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SOME SCHWARZ ALGORITHMS FOR THE P-VERSION FINITE ELEMENT METHOD

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Abstract. Domain decomposition methods based on the Schwarz framework were originally proposed for the h-version finite element method for elliptic problems. In this paper, we consider instead the p-version, in which increased accuracy is achieved by increasing the degree of the elements while the mesh is fixed. We consider linear, scalar, self adjoint, second order elliptic problems and quadrilateral elements in the finite element discretization. For a class of overlapping additive Schwarz methods, we prove a constant bound, independent of the degree p, the number of elements N and the subdomain size H, for the condition number of the iteration operator. This optimal result holds in two and three dimensions for additive and multiplicative schemes, as well as variants on the interface. We then study local refinement for the same class of overlapping methods in two dimensions. A constant bound holds under certain hypotheses on the refinement region, while in general an almost optimal bound with logarithmic growth in p is obtained.

1. Introduction. In this paper, we study some domain decomposition methods using p-version finite elements in the Schwarz framework developed by Dryja and Widlund, see [7], [8]. We consider linear, self-adjoint, second order elliptic problems and brick-shaped elements in the finite element discretization. In the p-version of the finite element method, the degree of the piecewise polynomial elements is increased in order to achieve the desired accuracy, while the mesh is fixed. This is in contrast to the standard h-version where fixed low order polynomial elements are used and the mesh is refined in order to obtain accuracy. For an overview and basic results about the p-version, see Babuška and Suri [3], Babuška and Szabo [4]. For other domain decomposition p-methods based on iterative substructuring ideas, see Babuška, Craig, Mandel, and Pitkäranta [1], and Mandel [9], [10], [11], [12]. The results of this paper were inspired by works for the h-version finite element method by Widlund [17], Dryja

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and Widlund [7], and Bramble, Ewing, Paranshkevov, and Pasciak [6], [5].

A brief description of the model problem and its discretization with the p-version finite element method is given in Section 2. In Section 3, an additive Schwarz method (ASM) with overlap is considered. It has been proved in Pavarino [14] that the condition number of the iteration operator of this method is bounded by a constant independent of p, the number N of subdomains and the subdomain size H. In section 4, this result is extended to the same method on the interface. In this case, the variables associated to the interior basis functions of each element are eliminated first, and the reduced linear system for the interface variables, known as the Schur complement, is solved by an overlapping ASM. In Section 5, local refinement for an overlapping ASM is considered. In this case, the degree p of the polynomial basis functions is increased only in selected subregions. Optimal and almost optimal bounds are obtained, depending on the choice of refinement subdomains. Numerical experiments confirming these results, as well as proofs and details, can be found in Pavarino [14], [13].

2. The Model Problem. We consider a model problem for linear, self adjoint, second order elliptic problems, on a bounded Lipschitz region Ω. Dirichlet boundary conditions are given on Γ_D , a closed subset of $\partial\Omega$ with positive measure, and Neumann conditions are given on $\Gamma_N = \partial\Omega \setminus \Gamma_D$. For simplicity, we suppose that the Dirichlet conditions are homogeneous. Neumann and mixed boundary conditions are considered in Pavarino [14]. In variational form, the model problem is:
find $u \in V = H^1_D(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$ such that

$$a(u,v) = f(v)\,, \qquad \forall\, v \in\; V\,,$$

where the bilinear form

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$$

defines a semi-norm $|u|_{H^1(\Omega)} = (a(u,u))^{1/2}$ in $H^1(\Omega)$, and a norm in $V = H^1_D(\Omega)$. Our analysis works equally well for any general self adjoint, continuous, coercive, bilinear form $b(\cdot,\cdot)$. The discrete problem is given by the p-version finite element method. A triangulation of the region Ω is introduced in terms of non-overlapping brick-like elements Ω_i , $i = 1, \cdots, N_e$. Using affine mappings onto the reference square or cube, our analysis also works for general quadrilateral elements. We suppose that the original region is a union of such elements and we denote the mesh size by H.

We define Q_p to be the set of polynomials of degree less then or equal to p in each variable, i.e. in three dimensions

$$Q_p = span\{x^i y^j z^k \,:\, 0 \leq i,j,k \leq p\}$$

and we discretize the problem with continuous, piecewise, degree p polynomial finite elements:

$$V^p = \{\phi \in C^0(\Omega) : \phi|_{\Omega_i} \in Q_p,\ i = 1, \cdots, N_e\}\,.$$

Then the discrete problem takes the form:
find $u_p \in V_D^p = \{v \in V^p : v = 0 \text{ on } \Gamma_D\}$ such that

$$a(u_p, v_p) = f(v_p)\,, \qquad \forall\ v_p \in\ V_D^p\,. \tag{1}$$

The choice of a basis for Q_p is an important computational issue, see Babuška, Griebel and Pitkäranta [2]. We follow the standard choice in the current literature of a hierarchical basis consisting of nodal, edge, face and internal functions. On the reference cube $[-1,1]^3$ in three dimensions, these are polynomials vanishing on three, four, five and all six faces of the cube, respectively.

