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Hybrid Domain Decomposition with Unstructured Subdomains*

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Abstract. We develop several new domain decomposition methods for solving large scale systems of symmetric, positive definite algebraic equations arising from discretizations of partial differential equations by conforming finite elements. First, the hybrid Schwarz alternating method is developed and analyzed. This method treats the coarse space in a multiplicative and the local spaces in an additive fashion, resulting in faster convergence at little extra cost. Then four methods based on reduction to interfaces and space splitting are presented, two using a coarse space consisting of linear functions and two with coarse space of piecewise constant functions on subdomain interfaces. Finally, we study an overlapping Schwarz method with a discrete harmonic coarse space, piecewise constant on interfaces. The condition numbers of all methods are proved to grow at most as $\log^2(H/h)$ and to be bounded independently of the number of subdomains when the subdomains form a shape regular coarse triangulation both in 2D and 3D. The methods with piecewise constant coarse space can be implemented as "black box" solvers without any reference to geometry and are suitable for subdomains of arbitrary shape.

1. Introduction. This paper is concerned with the analysis of a class of method of "interface decomposition" type and one method of overlapping Schwarz type. The interface decomposition methods studied here are essentially of the form proposed by Dryja and Widlund [8, Eq. (26)] with the addition of a coarse space. A formally closely related method was also used by the present author for the p-version finite element method [13]. The overlapping method is analogous to Dryja and Widlund [8], with the coarse space due to Cowsar [5]. A related method was recently studied by Sarkis [18]. The analysis tools used here are based on the results of Dryja [6] and Bramble, Pasciak, and Schatz [2, 3] as reformulated by Mandel and Brezina [15], where the same tools were used for the analysis of a different domain decomposition algorithm. See Dryja and Widlund [7] for other related domain decomposition methods and theoretical bounds.

The interface methods presented in this paper are based on computing with the global Schur complement on subdomain interfaces for the sake of robustness and increasing the locality of computations. The most expensive part of the calculation is computing the action of the inverse of a subdomain submatrix of the Schur complement on interfaces in each iteration. Since the submatrix

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contains contributions from neighboring subdomains, this computation cannot be reduced to subdomain solves, but rather requires the solution of a subproblem that is associated with all neighboring subdomains or an expensive explicit calculation of the Schur complement matrix. Balancing Domain Decomposition, introduced by the present author [12], only relies on subdomain solves to achieve the same asymptotic bounds on the condition number, but pays the price in less flexibility. The method of Farhat and Roux [9] is based on a Lagrange multiplier approach and also only requires subdomain solves, but is not asymptotically optimal.

Because of the page limit, some proofs are only sketched. Computational aspects and numerical results will be presented elsewhere.

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2. Abstract Hybrid Schwarz Method. First recall the formulation of abstract Schwarz methods, following [1]. For another exposition, see [8].

Let V be a real finite dimensional linear space with the inner product $\langle\cdot,\cdot\rangle$, A symmetric, positive definite linear operator on V, and V_i, $i=0,\dots,m$, subspaces of V such that

$$V = V_0 + \cdots + V_m.$$

Denote $a(u,v) = \langle Au, v\rangle$. The bilinear form $a(\cdot,\cdot)$ is called the energy inner product and $\|u\|_A = (a(u,u))^{1/2}$ is the energy norm.

We solve the problem $Au = f$, or, in the variational form,

$$u \in V: \quad a(u,v) = \langle r, v\rangle, \quad \forall v \in V, \tag{2.1}$$

by the preconditioned conjugate gradients method. In each iteration, this method requires approximate solution $\tilde{u}$ of the problem $Au = r$ in such manner that $\tilde{u} = Cr$, where C is a symmetric linear operator.

In the *Additive Schwarz Method*, this is accomplished by computing

$$\begin{aligned} &u_i \in V_i: \quad a(u_i, v_i) = \langle r, v_i\rangle, \quad \forall v_i \in V_i, \quad i = 0,\dots,m, \\ &\tilde{u} = \textstyle\sum_{i=0}^m u_i. \end{aligned} \tag{2.2}$$

Clearly, $\tilde{u} = \sum_{i=0}^m P_i u$, where $Au = r$ and P_i is energy orthogonal projection onto V_i, $i = 0,\dots,m$, and C is symmetric.

