Hybrid Domain Decomposition with Unstructured Subdomains

JAN MANDEL

Abstract. We develop several new domain decomposition methods for solving large scale systems of symmetric, positive definite algebraic equations arising from discretizations of partial differential equations by conforming finite elements. First, the hybrid Schwarz alternating method is developed and analyzed. This method treats the coarse space in a multiplicative and the local spaces in an additive fashion, resulting in faster convergence at little extra cost. Then four methods based on reduction to interfaces and space splitting are presented, two using a coarse space consisting of linear functions and two with coarse space of piecewise constant functions on subdomain interfaces. Finally, we study an overlapping Schwarz method with a discrete harmonic coarse space, piecewise constant on interfaces. The condition numbers of all methods are proved to grow at most as \( \log^2(H/h) \) and to be bounded independently of the number of subdomains when the subdomains form a shape regular coarse triangulations both in 2D and 3D. The methods with piecewise constant coarse space can be implemented as "black box" solvers without any reference to geometry and are suitable for subdomains of arbitrary shape.

1. Introduction. This paper is concerned with the analysis of a class of method of "interface decomposition" type and one method of overlapping Schwarz type. The interface decomposition methods studied here are essentially of the form proposed by Dryja and Widlund [8, Eq. (26)] with the addition of a coarse space. A formally closely related method was also used by the present author for the \( p \)-version finite element method [13]. The overlapping method is analogous to Dryja and Widlund [8], with the coarse space due to Cowsar [5]. A related method was recently studied by Sarkis [18]. The analysis tools used here are based on the results of Dryja [6] and Bramble, Pasciak, and Schatz [2, 3] as reformulated by Mandel and Brezina [15], where the same tools were used for the analysis of a different domain decomposition algorithm. See Dryja and Widlund [7] for other related domain decomposition methods and theoretical bounds.

The interface methods presented in this paper are based on computing with the global Schur complement on subdomain interfaces for the sake of robustness and increasing the locality of computations. The most expensive part of the calculation is computing the action of the inverse of a subdomain submatrix of the Schur complement on interfaces in each iteration. Since the submatrix

* 1991 Mathematics Subject Classification. 66N55.

This research was partially supported by the National Science Foundation under grants DMS-9015229, ASC-9227384, and by a grant from the MacNeal-Schwendler Corporation.

This paper is in final form and no version of it will be submitted for publication elsewhere.

© 1994 American Mathematical Society

0271-4132/94 $1.00 + $.25 per page
contains contributions from neighboring subdomains, this computation cannot
be reduced to subdomain solves, but rather requires the solution of a subproblem
that is associated with all neighboring subdomains or an expensive explicit
calculation of the Schur complement matrix. Balancing Domain Decomposition,
introduced by the present author [12], only relies on subdomain solves to achieve
the same asymptotic bounds on the condition number, but pays the price in less
flexibility. The method of Farhat and Roux [9] is based on a Lagrange multiplier
approach and also only requires subdomain solves, but is not asymptotically
optimal.

Because of the page limit, some proofs are only sketched. Computational
aspects and numerical results will be presented elsewhere.

Sincere thanks are due to Olof Widlund and Lawrence Cowsar for many
discussions and reading early versions of the paper, and to Olof Widlund for
making drafts of his papers available to the author.

2. Abstract Hybrid Schwarz Method. First recall the formulation of
abstract Schwarz methods, following [1]. For another exposition, see [8].

Let \( V \) be a real finite dimensional linear space with the inner product \((\cdot, \cdot)\), \( A \)
replaces the pseudo symmetric, positive definite linear operator on \( V \), and \( V_i, i = 0, \ldots, m \), subspaces
of \( V \) such that

\[
V = V_0 + \cdots + V_m.
\]

Denote \( a(u, v) = (Au, v) \). The bilinear form \( a(\cdot, \cdot) \) is called the energy inner
product and \( \|u\|_A = (a(u, u))^{1/2} \) is the energy norm.