3. An additive Schwarz method with overlap. For simplicity, we explain the method in dimension two using square elements Ω_i and we consider homogeneous Dirichlet boundary conditions only; see Pavarino [14] for generalizations to three dimensions using general quadrilateral elements and other types of boundary conditions. Let N be the number of interior nodes. Our finite element space is represented as the sum of $N+1$ subspaces

$$V_D^p = V_0^p + V_1^p + \cdots + V_N^p\,.$$

The first space V_0^p serves the same purpose as the coarse space in the h-version. Here we use $V_0^p = V_D^1$, and $V_i^p = V^p \bigcap H_0^1(\Omega_i')$. Ω_i' is the $2H \times 2H$ open square centered at the i-th vertex, see figure 1. As in the

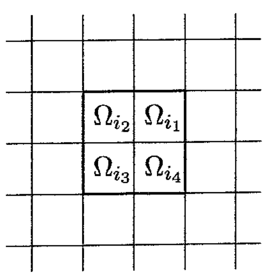

FIG. 1. *The substructure* Ω_i' = *interior of* $\overline{\Omega}_{i_1} \bigcup \overline{\Omega}_{i_2} \bigcup \overline{\Omega}_{i_3} \bigcup \overline{\Omega}_{i_4}$

h-version, the algorithm consists in solving, by an iterative method, the equation

$$Pv_p = (P_0 + P_1 + \cdots + P_N)v_p = g_p\,, \tag{2}$$

where the projections $P_i : V_D^p \to V_i^p$ are defined by

$$(3) \qquad a(P_i v_p, \phi_p) = a(v_p, \phi_p)\,, \qquad \forall \phi_p \in V_i^p$$

and $g = \sum_{i=0}^N P_i u_p$. The following theorem has been proved in [14]:

THEOREM 3.1. *The operator P of the additive algorithm defined by the spaces V_i^p satisfies the estimate $\kappa(P) \leq$ const. independent of p, H and N.*

Here we just remark that the lower bound is obtained via Lions' lemma by partitioning a finite element function in V_D^p as $v_p = \sum_{i=0}^N v_{p,i}$, $v_{p,i} \in V_i$. The first term $v_{p,0} = \hat{I}_1 v_p$ is the L^2-projection of v_p onto V_D^1 . It can also be defined by smoothing and interpolation as in Strang [16]. The other terms are $v_{p,i} = I_p(\theta_i(v_p - v_{p,0}))$, where $\{\theta_i\}$ is a special partition of unity consisting of linear combinations of the standard basis functions for Q_1 . Near the boundary, the construction of these θ_i is not trivial, especially where $\partial\Omega$ is L-shaped or has cracks. I_p is an interpolation operator defined locally from Q_{p+1} to Q_p as follows. On the reference square $[-1,1]^2$, the interpolation nodes are the $(p+1)^2$ points (x_n, x_m), where the $x_n's$ are the zeros of the polynomial

$$(4) \qquad \mathcal{L}_{p+1}(x) = \int_{-1}^{x} L_p(s)\, ds\,.$$

Here $L_p(s)$ is the Legendre polynomial of degree p. The proof is then based on the

LEMMA 3.2. *The interpolation operator*

$$I_p : Q_{p+1}([-1,1]^2) \to Q_p([-1,1]^2)$$

is uniformly bounded in the H^1- seminorm, i.e.

$$|I_p(f)|_{H^1} \leq const. |f|_{H^1}\,, \qquad \forall f \in Q_{p+1}([-1,1]^2)\,.$$

The proof of this lemma is technical: choosing an appropriate basis for Q_{p+1}, we bound the eigenvalues of a generalized eigenvalue problem. In the three dimensional version of the method, the main technical result is again the uniform boundness of the interpolation operator I_p : $\hat{Q}_{p+1}([-1,1]^3) \to \hat{Q}_p([-1,1]^3)$ in the H^1- seminorm.