The *Multiplicative Schwarz Method* used as a preconditioner starts from $u = 0$ and proceeds by replacements of the form $u \leftarrow u - u_i$, where

$$u_i \in V_i: \quad a(u_i, v_i) = \langle r, v_i\rangle - a(u, v_i), \quad \forall v_i \in V_i,$$

$i = 0,\dots,m$. One can perform the replacement steps once in forward and once in backward order to get a symmetric operator C.

In our application, the space V_0 is the *coarse space* that serves the purpose to "coordinate" the spaces V_i, $i = 1,\dots,m$. In the following hybrid variant, the space V_0 is treated in a multiplicative fashion, while all other spaces in an additive fashion.

ALGORITHM 1 (HYBRID SCHWARZ METHOD). *For given $r \in V$, compute $\tilde{u}$ as follows:*

$$\bar{u}_0 \in V_0 : \quad a(\bar{u}_0, v_0) = \langle r, v_0 \rangle, \quad \forall v_0 \in V_0 \tag{2.3}$$

$$u_i \in V_i : \quad a(u_i, v_i) = \langle r, v_i \rangle - a(\bar{u}_0, v_i), \quad \forall v_i \in V_i, \quad i = 1, \ldots, m, \tag{2.4}$$

$$\bar{u} = \sum_{i=1}^{m} u_i, \tag{2.5}$$

$$u_0 \in V_0 : \quad a(\bar{u} - u_0, v_0) = \langle r, v_0 \rangle, \quad \forall v_0 \in V_0, \tag{2.6}$$

$$\tilde{u} = \bar{u} - u_0. \tag{2.7}$$

REMARK 1. Algorithm 1 is just one step of a two-level variational multigrid method [17] for the problem $Au = r$, started with the initial approximation $u = 0$. Steps (2.4) and (2.5) play the role of smoothing, while steps (2.3) and (2.6) are coarse grid corrections.

REMARK 2. In practice, step (2.3) can be omitted if the initial approximation in the preconditioned conjugate gradients satisfies $a(u, v_0) = \langle f, v_0 \rangle, \quad \forall v_0 \in V_0$. This can be achieved by applying the correction (2.6), (2.7) to the initial approximation before the start of iterations, with $\bar{u}$ the given initial approximation and $\tilde{u}$ the corrected approximation used to start the iterations. Then the residual r in every step satisfies $\langle r, v_0 \rangle = 0, \forall v_0 \in V_0$, and so one has always $\bar{u}_0 = 0$ in (2.3).

REMARK 3. In the case of two subspaces, i.e., $m = 1$, the hybrid Schwarz method reduces to the multiplicative method used as a preconditioner.

3. Abstract Spectral Bounds. It is well known that the number of iterations of preconditioned conjugate gradients for a given reduction factor of the error in energy norm grows at most as $\sqrt{\kappa}$, where $\kappa = \kappa(CA) = \lambda_{\max}(CA)/\lambda_{\min}(CA)$ is the *condition number* and $\lambda_{\min}$ and $\lambda_{\max}$ are the least and the largest eigenvalue, respectively, cf., [10]. The maximal eigenvalue of the additive method is easy to estimate as the maximum number of intersecting subspaces,

$$\lambda_{\max}\left(\sum_{i=0}^{m} P_i\right) \leq \max_{i=1,\ldots,m} |\{V_j : V_j \cap V_i \neq \{0\}\}| \tag{3.1}$$

cf. also [7], and $\lambda_{\min}$ is bounded from the following lemma for the additive method.

