Contemporary Mathematics
Volume 157, 1994

Domain Decomposition and Multilevel PCG Method for Solving 3-D Fourth Order Problems

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ABSTRACT. We have studied some efficient algorithms for solving biharmonic equation in 3-D based on finding preconditioners on interface and on subdomains, respectively. Numerical results on Transputer T800 for some model problems are presented.

1. Introduction

It is well-known that the biharmonic equation arises from some 2-D elastic problems, such as plane stress functions and thin plate bending problem. However, even in 3-D elastic and structure mechanics fourth order partial differential equations appear often. As an example, when the static force field is constanst, all three components of displacements , six components of stress and strains satisfy biharmonic equation.

Consider the partial differential equation

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{1.1}$$

where Ω is decomposed into several regular domains, with boundary conditions

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

or

$$u = \Delta u = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

1991 Mathematics Subject Classification. Primary 65F10, 65N30.

This work was supported by National Natural Science Fundation of China and State Major Key Project for Basic Researches.

This paper is in final form and no version of it will be submitted elsewhere.

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0271-4132/94 $1.00 + $.25 per page

In theoritical point of view, one of difficulties for solving fourth order problems is that the well-known maximum principle for second order problem has to be modified now. In computing point of view, the ill-condition of resulting difference equation for fourth order problems is more serious than for second order problems.

In a single regular subdomain case, the problem in 2-D was solved numerically by using piecewise bicubic C^2 and piecewise bivariate cubic C^1 B-spline finite element method (cf. [5] and [6]). In this paper, we use tensor product of C^1 quadratic B-spline for 3-D problems.

For a given domain Ω, we have a 'coarse' rectangular partition $\tau_0 = \cup_i \tau_0^i$. Successively finer partition $\{\tau_k, k = 1, 2, \ldots, J\}$ are defined by connecting the midpoints of the edges. We denote by h_k the diameter of $\{\tau_k^i\}$, $H = h_0$, $h = h_J$. The subspace $\mathcal{M}_k$ is defined to be the C^1 continueous functions on Ω.

Let $B_{i_1}^k(x)$, $B_{i_2}^k(y)$ and $B_{i_3}^k(z)$ $(i_1 = 1, \ldots, m_1 - 1, i_2 = 1, \ldots, m_2 - 1, i_3 = 1, \ldots, m_3 - 1)$ be the quadratic B-spline functions centered at $x_{i_1+\frac{1}{2}}$, $y_{i_2+\frac{1}{2}}$ and $z_{i_3+\frac{1}{2}}$ respectively. Let

$$\phi_i^k(x, y, z) = B_{i_1}^k(x) B_{i_2}^k(y) B_{i_3}^k(z). \tag{1.4}$$

Then $\{\phi_i^k\}_{i=1}^{n_k}$ $(n_k = dim\mathcal{M}_k)$ are the tri-quadratic C^1 B-spline bases for the space $\mathcal{M}_k$. Namely

$$\mathcal{M}_k = span\{\phi_i^k\}_{i=1}^{n_k}.$$

Thus we may construct a nested sequence of finite-dimensional spaces

$$\mathcal{M}_0 \subset \mathcal{M}_1 \subset \ldots \subset \mathcal{M}_J \equiv \mathcal{M}, \quad J = log_2 \frac{H}{h} \geq 1$$

This finite dimensional variational problem leads to the following linear system

$$\hat{A}x = b. \tag{1.5}$$

where $\hat{A} = [a(\phi_i, \phi_j)]$, $b = [(f, \phi_j)]$,

$$a(u, v) = \int_\Omega \Delta u \Delta v dx,$$

and x is the vector of unknowns x_i (the coefficient of B-spline ϕ_i).