4. Overlapping ASM on the interface. As we mentioned in Section 2, the basis functions spanning V^p can be hierarchically ordered in groups of interior, face, edge and nodal functions. If the unknowns associated to the interior functions are eliminated, then the reduced Schur complement can be solved with the overlapping ASM introduced previously. More precisely, the discrete problem is now:
find $u_p \in \tilde{V}_D^p$ such that

$$(5) \qquad a(u_p, v_p) = f(v_p)\,, \qquad \forall\, v_p \in \tilde{V}_D^p\,,$$

where $\tilde{V}_D^p$ is the subspace of the discrete harmonic functions of V_D^p. A function $v \in V^p$ is said discrete harmonic if

$$a(v, \phi) = 0,$$

for every $\phi \in V^p$ that vanishes on the interface $\Gamma = \bigcup_i \partial\Omega_i$. The algorithm is defined by the following decomposition of $\tilde{V}_D^p$:

$$\tilde{V}_D^p = \tilde{V}_0^p + \tilde{V}_1^p + \cdots + \tilde{V}_N^p ,$$

where $\tilde{V}_i^p = \tilde{V}^p \bigcap H_0^1(\Omega_i')$ and Ω_i' and $\tilde{V}_0^p = V_0^p$ are defined as before. In terms of projections $P_i : \tilde{V}_D^p \to \tilde{V}_i^p$, the method solves iteratively the linear operator equation

$$Pv_p = (P_0 + P_1 + \cdots + P_N)v_p = g_p . \tag{6}$$

We can then prove a result analogous to Theorem 3.1:

THEOREM 4.1. *The operator P of the additive algorithm defined by the spaces $\tilde{V}_i^p$ satisfies the estimate $\kappa(P) \leq$ const. independent of p, H and N.*

Proof. The proof of the upper bound is the same as in Theorem 3.1. To obtain a lower bound via Lions' lemma, we use the previous decomposition of a function of $\tilde{V}^p \subset V^p$ obtained in the proof of 3.1:

$$\tilde{v}_p = \sum_{i=0}^{N} v_{p,i} \qquad \text{and} \qquad \sum_{i=0}^{N} a(v_{p,i}, v_{p,i}) \leq C_0^2 a(\tilde{v}_p, \tilde{v}_p).$$

We then restrict each component $v_{p,i}$ to Γ and extend it as a discrete harmonic function $\tilde{v}_{p,i} \in \tilde{V}_i^p$. Since the discrete harmonic extension minimizes the energy, we have obtained the desired decomposition. □

5. Local refinement in two dimensions. For standard h-version finite elements, local refinement can be introduced by selecting and refining some elements of a coarse triangulation. This process can be applied recursively and multilevel methods have been considered. For the p-version finite element method considered here, local refinement consists in increasing the order p of the polynomial basis functions only in selected elements of the fixed triangulation. This can be of interest in many applications where the accuracy of the numerical solution needs to be increased only in certain parts of the domain. In this section, we consider local refinement for the method introduced in Section 3.

With the same notations as before, we select $N^r < N$ interior nodes. Let I^r be the set of refinement indexes. With each selected interior node x_i, we associate a subdomain Ω_i', defined as before as the $2H \times 2H$ open square centered at x_i. The region of refinement is then

$$\Omega_r = \bigcup \Omega_i', \qquad i \in I^r$$

and the finite element space

$$V_r^p = V_0^p + V_1^p + \cdots + V_{N_r}^p .$$

Again, $V_0^p = V_D^1$ is the analog of the h-version coarse space and $V_i^p = V^p \bigcap H_0^1(\Omega_i')$ are the local spaces.
The discrete problem is now posed in the refinement space V_r^p :
find $u_p \in V_r^p$ such that

$$a(u_p, v_p) = f(v_p) , \qquad \forall\, v_p \in V_r^p . \tag{7}$$

The main result of this Section is the following:

THEOREM 5.1. *The operator P of the additive algorithm defined by the spaces V_i^p satisfies the estimate*

$$\kappa(P) \leq const.$$

if there are no isolated points on $\partial\Omega_r$, and

$$\kappa(P) \leq C(1 + \log p)^2$$

otherwise.
A point on $\partial\Omega_r$ is isolated if it is not a limit (accumulation) point of $\partial\Omega_r$. This result specifies which choices of refinement points lead to a bounded condition number. It is interesting to note that if a whole edge is isolated on $\partial\Omega_r$, then we will still have a constant bound. The proof of this theorem can be found in Pavarino [13] and is based on a series of technical results concerning the decomposition of discrete harmonic polynomials and on Theorem 3.1. The main tools used in the proof are Markov's theorem (see Rivlin [15]) and a p-version analog of the decomposition lemma 3.2 in Widlund [17]. A two dimensional extension theorem for polynomial finite elements (see Babuška, Craig, Mandel and Pitkäranta [1]) is needed to prove the logarithmic bound, while the constant bound can be obtained without it.

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