LEMMA 3.1 (P.L. LIONS [11]). *If there is a constant C_0 such that*

$$\forall v \in V \ \exists v_i \in V_i, \ i = 0, \ldots, m \ : \qquad \sum_{i=0}^{m} \|v_i\|_A^2 \leq C_0 \|v\|_A^2, \tag{3.2}$$

then $\lambda_{\min}\left(\sum_{i=0}^{m} P_i\right) \geq 1/C_0$.

Proof. For a proof of the lemma in this form, see [1, Theorem 3.2] or [21, Lemma 4]. ◻

The following lemma shows that the condition number for the hybrid method is smaller than for the additive method.

LEMMA 3.2. *Algorithm 1 returns $\tilde{u} = Cr$, where*

$$CA = (I - P_0) \sum_{i=1}^{m} P_i (I - P_0) + P_0, \tag{3.3}$$

is symmetric, positive definite, and

$$\lambda_{\min}(CA) \geq \lambda_{\min}\Big(\sum_{i=0}^{m} P_i\Big), \qquad \lambda_{\max}(CA) \leq \lambda_{\max}\Big(\sum_{i=1}^{m} P_i\Big). \tag{3.4}$$

In particular, $\kappa(CA) \leq \kappa\big(\sum_{i=0}^{m} P_i\big)$ with strict inequality if $V_0 \cap \sum_{i=1}^{m} V_i \neq \{0\}$.

Proof. Let $u = A^{-1}r$. Then from (2.3) to (2.7), we obtain in turn

$$\begin{aligned}
\bar{u}_0 &= P_0 u \\
u_i &= P_i(u - \bar{u}_0) = P_i(I - P_0)u, \qquad i = 1, \dots, m \\
\bar{u} &= \sum_{i=1}^{m} P_i (I - P_0) u \\
u_0 &= -P_0(u - \bar{u}) \\
\tilde{u} &= \bar{u} + P_0(u - \bar{u}) = (I - P_0)\bar{u} + P_0 u,
\end{aligned}$$

which gives (3.3). To prove (3.4), note that $P_0(I - P_0) = 0$, so the summation in (3.3) can be taken from $i = 0$; then (3.4) follows by a simple Rayleigh quotient argument in the energy inner product using the fact that the projection P_0 is energy orthogonal. □

4. Domain Decomposition on Interfaces. Assume the domain $\Omega \subset \Re^d$, $d = 2, 3$, is decomposed into non-overlapping subdomains Ω_i, $i = 1, \dots, m$ with characteristic size H, and assume that the subdomains Ω_i are shape and size regular. Similarly, assume that each Ω_i is decomposed into finite elements of characteristic size h, and the usual shape regularity and inverse assumptions are satisfied. Let W be the space of linear, conforming, finite element functions on those elements, and $W \subset H_0^1(\Omega)$. Let b be a bilinear form on $H_0^1(\Omega)$ given by

$$a(u, v) = \int_\Omega \sum_{i,j=1}^{d} b_{ij} \partial_i u \partial_j v, \tag{4.1}$$

the coefficient matrix $\{b_{ij}(x)\}$ being uniformly bounded and uniformly positive definite in Ω.

Denote the union of all interfaces of Ω_i as $\Gamma = \cup_{i=1}^{m} \partial\Omega_i \setminus \partial\Omega$. Let V be the space of all *discrete harmonic functions* $V = T(W) \subset W$, where T is the operator defined by trace on Γ and the discrete harmonic extension,

$$\begin{aligned}
&T : w \in W \mapsto v \in W, \qquad w|_\Gamma = v|_\Gamma, \\
&a(v, w_i) = 0, \quad \forall w_i \in W,\ \operatorname{supp} w_i \subset \bar{\Omega}_i, \quad \forall i = 1, \dots, m.
\end{aligned} \tag{4.2}$$

Note that functions in V are uniquely defined by their values on Γ.

