5) with the prescribed physical boundary conditions forms a well-posed problem. It is easier to work with the linearized equations (and in acoustics problems they are a very good approximation), so assume that we linearize locally (2.1)-(2.3) $\{\rho_0, u_0, p_0\}$. We also introduce the "entropy" variable: $\tilde{\rho} = \rho - p/C_0^2$, and the equations become:

$$u_t^1 + (u_0 \cdot \nabla)u^1 + \frac{1}{\rho_0}\nabla p = -g \tag{3.1}$$

$$p_t + (u^1 \cdot \nabla)p + C_0^2\rho_0\nabla \cdot u^1 = 0 \tag{3.2}$$

$$\tilde{\rho}_t + u_0 \cdot \nabla\tilde{\rho} = 0 \tag{3.3}$$

We will also use another form of the equations of linear elasticity by introducing the displacement vector s:

$$s_t = u^2 \tag{3.4}$$

$$\rho^2 u_t^2 = Div\,\sigma(s) - \rho^2 g \tag{3.5}$$

The two PDE sets combined can be written i

$$U_t + \mathcal{A}U = F \tag{3.6}$$

where $U = [u^1, p, \tilde{\rho}, s, u^2]^T$. We can use semigroup theory, see e.g. [**7**, ch. IV], to show that the operator $\mathcal{A}$ is dissipative in a Hilbert space setting, by defining an appropriate configuration space including the physical boundary conditions and a corresponding inner product. As configuration space we use

$$\mathcal{H} = L^2_{\rho_0}(\Omega_1) \times L^2_{\frac{1}{\rho_0 C_0^2}}(\Omega_1) \times L^2(\Omega_1) \times H^1(\Omega_2) \times L^2(\Omega_2)$$

equipped with the scalar product

$$\begin{aligned}(U, \hat{U})_{\mathcal{H}} &= \rho_0 \int_{\Omega_1} u_i\,\hat{u}_i\,dx + \frac{1}{\rho_0 C_0^2}\int_{\Omega_1} p\,\hat{p}\,dx + \int_{\Omega_1} \tilde{\rho}\,\hat{\tilde{\rho}}\,dx \\ &\quad + \int_{\Omega_2} \sigma_{ij}(s)\,\epsilon_{ij}(s)\,dx + \rho^2 \int_{\Omega_2} u_i^2\,\hat{u}_i^2\,dx\end{aligned}$$

where ϵ_{ij} is the strain tensor of the elastic medium.
Now computing $(\mathcal{A}U, U)_{\mathcal{H}}$ we get:

$$(\mathcal{A}U, U)_{\mathcal{H}} = \int_{\partial\Omega_2} \sigma_{ij} n_j u_i^2\,dx + \int_{\partial\Omega_1} p u_i^1\,dx \tag{3.7}$$

Because of the physical boundary conditions on Γ_I this reduces to

$$(3.8)\qquad (\mathcal{A}U,U)_{\mathcal{H}} = \int_{\Gamma_2} \sigma_{ij} n_j u_i^2\,dx + \int_{\Gamma_1} p u_i^1\,dx$$

so if there is no inflow form the boundaries (the normal vectors point outwards) or the boundaries are sufficiently far out so that $u_i^1 = 0$ and $u_i^2 = 0$, we then have $(\mathcal{A}U,U)_{\mathcal{H}} \geq 0$. Two other conditions will also have to be checked:

- $Range(\mathcal{A}) = \mathcal{H}$
- $\zeta = 0$ belongs to the resolvent set of $\mathcal{A}$.

These conditions are easily found to be satisfied in this case. Hence we have the result:

THEOREM 3.1. *The operator $\mathcal{A}$ as defined in (3.6) is the generator of a contractive semigroup in $\mathcal{H}$.*