We solve the problem \( Au = f \), or, in the variational form,

\[
(2.1) \quad u \in V : \quad a(u, v) = (r, v), \quad \forall v \in V,
\]

by the preconditioned conjugate gradients method. In each iteration, this
method requires approximate solution \( \tilde{u} \) of the problem \( Au = r \) in such manner
that \( \tilde{u} = Cr \), where \( C \) is a symmetric linear operator.

In the Additive Schwarz Method, this is accomplished by computing

\[
(2.2) \quad u_i \in V_i : \quad a(u_i, v_i) = (r, v_i), \quad \forall v_i \in V_i, \quad i = 0, \ldots, m,
\]

\[
\tilde{u} = \sum_{i=0}^{m} u_i.
\]

Clearly, \( \tilde{u} = \sum_{i=0}^{m} P_i u \), where \( Au = r \) and \( P_i \) is energy orthogonal projection
onto \( V_i, i = 0, \ldots, m \), and \( C \) is symmetric.

The Multiplicative Schwarz Method used as a preconditioner starts from
\( u = 0 \) and proceeds by replacements of the form \( u \leftarrow u - u_i \), where

\[
(2.3) \quad u_i \in V_i : \quad a(u_i, v_i) = (r, v_i) - a(u, v_i), \quad \forall v_i \in V_i,
\]

\( i = 0, \ldots, m \). One can perform the replacement steps once in forward and once
in backward order to get a symmetric operator \( C \).

In our application, the space \( V_0 \) is the coarse space that serves the purpose
to "coordinate" the spaces \( V_i, i = 1, \ldots, m \). In the following hybrid variant,
the space \( V_0 \) is treated in a multiplicative fashion, while all other spaces in an
additive fashion.
Algorithm 1 (Hybrid Schwarz Method). For given $r \in V$, compute $\bar{u}$ as follows:

(2.3) $\bar{u}_0 \in V_0 : \quad a(\bar{u}_0, v_0) = \langle r, v_0 \rangle, \quad \forall v_0 \in V_0$

(2.4) $u_i \in V_i : \quad a(u_i, v_i) = \langle r, v_i \rangle - a(\bar{u}_0, v_i), \quad \forall v_i \in V_i, \quad i = 1, \ldots, m,$

(2.5) $\bar{u} = \sum_{i=1}^{m} u_i,$

(2.6) $u_0 \in V_0 : \quad a(\bar{u} - u_0, v_0) = \langle r, v_0 \rangle, \quad \forall v_0 \in V_0,$

(2.7) $\bar{u} = \bar{u} - u_0.$

Remark 1. Algorithm 1 is just one step of a two-level variational multigrid method [17] for the problem $Au = r$, started with the initial approximation $u = 0$. Steps (2.4) and (2.5) play the role of smoothing, while steps (2.3) and (2.6) are coarse grid corrections.

Remark 2. In practice, step (2.3) can be omitted if the initial approximation in the preconditioned conjugate gradients satisfies $a(u, v_0) = \langle f, v_0 \rangle, \quad \forall v_0 \in V_0$. This can be achieved by applying the correction (2.6), (2.7) to the initial approximation before the start of iterations, with $\bar{u}$ the given initial approximation and $\bar{u}$ the corrected approximation used to start the iterations. Then the residual $r$ in every step satisfies $\langle r, v_0 \rangle = 0, \forall v_0 \in V_0$, and so one has always $\bar{u}_0 = 0$ in (2.3).

Remark 3. In the case of two subspaces, i.e., $m = 1$, the hybrid Schwarz method reduces to the multiplicative method used as a preconditioner.

3. Abstract Spectral Bounds. It is well known that the number of iterations of preconditioned conjugate gradients for a given reduction factor of the error in energy norm grows at most as $\sqrt{\kappa}$, where $\kappa = \kappa(CA) = \lambda_{\text{max}}(CA)/\lambda_{\text{min}}(CA)$ is the condition number and $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$ are the least and the largest eigenvalue, respectively, cf., [10]. The maximal eigenvalue of the additive method is easy to estimate as the maximum number of intersecting subspaces,

(3.1) $\lambda_{\text{max}}\left(\sum_{i=0}^{m} P_i\right) \leq \max_{i=1, \ldots, m} |\{V_j : V_j \cap V_i \neq \{0\}\}|$

cf. also [7], and $\lambda_{\text{min}}$ is bounded from the following lemma for the additive method.