Inspired by [6], define the scaled Sobolev norms

$$\begin{aligned}
\|u\|^2_{1,\Omega_i} &= |u|^2_{1,\Omega_i} + \tfrac{1}{H^2}\|u\|^2_{0,\Omega_i}, & \|u\|^2_{1/2,\partial\Omega_i} &= |u|^2_{1/2,\partial\Omega_i} + \tfrac{1}{H}\|u\|^2_{0,\partial\Omega_i} \\
|u|^2_{1,\Omega_i} &= \int_{\Omega_i} |\nabla u|^2, & |u|^2_{1/2,\partial\Omega_i} &= \int_{\partial\Omega_i}\int_{\partial\Omega_i} \frac{|u(t)-u(s)|^2}{|t-s|^d}dtds \\
|u|^2_{1,\Omega} &= \int_{\Omega} |\nabla u|^2, & |u|^2_{1/2,\Gamma} &= \sum_i |u|^2_{1/2,\partial\Omega_i}, \\
\|u\|^2_{1,\Omega} &= |u|^2_{1,\Omega} + \|u\|^2_{0,\Omega}, & \|u\|^2_{1/2,\Gamma} &= |u|^2_{1/2,\Gamma} + \|u\|^2_{0,\Gamma},
\end{aligned}$$

where, as usual, $\|\cdot\|_0$ is the L^2 norm. Note that then all components of $\|u\|_{1,\Omega_i}$ and $\|u\|_{1/2,\partial\Omega_i}$ scale in the same way under dilation, and, consequently, the constants in the trace and extension theorems are independent of H, cf., [15].

We wish to solve a variational problem on the finite element space W,

$$u \in W: \quad a(z,w) = \langle g, w \rangle, \quad \forall w \in W, \tag{4.3}$$

which is equivalent to a system of linear equations $Bz = g$. Eliminating the degrees of freedom in the interior of the subdomains Ω_i, we get the reduced system (2.1) set in the space V, with the bilinear form (4.1) restricted to V. The matrix of the reduced system is the *Schur complement* of the interior degrees of freedom.

The first principal observation is that our assumptions imply the equivalences of seminorms,

$$|u|^2_{1,\Omega_i} \approx |u|^2_{1/2,\partial\Omega_i}, \qquad |u|^2_{1,\Omega} \approx |u|^2_{1/2,\Gamma} \approx a(u,u), \qquad u \in V, \tag{4.4}$$

with the equivalence constants independent of H and h, cf., [3, 20]. For a detailed proof, see [15].

The second observation is that the tools from [3, 6] imply that a function on Γ can be "torn into pieces" with little penalty in the increase of energy; cf., also the proof of Theorem 4 in [7]. It is convenient to describe such tearing in terms of *globs*, defined as follows.

DEFINITION 4.1. *Any vertex, edge, and, in the 3D case, face, of the interfaces between subdomains $\{\Omega_i\}$ will be called a* **glob**. *A glob is understood to be relatively open; for example, an edge does not contain its endpoints. We will also identify a glob with the set of degrees of freedom associated with it.*

DEFINITION 4.2. *For glob G, define the selection operator $E_G : V \to V$ as follows: for $u \in V$, $E_G u$ is the unique function $v \in V$ that has the same values as u on the degrees of freedom in G, and all other degrees of freedom of v are zero.*

Note that the union of all globs is the set of all degrees of freedom on Γ, and the mappings E_G are projections that form a decomposition of unity on V, $\sum_G E_G = I$.

The following technical bound is the principal tool of our analysis. It is essentially a re-interpretation of the results of [2, 3, 6], with few extra ingredients in the 2D case. See [15] for a detailed proof.

LEMMA 4.3 (GLOB THEOREM). *For any glob G and for all $u \in V$,*

$$\|E_G u\|_{1/2,\Gamma}^2 \le C\big(1 + \log\big(\tfrac{H}{h}\big)\big)^2 \big(|u|_{1/2,\Gamma}^2 + \tfrac{1}{H}|u|_{0,\Gamma}^2\big),$$

where the constant does not depend on H and h, both in 2D and 3D.

4.1. Methods with Linear Coarse Space. The first method is a special case of the "vertex space" method of Smith [19], analyzed already in [7].