It is often convenient to scale B-spline basis as follows:in d-dimension

$$\bar{\phi}_i^k = \{h_k^{\frac{4-d}{2}} \phi_i^k\}, d = 1, 2, 3.$$

The corresponding stiffness matrix with respect to $\mathcal{M}_k$ is defined by

$$A^k = [a(\bar{\phi}_i^k, \bar{\phi}_j^k)]_{n_k \times n_k}.$$

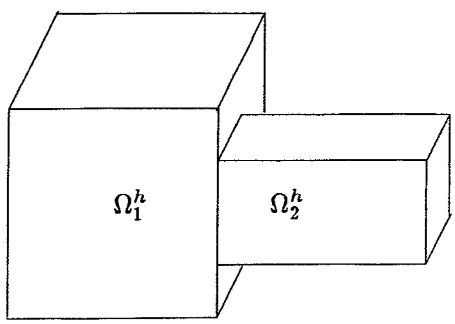
Suppose the defined domain Ω is decomposed into two different cubes Ω_1 and Ω_2 with a common inner surface interface, for examples, L-Shape or T-Shape in 3-D, see Fig.1. The coefficient matrix $\hat{A}$ in (1.5) can be expressed in block form as

$$\hat{A} = \begin{bmatrix} A_{11} & 0 & A_{13} & A_{14} \\ 0 & A_{22} & A_{23} & A_{24} \\ A_{13}^T & A_{23}^T & A_{33} & A_{34} \\ A_{14}^T & A_{24}^T & A_{34}^T & A_{44} \end{bmatrix} \tag{1.6}$$

2. An analysis on fourth order interface

The Schur complement of the resulting matrix (1.6) which is corresponding to the reduced surface interface operator can be written as

$$C = \begin{bmatrix} A_{33} & A_{34} \\ A_{34}^T & A_{44} \end{bmatrix} - \begin{bmatrix} A_{13} & A_{14} \\ A_{23} & A_{24} \end{bmatrix}^T \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix} \begin{bmatrix} A_{13} & A_{14} \\ A_{23} & A_{24} \end{bmatrix} \tag{2.1}$$



Figl. A model of T-Shape in 3-D

It is clear that the Schur complement C in (2.1) is dense and expansive to form explicitly. Moreover we have proved in [2] that in 2-D case C is serious ill-condition in the sense the condition number

$$\kappa(C) = O(h^{-3})$$

It is not difficult to claim the above conclusion is still valid in 3-D case.

There are two different way to analyse the behaviour of the fourth order interface. One is to do similar analysis directly, by using the known technique for related second order interface. The other is to use the known results on second order interface and establish a relationship between the two interfaces. This means a good PCG interface precondiitoner for fourth order problems can be converted from the related second order problems.

THEOREM 1. *If M_2 is a good interface preconditioner for a second order elliptic operator, then*

$$M_4 = \begin{bmatrix} M_2(I + M_2^2) & -M_2(I - M_2^2) \\ -M_2(I - M_2^2) & M_2(I + M_2^2) \end{bmatrix} \tag{2.2}$$

is also a good interface preconditioner.

Now we may extend some analysis, for examples Dryja's, Golub-Mayers' or Tony Chan's work for second order interface from 2-D to 3-D. For 3-D Dryja-like interface preconditioner may be simply written in the form

$$M_2 = \sqrt{-\Delta} \tag{2.3}$$

To extend Golub-Mayers' [4] analogue to 3D, we consider Laplace equation over upper half space. A resulting difference scheme can be written as

$$\{(E_1 + E_1^{-1})(E_2 + E_2^{-1})(E_3 + E_3^{-1}) + 2[(E_1 + E_1^{-1})(E_2 + E_2^{-1}) + (E_2 + E_2^{-1})(E_3 + E_3^{-1}) + (E_3 + E_3^{-1})(E_1 + E_1^{-1})] - 32I\}u_{rst} = 0$$

where E_i is shifting operators: $E_1 u_{rst} = u_{r+1,s,t}$, $E_1^{-1} u_{rst} = u_{r-1,s,t}$, ect.