Lemma 3.1 (P.L. Lions [11]). If there is a constant $C_0$ such that

(3.2) $\forall u \in V \exists u_i \in V_i, \quad i = 0, \ldots, m : \sum_{i=0}^{m} \|v_i\|^2_{\lambda_i} \leq C_0 \|u\|^2_A,$

then $\lambda_{\text{min}}\left(\sum_{i=0}^{m} P_i\right) \geq 1/C_0.$

Proof. For a proof of the lemma in this form, see [1, Theorem 3.2] or [21, Lemma 4]. $\Box$
The following lemma shows that the condition number for the hybrid method is smaller than for the additive method.

**Lemma 3.2.** Algorithm 1 returns $\bar{u} = Cr$, where

$$CA = (I - P_0) \sum_{i=1}^{m} P_i (I - P_0) + P_0,$$

is symmetric, positive definite, and

$$\lambda_{\min}(CA) \geq \lambda_{\min} \left( \sum_{i=1}^{m} P_i \right), \quad \lambda_{\max}(CA) \leq \lambda_{\max} \left( \sum_{i=1}^{m} P_i \right).$$

In particular, $\kappa(CA) \leq \kappa(\sum_{i=1}^{m} P_i)$ with strict inequality if $V_0 \cap \sum_{i=1}^{m} V_i \neq \{0\}$.

**Proof.** Let $u = A^{-1}r$. Then from (2.3) to (2.7), we obtain in turn

$$\begin{align*}
\bar{u}_0 & = P_0 u \\
\bar{u}_i & = P_i (u - \bar{u}_0) = P_i (I - P_0) u, \quad i = 1, \ldots, m \\
\bar{u} & = \sum_{i=1}^{m} P_i (I - P_0) u \\
\bar{u}_0 & = -P_0 (u - \bar{u}) \\
\bar{u} & = \bar{u} + P_0 (u - \bar{u}) = (I - P_0) \bar{u} + P_0 u,
\end{align*}$$

which gives (3.3). To prove (3.4), note that $P_0 (I - P_0) = 0$, so the summation in (3.3) can be taken from $i = 0$; then (3.4) follows by a simple Rayleigh quotient argument in the energy inner product using the fact that the projection $P_0$ is energy orthogonal. \( \square \)

4. **Domain Decomposition on Interfaces.** Assume the domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is decomposed into non-overlapping subdomains $\Omega_i$, $i = 1, \ldots, m$ with characteristic size $H$, and assume that the subdomains $\Omega_i$ are shape and size regular. Similarly, assume that each $\Omega_i$ is decomposed into finite elements of characteristic size $h_i$ and the usual shape regularity and inverse assumptions are satisfied. Let $W$ be the space of linear, conforming, finite element functions on those elements, and $W \subset H^1_0(\Omega)$. Let $b$ be a bilinear form on $H^1_0(\Omega)$ given by

$$a(u, v) = \int_{\Omega} \sum_{i,j=1}^{d} b_{ij} \partial_i u \partial_j v,$$

the coefficient matrix $\{b_{ij}(x)\}$ being uniformly bounded and uniformly positive definite in $\Omega$.

Denote the union of all interfaces of $\Omega_i$ as $\Gamma = \cup_{i=1}^{m} \partial \Omega_i \setminus \partial \Omega$. Let $V$ be the space of all discrete harmonic functions $V = T(W) \subset W$, where $T$ is the operator defined by trace on $\Gamma$ and the discrete harmonic extension,

$$T: w \in W \mapsto v \in W, \quad w|_{\Gamma} = v|_{\Gamma},$$

$$a(v, w_i) = 0, \quad \forall w_i \in W, \supp w_i \subset \Omega_i, \quad \forall i = 1, \ldots, m.$$
Note that functions in $V$ are uniquely defined by their values on $\Gamma$.