THEOREM 4.4. *Let V_0 be the space of piecewise linear functions on the interfaces of the triangulation defined by the subdomains $\{\Omega_i\}$, and the spaces V_i associated with globs, $V_i = \operatorname{Range} R_{G_i}$, where $G_1, \dots, G_m$ are all globs. Then $\kappa \le C(1 + \log(H/h))^2$.*

Proof. It is well known that there exists a linear mapping $\Pi_0 : H^1(\Omega) \to W_0$ such that for all $v \in H^1(\Omega)$,

$$\|\Pi_0 v\|_{1,\Omega}^2 \le C\|v\|_{1,\Omega}^2, \qquad \|v - \Pi_0 v\|_{0,\Omega}^2 \le CH^2\|v\|_{1,\Omega}^2, \tag{4.5}$$

cf., [4]. For $u \in V$, define $u_0 = \Pi_0 u \in V_0$ and $u_i = E_{G_i}(u - u_0)$. From (4.5), $\|u_0\|_{1/2,\Gamma} \le C\|u\|_{1/2,\Gamma}$. To estimate u_i, $i = 1, \dots, m$,

$$\begin{aligned} \|u_i\|_{1/2,\Gamma}^2 &= \|E_{G_i}(u - u_0)\|_{1/2,\Gamma}^2 \\ &\le C\big(1 + \log\big(\tfrac{H}{h}\big)\big)^2 \textstyle\sum_{j:G_i \subset \Omega_j} \|u - u_0\|_{1/2,\partial\Omega_j}^2 \\ &\le C\big(1 + \log\big(\tfrac{H}{h}\big)\big)^2 \textstyle\sum_{j:G_i \subset \Omega_j} |u - u_0|_{1,\Omega_j}^2 + \tfrac{1}{H^2}|u - u_0|_{0,\Omega_j}^2. \end{aligned} \tag{4.6}$$

Using (4.5) and the equivalence of norms (4.4), we have $C_0 \le C(1 + \log(H/h))^2$ in Lemma 3.1, giving $\lambda_{\min} \ge 1/(C(1 + \log(H/h))^2)$. From (3.1), it is easy to see that $\lambda_{\max} \le C$. □

Next we consider a method where the subspaces V_i are associated with subdomains instead of globs. We expect such a method to be more robust because merging the glob subspaces has proved to be successful means to treat ill-conditioning caused by high aspect ratios of subdomains in related investigations for the p-version finite element method [14, 16].

Define the operators $\Pi_i : V \to V$ by $\Pi_i v = v_i \in V$, $v = v_i$ on the nodes of $\partial\Omega_i$ and $v = 0$ on all other nodes. Since

$$\Pi_i v = \sum_{G \subset \partial\Omega_i} E_G v, \tag{4.7}$$

Lemma 4.3 immediately implies the following bound.

LEMMA 4.5. *For any $v \in V$, $|\Pi_i v|_{1/2,\Gamma}^2 \le C(1 + \log(H/h))^2\|v\|_{1/2,\partial\Omega_i}^2$.*

The condition number estimate follows.

THEOREM 4.6. *Let V_0 be the space of piecewise linear functions on the interfaces of the triangulation defined by the subdomains $\{\Omega_i\}$, and the spaces V_i associated with subdomains Ω_i, $V_i = \operatorname{Range} P_i$. Then $\kappa \le C(1 + \log(H/h))^2$.*

Proof. The proof is same as that of Theorem 4.4 except that $u_i = \Pi_i(u - u_0)$ and Lemma 4.5 is used to obtain an estimate analogous to (4.6). □

5. Methods with Piecewise Constant Coarse Space. For our second family of methods, define the operator $I_0 : V \to V$ as follows. For $v \in V$, define c_i as the average of v on $\partial\Omega_j$, and let $I_0 v$ at a node equal to the average of the numbers c_i for all subdomains Ω_i that share that node. That is,

$$I_0 u = \sum_G \frac{1}{n_G} \sum_{j: G \subset \partial\Omega_j} E_G Q_j u, \tag{5.1}$$

where $Q_j : u \mapsto \frac{1}{|\partial\Omega_j|} \int_{\partial\Omega_j} u$ and n_G is the number of subdomains Ω_j such that $G \subset \partial\Omega_j$.