Substituting the following generating function

$$\phi_t(z_1, z_2) = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} z_1^r z_2^s u_{rst}$$

with $\phi_t - > 0$ as $t - > \infty$ into the discretization scheme leads to a second order difference equation with respect to t:

$$\{[2(z_1 + z_1^{-1})(z_2 + z_2^{-1}) - 32]I + [2(z_1 + z_1^{-1} + z_2 + z_2^{-1}) + (z_1 + z_1^{-1})(z_2 + z_2^{-1})](E_1 + E_1^{-1})\}\phi_t = 0$$

The solution for upper space is equal to

$$\{\phi_t(z_1, z_2)\}^{1/t} = 16 - (z_1 + z_1^{-1})(z_2 + z_2^{-1}) - \{[16 - (z_1 + z_1^{-1} + z_2 + z_2^{-1})]^2 - [2(z_1 + z_1^{-1} + z_2 + z_2^{-1}) + (z_1 + z_1^{-1})(z_2 + z_2^{-1})]^2\}^{1/2}$$

By symmetry there is a similar solution for lower half space. Put the two solution together we obtain the solution for whole space

$$\phi_t(z_1, z_2) = [2(z_1 + z_1^{-1} + z_2 + z_2^{-1}) - 32]\phi_0 + 2\phi_1$$

Let $z_1 = e^{i\theta_1}, z_2 = e^{i\theta_2}$ Thus we have an estimate for eigenvalues on the interface of second order elliptic problems in 3-D. Finally a interface preconditioner can be taken as

$$\lambda(\theta_1, \theta_2) = \phi_t(z_1, z_2)$$

$$M_2 = W\Lambda W \quad with \;\; \Lambda = Diag\{\lambda\} \tag{2.4}$$

3. Parallel multilevel B-spline preconditioners

We shall use the following notation as shown in [1]. For each $k = 0, 1, \ldots, J$, we introduced the following operations:

1^o The operator $A_k : \mathcal{M}_k \to \mathcal{M}_k$ is defined for $u \in \mathcal{M}_k$ by

$$a(A_k u, v) = a(u, v), \text{ for all} v \;\; \in \mathcal{M}_k$$

If $k = J$, we denote $A = A_J$.

2^o The projection $Q_k : \mathcal{M} \to \mathcal{M}_k$ is defined for $u \in \mathcal{M}$ by

$$a(Q_k u, v) = (u, v), \text{ for all } \; v \in \mathcal{M}_k$$

THEOREM 2.

$$\kappa(A_k) \asymp h_k^{-4}.$$

Similarly to [1], we shall study the following preconditioner

$$\mathcal{B} = A_0^{-1} Q_0 + \sum_{k=1}^{J} \lambda_k^{-1} Q_k$$

for fourth order problems, where $\lambda_k = \lambda_{max}(A_k)$.

In order to obtain some main results, we need introduce a spline interpolant operator $I_k : C(\Omega) \to \mathcal{M}_k$ defined on midpoints. This spline interpolant is unique. By using the B-spline properties, it is not difficult to verify that for $u \in \mathcal{M}$,

$$||u - I_k u||_{L^2(\Omega)} \preceq h_k^2 |u|_{H^2(\Omega)}.$$

Thus, we obtain the following results [7].

LEMMA. 1^o*For* $u \in \mathcal{M}$, $||u - Q_k u||_{L^2(\Omega)} \preceq h_k^2 |u|_{H^2(\Omega)}$.

2^o*For* $u \in \mathcal{M}_k$, $||(I - Q_{k-1})u||_{L^2(\Omega)}^2 \preceq \lambda_k^{-1} a(u, u)$

3^o $|Q_k u|_{H^2(\Omega)} \preceq |u|_{H^2(\Omega)}, \;\; \forall u \in \mathcal{M}$.

4^o $a(Q_k v, Q_k v) \preceq a(v, v), \; \forall v \in \mathcal{M}$.

