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Domain Decomposition Method Coupling Finite Elements and Preconditioned Chebyshev Collocation to Solve Elliptic Problems

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ABSTRACT. This paper presents a mixed discretization method based on domain decomposition technique which couples the preconditioned Chebyshev collocation to the finite element method. Test problems show that the accuracy of the spectral method is regained if the finite element part of the domain has limited extent.

1. Introduction

Various spectral discretization methods, known to be very accurate when the solution to be approximated is very smooth, are currently developed and used to solve partial differential equations [1]. Difficulties to apply such high-order approximations to real-life problems arise however when the function to be approximated presents singularities. On the contrary, the finite element methods, which restrict the approximation to piecewise low-order polynomials defined over very small domains, are consequently well suited to produce acceptable solution in the presence of singularities.

Some methods combine the advantages of each one, such as the p-version of the finite elements method [2] or the spectral element method [6]. However, these methods use the same approximation all over the discretized domain. A totally different approach is to couple distinct finite element and spectral discretizations together in the same solution procedure. The approximation functions will thus be piecewise polynomials on one part of the computational domain and high order polynomials on the remaining part with a matching condition at the interface. This way has been investigated for the spectral elements method [3].

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In the context of the preconditionned Chebyshev collocation, we developed a general multi-domain decomposition method to solve elliptic equations based simultaneously on both discretizations : in some subdomains, the problem will be approximated by a finite element method in its weak Galerkin formulation while in the remainder of the computational domains the solution will satisfy a Chebyshev collocation technique. The preconditioned Chebyshev collocation presents major numerical advantages [4,5] and has been demonstrated to be an efficient way to attain the attractive properties of a spectral method. Since the initial step of its preconditioned iterative algorithm consists in the computation of a global finite element solution, it appeared natural in such a framework to embed in the same iterative procedure both finite element and spectral discretizations.

In this paper, we present the first results of this method applied to the solution of second-order elliptic problems. For the sake of simplicity, the domain of interest is decomposed into rectangles but it can be decomposed into curved quadrilaterals if needed [7]. The interface condition [7], naturally incorporated into the FE preconditioning of the full Chebyshev collocation, relates neighbour subdomains through the normal jump of spectral fluxes across internal boundaries. A weak C^1 continuity is consequently achieved at the interface. While coupling finite elements, the jump of spectral fluxes is replaced by a difference between the weak flux induced at the interface by the Galerkin finite element discretisation and between the spectral flux from the subdomain where the collocation technique applies. After a short description of the principles of our method, we show on test problems that global solutions may be fairly more accurate by using the mixed FE/SP discretization instead of simple full FE.

2. The Method

Given a function $f \in L^2(\Omega)$, we consider the 2-D Helmholtz problem :

$$(1.1) \qquad \begin{cases} -\Delta u + u = f\,, & \text{on } \Omega\,, \\ u = 0\,, & \text{on the boundary } \partial\Omega\,. \end{cases}$$

Let us denote by N the couple $(N_{x_1}, N_{x_2}) \in \mathbb{N} \times \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. The domain Ω is broken up into several non-overlaping subdomains Ω_p. The decomposition imposes that the number of degrees of freedom (d.o.f.) in the direction of the common side must be the same in each adjacent subdomain, leading to conforming finite elements in the preconditioner.
For each one of those subdomains Ω_p, a suitable mapping associates the collocation grid defined by the tensor product of one-dimensional N_{x_i} Gauss-Lobatto-Chebyshev quadratures in the reference square to the locally corresponding physical collocation (sub-)grid. The collection of the local sub-grids composes the overall collocation grid and FE mesh.

The matrix system corresponding to the Chebyshev collocation discretization of (1.1) over the entire domain Ω takes the form :

$$(1.2) \qquad L_c\, \underline{x} = \underline{b}\ .$$

In Eq. (1.2), $\underline{x}$ denotes the vector of collocation values of u over $\Omega \cup \partial\Omega$ and the collocation operator L_c includes the collocation approximation of the boundary conditions. Thus, the term $\underline{b}$ also incorporates the imposed boundary values.