First we need to show that the operator I_0 does not increase energy too much.

LEMMA 5.1. *For all $u \in V$*

$$|I_0 u|^2_{1/2,\Gamma} \le C\big(1 + \log\left(\tfrac{H}{h}\right)\big)^2 |u|^2_{1/2,\Gamma}, \tag{5.2}$$

Proof. Let $G \subset \partial\Omega_i$ be a glob. Define $\Gamma_i = \bigcup_{j:\partial\Omega_j \cap \partial\Omega_i \neq \emptyset} \partial\Omega_j$. From Lemma 4.3 and from (5.1), it holds for all $u \in V$ that $|I_0 u|^2_{1/2,\partial\Omega_i} \le C(1 + \log(H/h))^2 |u|^2_{1/2,\Gamma_i}$ since the values of $I_0 u$ on $\partial\Omega_i$ depend only on the values of u on Γ_i. Then (5.2) follows by summation. □

The following theorem bounds the condition number of the method with the space V_0 determined by one number per subdomain and the spaces V_i associated with globs.

THEOREM 5.2. *Let $V_0 = \operatorname{Range} I_0$ and $V_i = \operatorname{Range} E_{G_i}$, where $G_1, \dots, G_m$ are all globs. Then $\kappa \le C(1 + \log(H/h))^2$.*

Proof. Again, we verify the existence of a decomposition needed for Lemma 3.1. Let $u_0 = I_0 u$ and $u_i = E_{G_i}(u - u_0)$, $i = 1, \dots, m$. The needed bound on u_0 is given by (5.2). To estimate u_i, note that from (5.1),

$$E_G(u - I_0 u) = \frac{1}{n_G} \sum_{j: G \subset \partial\Omega_j} E_G(u - Q_i u). \tag{5.3}$$

Additionally, from Lemma 4.3 we have for any $u \in V$,

$$|E_G(u - Q_i u)|^2_{1/2,\partial\Omega_i} \le C\big(1 + \log\left(\tfrac{H}{h}\right)\big)^2 \left(|u - Q_i u|^2_{1/2,\partial\Omega_i} + \tfrac{1}{H}|u - Q_i u|^2_{0,\partial\Omega_i}\right),$$

where $|Q_i u|^2_{1/2,\partial\Omega_i} = 0$ since $Q_i u$ is constant on $\partial\Omega_i$, and $|u - Q_i u|^2_{0,\partial\Omega_i} \le CH|u|^2_{1/2,\partial\Omega_i}$ by mapping to a reference domain size $H = 1$ and scaling to subdomain Ω_i size H. Noting that for any $v \in V$ and $G \subset \partial\Omega_i \cap \partial\Omega_j$, one has $|E_G v|^2_{1/2,\partial\Omega_j} \le C|E_G v|^2_{1/2,\partial\Omega_i}$, it follows that

$$|E_G(u - I_0 u)|^2_{1/2,\Gamma} \le C\big(1 + \log\left(\tfrac{H}{h}\right)\big)^2 |u|^2_{1/2,\Gamma}, \tag{5.4}$$

which concludes the proof in view of (5.3). □

In the next method, the subspaces are again associated with subdomains rather than globs.

THEOREM 5.3. *Let* $V_0 = \operatorname{Range} I_0$ *and* V_i *be associated with subdomains* Ω_i, $V_i = \operatorname{Range} \Pi_i$, $i = 1, \ldots, m$, *cf. (4.7). Then* $\kappa \leq C(1 + \log(H/h))^2$.

Proof. The proof is same as the proof of Theorem 5.2 except that $u_i = \Pi_i(u - u_0)$ and (4.7) is used along with (5.4) to bound $|u_i|^2_{1/2,\Gamma}$. □

The last method, due to Cowsar [5], is set in the original space W of functions on Ω, but it uses as the coarse space the same space as above, consisting of discrete harmonic functions determined by their piecewise constant values on interfaces.

