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Finite Volume Variational Formulation. Application to Domain Decomposition Methods.

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ABSTRACT. The finite volume element method for diffusion equations is a discretization technique for partial differential equations formulated in divergence form. With this presentation, the convergence proof use some tools which are well known for the study of the finite difference methods. The generalization to complex geometry lead to some difficulties and the proposed analysis seems to be artful but not natural. We give here a new formulation of the finite volume element method as a particular case of a generalized mixed formulation with two fields but four distinct vector spaces: the trial fonctions are in a finite dimensional subspace of $H(div) \times H^1$ while the test functions are in a finite dimensional subspace of $(L^2)^n \times L^2$.
So the approximation solution by this method will be an internal approximation of the solution (p, u) *in* $H(div) \times H^1$. Note that the (standard = dual) mixed method gives an approximate solution in $H(div) \times L^2$ while the primal mixed method gives an approximate solution in $(L^2)^n \times H^1$.
For such a variational formulation we have three Babuska-Brezzi conditions to verify for obtaining the existence-uniqueness results and then the a priori error bounds. Examples of choices of the four finite dimensional vector spaces which satisfy the three inf-sup conditions will be given.

Then in this framework of the finite volume method, we develop some domain decomposition algorithms and we analyse them.

1. Presentation of finite volume methods

Let us begin first by presenting finite volume methods. Consider the homogenous Dirichlet problem:

$$(1) \qquad \begin{cases} -div(Kgradu) = f & in\ \Omega \\ u = 0 & on\ \partial\Omega \end{cases}$$

with K positive, $f \in L^2(\Omega)$ and Ω an open bounded polygonal domain of $\mathbf{R}^2$, whose boundary will be denoted by $\partial\Omega$.
The finite volume formulation consists to find $u \in H_0^1(\Omega)$ with $div(Kgradu) \in L^2(\Omega)$ such that

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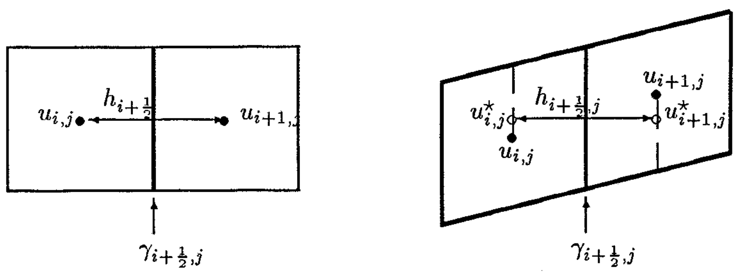
$$\forall \omega \subset \Omega, \quad -\int_\omega div(Kgradu)dx = \int_\omega f dx \ .$$

This can also be written as:

$$(2) \qquad \forall \omega \subset \Omega, \quad -\int_{\partial\omega} Kgradu.n_\omega dx = \int_\omega f dx$$

where n_ω is the unit outward normal vector on the boundary $\partial\omega$ of ω: $gradu.n_\omega = \frac{\partial u}{\partial n_\omega}$ is the outward normal derivative.
The equation (2) is now discretized using a finite difference method, with one unknow by cell (or "finite volume").
Example in a 2 dimensional case on a structured mesh:

Rectangular case:(5 pts scheme) Quadrangular case: (9 pts scheme)



On $\gamma_{i+\frac{1}{2},j}$, we take:

$$5 \ pts \ scheme: \ \frac{\partial u}{\partial n} \simeq \frac{u_{i+1,j}-u_{i,j}}{h_{i+\frac{1}{2}}} \quad ; \quad 9 \ pts \ scheme: \ \frac{\partial u}{\partial n} \simeq \frac{u^\star_{i+1,j}-u^\star_{i,j}}{h_{i+\frac{1}{2},j}}$$

($u^\star_{i,j}$ is in this example a linear interpolation of $u_{i,j}$ and $u_{i,j+1}$.)
Convergence is not proved in general case (cf Faille [1]).

2. Finite volume methods analysis using a mixed formulation

The problem (1) can be rewritten as,

$$\begin{cases} p = Kgradu & in \ \Omega \\ divp + f = 0 & in \ \Omega \\ u = 0 & on \ \partial\Omega \end{cases}$$

We recall in the following the two classical mixed formulations of problem (1) (cf Roberts and Thomas [4]).

2.1. Primal mixed formulation.

$$\begin{cases} find \quad (p,u) \in (L^2(\Omega))^2 \times H^1_0(\Omega) & solution \ \ of \\ \forall q \in (L^2(\Omega))^2, & \int_\Omega \frac{1}{K} p.q \ dx - \int_\Omega gradu.q \ dx = 0 \\ \forall v \in H^1_0(\Omega), & \int_\Omega gradv.p \ dx = \int_\Omega fv \ dx \end{cases}$$

2.2. Dual mixed formulation.

$$\begin{cases} find \quad (p,u) \in H(div,\Omega) \times L^2(\Omega) & solution \quad of \\ \forall q \in H(div,\Omega), & \int_\Omega \frac{1}{K} p.q \; dx \; + \; \int_\Omega u div q \; dx \; = \; 0 \\ \forall v \in L^2(\Omega), & \int_\Omega v div p \; dx \; = \; - \int_\Omega f v \; dx \end{cases}$$