To improve the numerical solution of the resulting system ($O(N^4)$ conditioning of the discrete operator and very large bandwidth), the collocation system (1.2) is preconditioned by finite elements (FE). The numerical scheme is based on the preconditioned Richardson iteration technique:

$$L_{FE}(\underline{x}^{k+1} - \underline{x}^k) = + \alpha\,(\underline{b} - L_c\,\underline{x}^k) \;, \qquad (1.3)$$

where L_{FE} is a finite element approximate operator and α a relaxation factor. In full Chebyshev collocation approximation of elliptic problems, a relaxation factor α set to 1 (one) drives optimal convergence rates. In (1.3), the superscript k denotes the Richardson index. The initial guess of this iterative procedure is the solution of the problem (1.1) resulting from the associated FE Galerkin approximation.

When coupling FE to collocation discretization (SP), we split the set of subdomains into the subset $\{\Omega_p^{FE}\}$ where the solution will be sought in the FE space and into the subset $\{\Omega_p^{SP}\}$ where a Chebyshev collocation solution is to be computed. The general iteration procedure (1.3) still holds and may, for the sake of simplicity, be expressed as

$$\underline{x}^{k+1} = \underline{x}^k + \alpha\,(L_{FE})^{-1}\left\{\underline{r}_{FE}\right\} \;, \qquad \text{in } \{\Omega_p^{FE}\} \;, \qquad (1.4.a)$$

$$\underline{x}^{k+1} = \underline{x}^k + \alpha\,I_N\,(L_{FE})^{-1}\left\{\underline{r}_{FE}\right\} \;, \qquad \text{in } \{\Omega_p^{SP}\} \;, \qquad (1.4.b)$$

where I_N is a projector in the space of polynomials of degree N_{x_i} in each direction.

The term $\underline{r}_{FE}$ is composed differently, depending on the region considered. For any unknown in $\{\Omega_p^{FE}\}$, the corresponding $\underline{r}_{FE}$ is computed directly from the FE residual $\underline{f}$-$L_{FE}\,\underline{x}^k$ ($\underline{f}$ denotes the FE right-hand side of the Galerkin problem). Since $\underline{x}^k$ is the FE solution of the weak form of (1.1), whose operator is identical to L_{FE}, the FE residual vanishes everywhere over the FE sub-domain(s) but at the interface where it provides the weak flux (Neumann contribution) from the FE current solution. In the SP part of the domain, $\underline{r}_{FE}$ is the finite element right-hand side obtained through the projection of the collocation spectral residual $\underline{b}$-$L_c\underline{x}^k$. At any interface, this term also incorporates the integrated spectral flux across the interface.

Consequently, for any unknown of the global system which lies on the interface, the corresponding right-hand side accumulates the weak FE flux from the FE part and the integrated spectral flux together with the residual from the SP part. In a global iterative solution procedure, the solution in the FE part(s) of the computational domain is slightly influenced by the accurate spectral interpolation through the common interface. More details on the method can be found in [7].

3. Numerical Results

In order to test our method, the homogeneous problem (1.1) in the [0,1]x[0,1] square was solved with a function f corresponding to the exact solution $u = xy(1-x)(1-y)\ln(0.1+x+y)$ which presents steep variations close to the origin (problem tp1). In one of the partitionings considered (see [7]), the domain is decomposed into nine subdomains where four FE subdomains fill the corners and the remainder are devoted to full SP collocation. The final solution obtained through the iterative procedure (1.3) is compared in Figure 1 to the initial FE solution and the full SP solution of the classical procedure in the case where the corner subdomains are relatively small (intermediate spatial discretization 7x7 d.o.f. each with 17x17 d.o.f in the center subdomain, thus 29x29 global equivalent d.o.f.). In general, we observe that the FE approximation limits in the mixed method the accuracy in the spectral subdomain. However, the smaller the extent of the FE part is, the larger the gain in accuracy for the SP approximation will be.