Inspired by [6], define the scaled Sobolev norms

$$
\begin{align*}
&\|u\|_{1,\Omega}^2 = \|u\|_{0,\Omega}^2 + \frac{1}{H} \|u\|_{0,\Omega}^2,
&\|u\|_{1/2,\partial\Omega}^2 = \|u\|_{1,\Omega}^2 + \frac{1}{H} \|u\|_{0,\partial\Omega}^2,
&\|u\|_{2,\partial\Omega}^2 = \int_{\partial\Omega} |\nabla u|^2, \\
&\|u\|_{3/2,\partial\Omega}^2 = \int_{\partial\Omega} \int_{\partial\Omega} \frac{|u(t) - u(s)|^2}{|t - s|^2} dt ds,
&\|u\|_{3/2,\Omega}^2 = \int_{\partial\Omega} |\nabla u|^2, \\
&\|u\|_{5/2,\Omega}^2 = \|u\|_{3/2,\Omega}^2 + \|u\|_{0,\Omega}^2.
\end{align*}
$$

where, as usual, $\| \cdot \|_{0}$ is the $L^2$ norm. Note that then all components of $\|u\|_{1,\Omega}$ and $\|u\|_{1/2,\partial\Omega}$ scale in the same way under dilation, and, consequently, the constants in the trace and extension theorems are independent of $H$, cf., [15].

We wish to solve a variational problem on the finite element space $W$,

$$u \in W : \ a(x, w) = \langle g, w \rangle, \ \forall w \in W,$$

which is equivalent to a system of linear equations $Bz = g$. Eliminating the degrees of freedom in the interior of the subdomains $\Omega_i$, we get the reduced system (2.1) set in the space $V$, with the bilinear form (4.1) restricted to $V$. The matrix of the reduced system is the Schur complement of the interior degrees of freedom.

The first principal observation is that our assumptions imply the equivalences of seminorms,

$$
\|u\|_{1,\Omega}^2 \approx \|u\|_{1/2,\partial\Omega}^2, \quad \|u\|_{1,\Omega}^2 \approx \|u\|_{1/2,\Gamma}^2 \approx a(u, u), \quad u \in V,
$$

with the equivalence constants independent of $H$ and $h$, cf., [3, 20]. For a detailed proof, see [15].

The second observation is that the tools from [3, 6] imply that a function on $\Gamma$ can be "torn into pieces" with little penalty in the increase of energy; cf., also the proof of Theorem 4 in [7]. It is convenient to describe such tearing in terms of globs, defined as follows.

**Definition 4.1.** Any vertex, edge, and, in the 2D case, face, of the interfaces between subdomains $\{\Omega_i\}$ will be called a glob. A glob is understood to be relatively open; for example, an edge does not contain its endpoints. We will also identify a glob with the set of degrees of freedom associated with it.

**Definition 4.2.** For glob $G$, define the selection operator $E_G : V \rightarrow V$ as follows: for $u \in V$, $E_G u$ is the unique function $v \in V$ that has the same values as $u$ on the degrees of freedom in $G$, and all other degrees of freedom of $v$ are zero.

Note that the union of all globs is the set of all degrees of freedom on $\Gamma$, and the mappings $E_G$ are projections that form a decomposition of unity on $V$, $\sum_G E_G = I$.

The following technical bound is the principal tool of our analysis. It is essentially a re-interpretation of the results of [2, 3, 6], with few extra ingredients in the 2D case. See [15] for a detailed proof.
Lemma 4.3 (Glob Theorem). For any glob $G$ and for all $u \in V$,

$$\|E_Gu\|_{1/2,\Gamma}^2 \leq C (1 + \log \left( \frac{H}{h} \right))^2 (\|u\|_{1/2,\Gamma}^2 + \frac{1}{H} \|u\|_{0,\Gamma}^2),$$

where the constant does not depend on $H$ and $h$, both in 2D and 3D.

4.1. Methods with Linear Coarse Space. The first method is a special case of the "vertex space" method of Smith [19], analyzed already in [7].