Thus, follow the idea of [1], we may obtain

Theorem 3. *For all $v \in \mathcal{M}$,*

$$(J+1)^{-1}a(v,v) \preceq a(\mathcal{B}Av,v) \leq (J+1)a(v,v).$$

Hence

$$\kappa(\mathcal{B}A) \preceq (J+1)^2.$$

In the above, the preconditioners for biharmonic problem are presented in terms of operators on the spline finite element spaces. Now we turn to get a preconditioner in terms of matrix. For $k \leq l$, we note that $\mathcal{M}_k \subset \mathcal{M}_l$, hence each ϕ_i^k is a linear combination of some ϕ_i^l. Let T_k^l be the representation matrix of the spline basis $\{\phi_i^k\}$ of $\mathcal{M}_k$ in terms of the spline basis$\{\phi_i^l\}$ of $\mathcal{M}_l$. Namely

$$\Phi^k = T_k^l \Phi^l$$

where $\Phi^k = (\phi_1^k, \ldots, \phi_{n_k}^k)^t$ and "t" denotes the usual transpose operation.

The above preconditioner $\mathcal{B}$ in terms of matrix is given by

$$\hat{B} = \sum_{k=0}^{J} h_k^{4-d} T_k^t T_k \tag{3.1}$$

where $T_k = T_k^J$.

It can be shown that

$$\kappa(\hat{B}\hat{A}) = \kappa(\mathcal{B}A)$$

Hence we can conclude that the matrix $\hat{B}$ given by (3.1) is a good preconditioner of the stiffness matrix $\hat{A}$ in (1.5).

4. Numerical Results

The standard uniform mesh and bi-quadratic and tri-quadratic spline finite element discretization are used for solving 2-D and 3-D biharmonic equation, respectively. The domain Ω in the following computation is considered to be decomposed into two cubes with the same or different size. In all CG iteration procedures the stopping criterion is used when the relative 2-norm of the residual fall below 10^{-5}. The initial guess used is $x^{(0)} = 0$.

Table 4.1 lists iteration number of CG and PCG required on the interface with two same cubes $[0,1]^3$, where h is the mesh size, N is the total number of unknowns. PCG1 and PCG2 are the preconditioner (2.2) combining (2.3) or (2.4), respectively. The numbers in round brackets record the speedup of PCG CPU time with respect to CG iteration on transputer T800/25. More detailed iteration record for PCG2 which is acsses to Theorem 1 is in Table 4.2. The iteration number tends a constant as the mesh size is getting smaller and smaller. The domain in Table 4.3 is so-called 3-D T-shape region. The bigger subdomain is $[0,1]^3$, the smaller is $[0.25, 0.75]^3$. It shows the interface PCG still works well even for 3-D T-shape domain.

In each subdomain we use multilevel B-spline described in section 3. Table 4.4 and 4.5 record the iteration number and CPU time for a given step size h. We observe that the iteration counts is at worst linearly dependent on $O(log(\frac{H}{h}))$ for the model problem.

Table 4.1. Interface CG and PCG Comparision in 3-D
Two Cubic Domain:Iteration count (CPU Speedup)

Method 1/h	8	16	32	64
N	1024	8192	64K	512K
CG	6	18	64 (1.00)	
PCG1	6	10	13 (4.05)	
PCG2	5	9	10 (5.30)	12

Table 4.2. Interface PCG for 3-D biharmonic equation
Cubic domain: Iteration count

1/h	12	16	24	32	48	64
N	3456	8192	27K	64K	221K	512K
PCG2	8	9	10	10	11	12

Table 4.3. Interface CG and PCG comparision
3-D biharmonic equation over T-Shape Domain
Iteration count (CPU Speedup)

Method	1/h=	6	8	16
CG		7	13	49 (1.00)
PCG		7	11	14 (3.60)

Table 4.4. Multilevel B-spline PCG for 3-D
Iteration count versus h and J(number of level)

J	1/h	8	16	32	64
0		8	29	112	436
1		7	16	36	125
2			14	21	42
3				20	27

Table 4.5 Multilevel B-spline PCG for 3-D
CPU Speedup on Transputer T800/25

J	1/h	8	16	32	64
0		1.00	1.00	1.00	1.00
1		1.00	1.64	2.72	3.02
2			1.74	4.30	8.26
3				4.20	11.81

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