We also solved over [0,1]x[0,1] the Laplace equation with the function f set to 1 (one) and homogeneous Dirichlet boundary conditions. This problem (tp2) presents geometric singularities in the four corners where the differential equation is enforced while the boundary conditions lead to a vanishing Δu. It will serve us to test our coupling method in the presence of boundary singularities, which are known to alter significantly the rate of convergence of a spectral method. This problem was already solved by classical preconditioned Chebyshev collocation and compared to a singular FE analysis [4]. We resorted to the same 9 subdomains decomposition than above, since this combination was demonstrated as the most advantageous one. A comparison of the absolute analytical errors (for 25x25 eq. d.o.f.) can be found in Figure 2 which illustrates the gain achievable by our method. While the discrete L_∞ norm of the error for full FE converges as N^{-2}, the FE/SP error decays as N^{-5}. In all the test cases, the increments of the Richardson process reached machine round-off after roughly 20 iterations.

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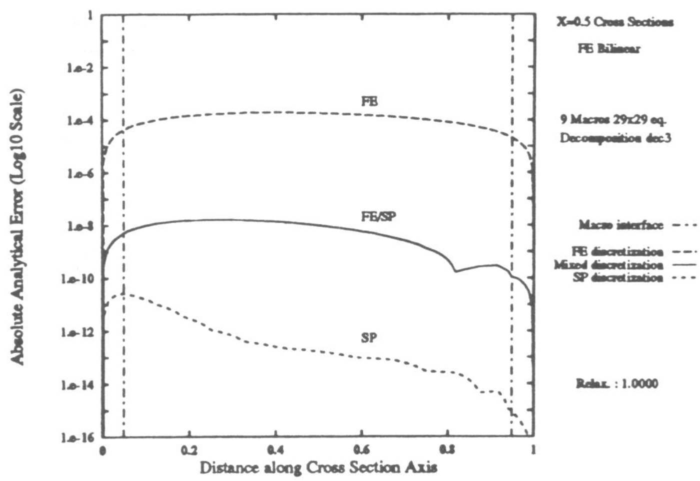
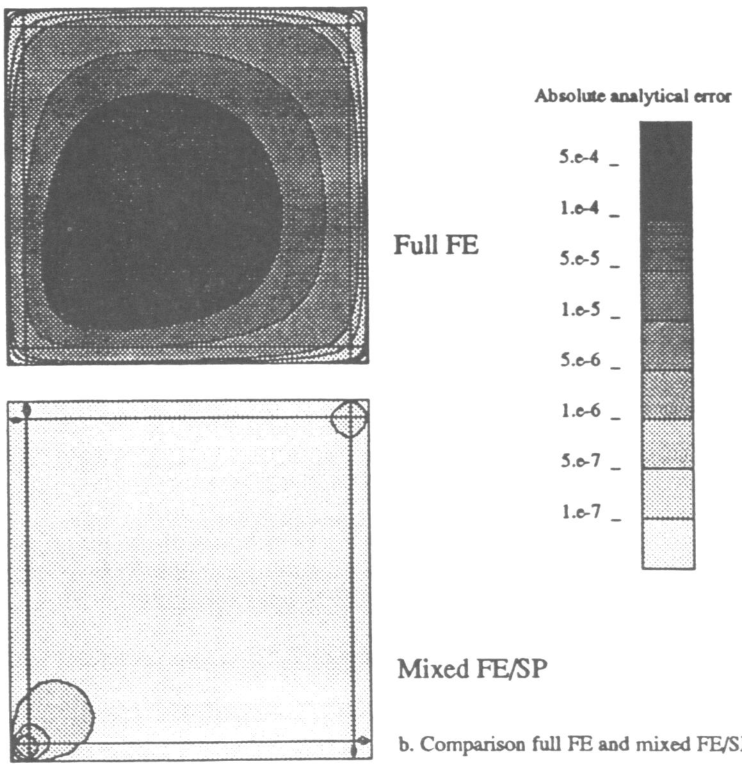
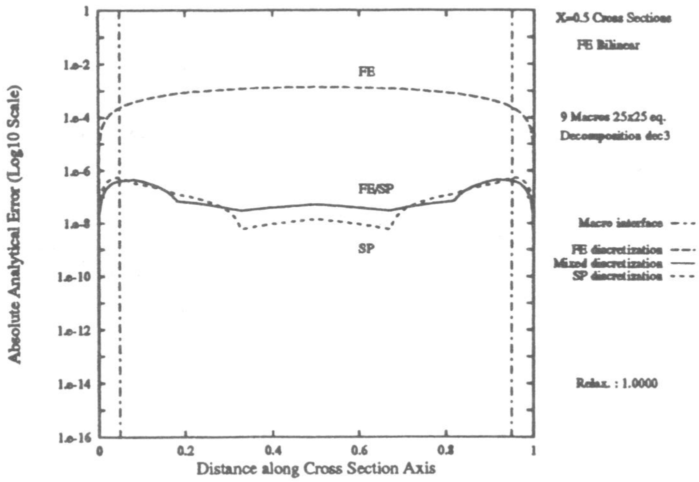