Theorem 4.4. Let $V_0$ be the space of piecewise linear functions on the interfaces of the triangulation defined by the subdomains $\{\Omega_i\}$, and the spaces $V_i$ associated with globs, $V_i = \text{Range} R_{G_i}$, where $G_1, \ldots, G_m$ are all globs. Then $\kappa \leq C (1 + \log (H/h))^2$.

Proof. It is well known that there exists a linear mapping $\Pi_0 : H^1(\Omega) \to W_0$ such that for all $v \in H^1(\Omega)$,

$$\|\Pi_0v\|_{2,\Omega}^2 \leq C \|v\|_{2,\Omega}^2, \quad \|v - \Pi_0v\|_{2,\Omega}^2 \leq CH^2 \|v\|_{2,\Omega}^2,$$

(cf., [4]). For $u \in V$, define $u_0 = \Pi_0u \in V_0$ and $u_i = E_{G_i}(u - u_0)$. From (4.5),

$$\|u_0\|_{1/2,\Gamma} \leq C \|u\|_{1/2,\Gamma}. \quad \text{To estimate } u_i, \text{ } i = 1, \ldots, m,$

$$\|u_i\|_{1/2,\Gamma}^2 = \|E_{G_i}(u - u_0)\|_{1/2,\Gamma}^2 \leq C (1 + \log \left( \frac{H}{h} \right))^2 \sum_{j: G_i \cap \Omega_j} \|u - u_0\|_{1/2,\Omega_j}^2 \leq C (1 + \log \left( \frac{H}{h} \right))^2 \sum_{j: G_i \cap \Omega_j} \|u - u_0\|_{1,\Omega_j}^2 + \frac{1}{H^2} \|u - u_0\|_{0,\Omega_j}^2.$$

Using (4.5) and the equivalence of norms (4.4), we have $C_0 \leq C (1 + \log (H/h))^2$ in Lemma 3.1, giving $\lambda_{\min} \geq 1/(C (1 + \log (H/h))^2)$. From (3.1), it is easy to see that $\lambda_{\max} \leq C$. \(\square\)

Next we consider a method where the subspaces $V_i$ are associated with subdomains instead of globs. We expect such a method to be more robust because merging the glob subspaces has proved to be successful means to treat ill-conditioning caused by high aspect ratios of subdomains in related investigations for the $p$-version finite element method [14, 16].

Define the operators $\Pi_i : V \to V$ by $\Pi_i v = v_i \in V$, $v = v_i$ on the nodes of $\partial \Omega_i$ and $v = 0$ on all other nodes. Since

$$\Pi_i v = \sum_{G \in \partial \Omega_i} E_G v,$$

Lemma 4.3 immediately implies the following bound.

Lemma 4.5. For any $v \in V$, $\|\Pi_i v\|_{1/2,\Gamma}^2 \leq C (1 + \log (H/h))^2 \|v\|_{1/2,\Omega_i}^2$.

The condition number estimate follows.

Theorem 4.6. Let $V_0$ be the space of piecewise linear functions on the interfaces of the triangulation defined by the subdomains $\{\Omega_i\}$, and the spaces $V_i$ associated with subdomains $\Omega_i$, $V_i = \text{Range} P_i$. Then $\kappa \leq C (1 + \log (H/h))^2$.

Proof. The proof is same as that of Theorem 4.4 except that $u_i = \Pi_i(u - u_0)$ and Lemma 4.5 is used to obtain an estimate analogous to (4.6). \(\square\)
5. Methods with Piecewise Constant Coarse Space. For our second family of methods, define the operator $I_0 : V \to V$ as follows. For $v \in V$, define $c_4$ as the average of $v$ on $\partial \Omega_j$, and let $I_0v$ at a node equal to the average of the numbers $c_4$ for all subdomains $\Omega_i$ that share that node. That is,

$$I_0u = \sum_G \frac{1}{n_G} \sum_{j:G \subset \partial \Omega_j} E_G Q_j u,$$

where $Q_j : u \mapsto \frac{1}{|\partial \Omega_j|} \int_{\partial \Omega_j} u$ and $n_G$ is the number of subdomains $\Omega_j$ such that $G \subset \partial \Omega_j$.