In order to show the time-stability of the interface procedure, it is convenient to work with the PDE sets in characteristic form. (We now return to use (2.4) and (2.5) again). If we consider a horizontal interface the locally one-dimensional version of the two sets are:

$$(3.9)\qquad W_t^i + \Lambda^i W_y^i = G^i \qquad i = 1,2$$

where $W^1 = \{p+\rho_0 C v^1, u^1, \rho - p/C^2, p - \rho_0 C v^1\}$, $W^2 = \{(\lambda+2\mu)\sigma_{11} - \lambda\sigma_{22}, \sigma_{22} \mp \rho^2 c_P V^2, \sigma_{12} \mp \rho^2 c_S u^2\}$, $\Lambda^1 = diag(C,0,0,-C)$, $\Lambda^2 = diag(0, \pm c_P, \pm c_S)$. We discretize in space by Chebyshev spectral collocation, and apply the interface procedure as described briefly above (for details see [**10**]): Outgoing variables are unchanged, incoming variables are determined by a linear combination of the corresponding outgoing variable (reflection) and the corresponding variable in the other domain (transmission). We want to indicate how we can show that the discretized version of $(\|W^1\| + \|W^2\|)$ in an appropriate norm is bounded in time. The discretized variables are denoted by W_N^i. The original problem is reduced to a (locally) one-dimensional problem, and this is reasonable since our interface conditions are locally one-dimensional. Since we consider linear or linearized equations, we can apply the theory in [**6**]. We will use the terminology from [**6**] in the following.

Consider first Ω_2 where we have linear PDEs and the following boundary conditions:

$$(3.10)\qquad W_N^{2,I}(-1,t) = g^I(t)$$

$$(3.11)\qquad W_N^{2,II}(1,t) = R_2 W^{2,I}(1,t) + g^{II}(t)$$

where superscripts I and II denotes parts of a vector corresponding to negative and positive eigenvalues in (3.9). Then from [**6**] we have the following stability estimate:

$$(\eta - \eta_0)\|\hat{W}^2(y,s)\|_\omega \le C_1 N^{2\alpha}(|\hat{g}^I(s)| + |\hat{g}^{II}(s)|) \tag{3.12}$$

for $\Re s = \eta > \eta_0 > 0$. Here $\hat{W}_N(y,s) = \mathcal{F}(e^{-\eta t}W_N(y,t))$ is the Fourier-Laplace transform of W_N and $s = \eta + i\xi$. The ω-norm is defined as:

$$\|u\|_\omega^2 = \int_{-1}^{1} |u^I|^2 \omega^I(x)dx + \int_{-1}^{1} |u^{II}|^2 \omega^{II}(x)dx$$

for some positive weight functions ω^I and ω^{II}, for details see [**6**].

Consider then Ω_1 and the locally linearized equations there. The same procedure can be applied here, and we have the following boundary conditions:

$$W^{1,I}(-1,t) = L\,W^{1,II}(-1,t) + h^I(t) \tag{3.13}$$

$$W^{1,II}(1,t) = R_1 W^{1,I}(1,t) + h^{II}(t) \tag{3.14}$$

and we have a similar estimate:

$$(\eta - \eta_0)\|\hat{W}_N^1(y,s)\| \le C_2 N^{2\alpha}(|\hat{h}^I(s)| + |\hat{h}^{II}(s)|) \tag{3.15}$$

We can safely assume that $R_i < 1$ and $L < 1$ because of the transmission of energy into neighbouring domains. The coupling between the domains is materialized by $g^{II}(t)$ and $h^I(t)$. We will assume that the following holds:

$$|W_N^{1,I}(-1,t)| \le C_1\|W_N^1\|_\omega \qquad |W_N^{1,II}(-1,t)| \le C_2\|W_N^1\|_\omega$$

and the same at $(1,t)$. We also assume that the same type of relations are valid for W_N^2 at both boundary points.

By simple algebraic manipulations of the estimates (3.12) and (3.15) we can now deduce that $(\eta - \eta_0)(\|W_N^1()\|_\omega + \|W_N^2()\|_\omega)$ is bounded. Hence we have the following result:

LEMMA 3.1. *The Chebyshev spectral collocation discretization of the ocean/bottom acoustics problem described by the PDE systems (2.1)-(2.3) and (2.4)-(2.5) with interface conditions as described in section 2 is time-stable.*

4. Conclusion

We have presented a procedure of correction type for construction of interface conditions between domains with different governing equations of hyperbolic type. This procedure has been used with success in a model for ocean/bottom acoustics as well as a model for the study of waves in the atmosphere. In the case of the ocean/bottom acoustic model we have shown that the interface conditions based on the physical boundary conditions between the media, is a well-posed problem. The locally one-dimensional interface procedure is shown to form a stable problem when we discretize by Chebyshev spectral collocation. The interface procedure can also be used in the homogeneous domain decomposition case, e.g. in problems with variable coefficients. A good example of this is the simulation of seismic waves in a layered medium.

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