a. Comparison for all the discretization methods (FE, FE/SP and SP) on the X=0.5 cross section



b. Comparison full FE and mixed FE/SP

Figure 1 Comparison of the absolute analytical errors for test problem tp1
(9 subdomains bilinear FE preconditioner, 29x29 eq. d.o.f.)



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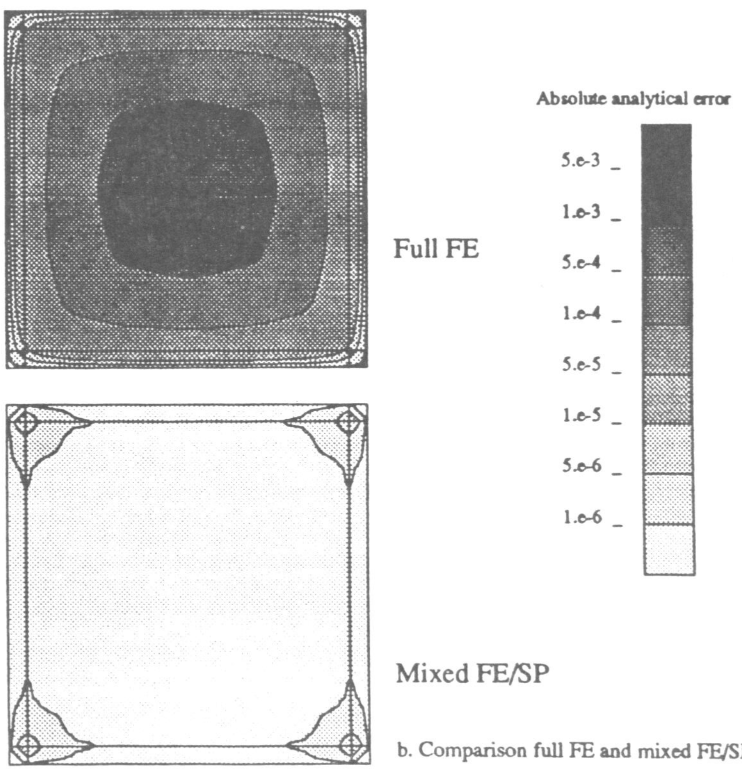

b. Comparison full FE and mixed FE/SP

Figure 2 Comparison of the absolute analytical errors for test problem tp2
(9 subdomains bilinear FE preconditioner, 25x25 eq. d.o.f.)