First we need to show that the operator $I_0$ does not increase energy too much.

**Lemma 5.1.** For all $u \in V$

$$|I_0u|_{1/2, \Gamma}^2 \leq C \left( 1 + \log \left( \frac{H}{h} \right) \right)^2 |u|_{1/2, \Gamma}^2,$$

**Proof.** Let $G \subset \partial \Omega_i$ be a glob. Define $\Gamma_i = \bigcup_{j: \partial \Omega_j \cap \partial \Omega_i \neq \emptyset} \partial \Omega_j$.
From Lemma 4.3 and from (5.1), it holds for all $u \in V$ that $|I_0u|_{1/2, \partial \Omega_i}^2 \leq C (1 + \log (H/h))^2 |u|_{1/2, \Gamma}^2$, since the values of $I_0u$ on $\partial \Omega_i$ depend only on the values of $u$ on $\Gamma_i$. Then (5.2) follows by summation.

The following theorem bounds the condition number of the method with the space $V_0$ determined by one number per subdomain and the spaces $V_i$ associated with globs.

**Theorem 5.2.** Let $V_0 = \text{Range } I_0$ and $V_i = \text{Range } E_{G_i}$, where $G_1, \ldots, G_m$ are all globs. Then $\kappa \leq C (1 + \log (H/h))^2$.

**Proof.** Again, we verify the existence of a decomposition needed for Lemma 3.1. Let $u_0 = I_0u$ and $u_i = E_{G_i}(u - u_0)$, $i = 1, \ldots, m$. The needed bound on $u_0$ is given by (5.2). To estimate $u_i$, note that from (5.1),

$$E_G (u - I_0u) = \frac{1}{n_G} \sum_{j:G \subset \partial \Omega_j} E_G (u - Q_j u).$$

Additionally, from Lemma 4.3 we have for any $u \in V$,

$$|E_G (u - Q_i u)|_{1/2, \partial \Omega_i}^2 \leq C \left( 1 + \log \left( \frac{H}{h} \right) \right)^2 \left( |u - Q_i u|_{1/2, \partial \Omega_i}^2 + \frac{1}{H} |u - Q_i u|_{0, \partial \Omega_i}^2 \right),$$

where $|Q_i u|_{1/2, \partial \Omega_i}^2 = 0$ since $Q_i u$ is constant on $\partial \Omega_i$, and $|u - Q_i u|_{0, \partial \Omega_i}^2 \leq C H |u|_{1/2, \partial \Omega_i}^2$ by mapping to a reference domain size $H = 1$ and scaling to subdomain $\Omega_i$ size $H$. Noting that for any $v \in V$ and $G \subset \partial \Omega_i \cap \partial \Omega_j$, one has $|E_G v|_{1/2, \partial \Omega_j}^2 \leq C |E_G v|_{1/2, \partial \Omega_i}^2$, it follows that

$$|E_G (u - I_0u)|_{1/2, \Gamma}^2 \leq C \left( 1 + \log \left( \frac{H}{h} \right) \right)^2 |u|_{1/2, \Gamma}^2,$$

which concludes the proof in view of (5.3). \qed

In the next method, the subspaces are again associated with subdomains rather than globs.
THEOREM 5.3. Let $V_0 = \text{Range} I_0$ and $V_i$ be associated with subdomains $\Omega_i$, $V_i = \text{Range} \Pi_i$, $i = 1, \ldots, m$, cf. (4.7). Then $\kappa \leq C(1 + \log(H/h))^2$.

Proof. The proof is same as the proof of Theorem 5.2 except that $u_i = \Pi_i(u - u_0)$ and (4.7) is used along with (5.4) to bound $|u_i|_{L^2,0}^2$. □

The last method, due to Cowar [5], is set in the original space $W$ of functions on $\Omega$, but it uses as the coarse space the same space as above, consisting of discrete harmonic functions determined by their piecewise constant values on interfaces.