THEOREM 5.4. *Let* $\Omega_i \subset \Omega_i'$ *so that* $\operatorname{dist}(\Omega_i, \partial\Omega_i') \geq CH$ *and* Ω_i' *are shape regular and* $\Omega_i' \cap \Omega$ *consist of unions of elements. Define* $W_i = W \cap H_0^1(\Omega_i')$, $i = 1, \ldots, n$, *and* $W_0 = \operatorname{Range} I_0$. *Then the Schwarz method based on the decomposition* $W = W_0 + W_1 + \cdots + W_n$ *satisfies* $\kappa \leq C(1 + \log(H/h))^2$.

Proof. The upper bound is immediate from (3.1). We verify the lower bound by Lemma 3.1. For $v \in W$, let $v_0 = I_0 T v$, where I_0 and T were defined in (5.1) and (4.2), respectively. From (5.2) and the equivalence (4.4), $|v_0|^2_{1,\Omega} \leq C(1 + \log(H/h))^2 \|v\|^2_{1,\Omega}$. By mapping to the reference domain with $H = 1$ and scaling, we get for any glob $G \subset \partial\Omega_i$ that $\|E_G(Tv - I_0 T v)\|^2_{0,\Omega_i} \leq CH^2 \|v\|^2_{1,\Omega_i}$, which implies

$$\|v - v_0\|^2_{0,\Omega} \leq CH^2 \|v\|^2_{1,\Omega}.$$

From [8], there are $v_i \in W_i$ such that $v_1 + \cdots + v_n = v - v_0$ and

$$\|v_1\|^2_{1,\Omega} + \cdots + \|v_n\|^2_{1,\Omega} \leq C\left(\tfrac{1}{H^2}\|v - v_0\|^2_{0,\Omega} + \|v - v_0\|^2_{1,\Omega}\right),$$

completing the proof. □

Note that Dryja and Widlund [8] proved that $\kappa \leq C$ for an analogous method that differs only in the use of coarse linear functions as the coarse space. That is, the use of the piecewise constant on interfaces coarse space increases the condition number bound by the factor of $C(1 + \log(H/h))^2$.

6. Concluding Remarks. Note that in all interface decomposition methods considered here, the support of the functions from the spaces V_i corresponding to adjacent subdomains overlap with width of at least $2h$. The glob spaces overlap with width of at least h.

The "piecewise constant" interpolation allows for unstructured domains. This means that there is no need for the subdomains to form a coarse triangulation and we do not need to have the concepts of an edge or a vertex of a subdomain if subdomain spaces are used rather than glob spaces. Even for completely unstructured subdomains, the globs can be defined as the basis of the set algebra generated by the sets of degrees of freedom in subdomains, which can be implemented using simple graph theoretical algorithms on the element connectivity data.

In principle, the linear interpolation need not be related to subdomains as well. The only properties we need are the H^1 stability and the L^2 optimal approximation property (4.5). Thus one could use unstructured domains and define the operator Π_0 on a set of nodes unrelated to the subdomains.

The algorithms developed here can be applied to completely unstructured meshes and subdomains, but the analysis and performance of the algorithms in the general case remains to be investigated. The theory presented here uses a simple adaptation of earlier work for different domain decomposition methods in the case of regular subdomains, so it applies only when the mesh and the subdomains are "reasonable".

For piecewise constant interpolation, examination of the proofs in Section 5 shows that it needs only be assumed that the ratio of the sizes of neighboring subdomains is bounded; it is not necessary that all subdomains are about of the same size H.

For the glob based methods, one can replace the solution of subproblems in the glob spaces V_i by more efficient approximate solvers, analogous to Bramble, Pasciak, and Schatz [2, 3]. The analysis of the additive Schwarz method with approximate solvers by Dryja and Widlund [7] can be used to show that the asymptotic bounds $(1 + \log(H/h))^2$ on the condition number are retained. However, the decreased computational complexity comes at the cost of likely loss of robustness.

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