THEOREM 5.4. Let $\Omega_i \subset \Omega_i'$ so that $\text{dist}(\Omega_i, \partial \Omega_i') \geq CH$ and $\Omega_i'$ are shape regular and $\Omega_i' \cap \Omega$ consist of unions of elements. Define $W_i = W \cap H^1_0(\Omega_i')$, $i = 1, \ldots, n$, and $W_0 = \text{Range} I_0$. Then the Schwarz method based on the decomposition $W = W_0 + W_1 + \cdots + W_n$ satisfies $\kappa \leq C(1 + \log(H/h))^2$.

Proof. The upper bound is immediate from (3.1). We verify the lower bound by Lemma 3.1. For $v \in W$, let $v_0 = I_0Tv$, where $I_0$ and $T$ were defined in (5.1) and (4.2), respectively. From (5.2) and the equivalence (4.4), $|v_0|_{L^2,0}^2 \leq C(1 + \log(H/h))^2|v|_{L^2,0}^2$. By mapping to the reference domain with $H = 1$ and scaling, we get for any glob $G \subset \partial \Omega_i$ that $\|E_G(Tv - I_0Tv)\|_{L^2,0}^2 \leq CH^2\|v\|_{L^2,0}^2$, which implies

$$\|v - v_0\|_{L^2,0}^2 \leq CH^2\|v\|_{L^2,0}^2.$$  

From [8], there are $v_i \in W_i$ such that $v_1 + \cdots + v_n = v - v_0$ and

$$|v_1|_{L^2,0}^2 + \cdots + |v_n|_{L^2,0}^2 \leq C\left(\frac{1}{H^2}\|v - v_0\|_{L^2,0}^2 + \|v - v_0\|_{L^2,0}^2\right),$$

completing the proof. □

Note that Dryja and Widlund [8] proved that $\kappa \leq C$ for an analogous method that differs only in the use of coarse linear functions as the coarse space. That is, the use of the piecewise constant on interfaces coarse space increases the condition number bound by the factor of $C(1 + \log(H/h))^2$.

6. Concluding Remarks. Note that in all interface decomposition methods considered here, the support of the functions from the spaces $V_i$ corresponding to adjacent subdomains overlap with width of at least $2h$. The glob spaces overlap with width of at least $h$.

The “piecewise constant” interpolation allows for unstructured domains. This means that there is no need for the subdomains to form a coarse triangulation and we do not need to have the concepts of an edge or a vertex of a subdomain if subdomain spaces are used rather than glob spaces. Even for completely unstructured subdomains, the globs can be defined as the basis of the set algebra generated by the sets of degrees of freedom in subdomains, which can be implemented using simple graph theoretical algorithms on the element connectivity data.

In principle, the linear interpolation need not be related to subdomains as well. The only properties we need are the $H^1$ stability and the $L^2$ optimal approximation property (4.5). Thus one could use unstructured domains and define the operator $\Pi_0$ on a set of nodes unrelated to the subdomains.

The algorithms developed here can be applied to completely unstructured meshes and subdomains, but the analysis and performance of the algorithms in
the general case remains to be investigated. The theory presented here uses a simple adaptation of earlier work for different domain decomposition methods in the case of regular subdomains, so it applies only when the mesh and the subdomains are “reasonable”.

For piecewise constant interpolation, examination of the proofs in Section 5 shows that it needs only be assumed that the ratio of the sizes of neighboring subdomains is bounded; it is not necessary that all subdomains are about of the same size $H$.

For the glob based methods, one can replace the solution of subproblems in the glob spaces $V_i$ by more efficient approximate solvers, analogous to Bramble, Pasciak, and Schatz [2, 3]. The analysis of the additive Schwarz method with approximate solvers by Dryja and Widlund [7] can be used to show that the asymptotic bounds $(1 + \log(H/h))^2$ on the condition number are retained. However, the decreased computational complexity comes at the cost of likely loss of robustness.

REFERENCES


Computational Mathematics Group, University of Colorado at Denver, Denver CO 80217-3364

E-mail address: jmandel@colorado.edu