2.3. A new mixed formulation. This new formulation can be seen as a primal-dual method. We write the problem (1) in the weak form

$$(3) \quad \begin{cases} find \quad p \in H(div,\Omega), \quad u \in H_0^1(\Omega) & solution \quad of \\ \forall q \in (L^2(\Omega))^2, & \int_\Omega (\frac{1}{K} p - grad u).q \; dx \; = 0 \\ \forall v \in L^2(\Omega), & \int_\Omega v div p \; dx \; = \; - \int_\Omega f v \; dx \end{cases}$$

It is easy to prove that there exists a unique solution of (3).
Let be

$$a(p,q) \; = \; \int_\Omega \frac{1}{K} p.q \; dx$$

and

$$b_1(p,v) \; = \; \int_\Omega v div p \; dx \quad , \quad b_2(q,u) \; = \; - \int_\Omega q.grad u \; dx \; .$$

In order to have a more general framework, we introduce the next notations:

$$W_1 \; = \; H(div,\Omega) \; , \; W_2 \; = \; (L^2(\Omega))^2 \; , \; M_1 \; = \; L^2(\Omega) \; , \; M_2 \; = \; H_0^1(\Omega).$$

$||.||_{W_i}$ and $||.||_{M_i}$ $(i = 1,2)$ will denote the natural norm of these Hilbert spaces. Let us now assume that W_{1h}, W_{2h}, M_{1h} *and* M_{2h} are finite dimensional spaces.

$$W_{1h} \; \subset H(div,\Omega), \; W_{2h} \; \subset (L^2(\Omega))^2, \; M_{1h} \; \subset L^2(\Omega), \; M_{2h} \; \subset H_0^1(\Omega).$$

We study the finite dimensional problem:

$$(4) \quad \begin{cases} find \quad p_h \in W_{1h}, \; u_h \in M_{2h} & solution \quad of \\ \forall q_h \in W_{2h}, & a(p_h,q_h) \; + \; b_2(q_h,u_h) \; = 0 \\ \forall v_h \in M_{1h}, & b_1(p_h,v_h) = \; - \int_\Omega f v_h \; dx \; . \end{cases}$$

Let be

$$V_{1h} \; = \; \{ \; p_h \; \in W_{1h}; \; \forall v_h \; \in M_{1h}, \; b_1(p_h,v_h) \; = \; 0 \; \}$$
$$V_{2h} \; = \; \{ \; q_h \; \in W_{2h}; \; \forall u_h \; \in M_{2h}, \; b_2(q_h,u_h) \; = \; 0 \; \}$$

The next theorem (cf Nicolaïdes [3]) guarantees the existence and the unicity of the solution of (4).

Theorem 1 : *Assume that the next three Babuska-Brezzi conditions are verified:*

$$\inf_{v_h \in M_{1h}} \; \sup_{p_h \in W_{1h}} \frac{b_1(p_h,v_h)}{||p_h||_{W_1} ||v_h||_{M_1}} \; \geq \; \beta_1 \; > \; 0$$

$$\inf_{u_h \in M_{2h}} \; \sup_{q_h \in W_{2h}} \frac{b_2(q_h,u_h)}{||q_h||_{W_2} ||u_h||_{M_2}} \; \geq \; \beta_2 \; > \; 0$$

$$\inf_{p_h \in V_{1h}} \; \sup_{q_h \in V_{2h}} \frac{a(p_h,q_h)}{||p_h||_{W_1} ||q_h||_{W_2}} \; \geq \; \alpha \; > \; 0$$

and moreover, $dim(W_{1h}) \; + \; dim(M_{2h}) \; = \; dim(W_{2h}) \; + \; dim(M_{1h})$. *Then the problem (4) has a unique solution* (p_h,u_h).

We now add to this theorem a result of error estimates for approximate solution.

Theorem 2 : *Let (p,u) be the solution of (3) and (p_h,u_h) be the solution of (4), under assumptions of Theorem 1 with α, β_1 and β_2 independent of h, we have*

$$||p-p_h||_{W_1} + ||u-u_h||_{M_2} \leq C\{ \inf_{r_h \in W_{1h}} ||p-r_h||_{W_1} + \inf_{w_h \in M_{2h}} ||u-w_h||_{M_2} \}$$

where C is a positive constant independent of h.

3. Example in a rectangular case

In this section, we would like to present an example of discretization spaces which satisfy the conditions of Theorem (1) and Theorem (2). We define

$$W_{1h} = \{ p_h \in H(div,\Omega),\ \forall K\ (centered\ on\ i,j)\ p_h/_K \in RT_1 \}\ \ (see\ figure\ 1)$$

where RT_1 is the space of the vectorial functions of the form $\begin{pmatrix} a_0 + a_1 x \\ b_0 + b_1 y \end{pmatrix}$

Let $E(L)$ be the space of the vectorial functions defined on L constant and proportional to e. (see figure 2)

$$W_{2h} = \{ q_h \in (L^2(\Omega))^2,\ \forall L\ (centered\ on\ i+\frac{1}{2},j\ or\ on\ i,j+\frac{1}{2})\ q_h/_L \in E(L) \}$$

$$M_{1h} = \{ v_h \in L^2(\Omega);\ \forall K\ centered\ on\ (i,j)\ v_h/_K \in P_0 \}\ \ (see\ figure\ 1)$$

$$M_{2h} = \{ u_h \in H_0^1(\Omega);\ \forall T\ u_h/_T \in P_1 \}\ \ (see\ figure\ 3)$$

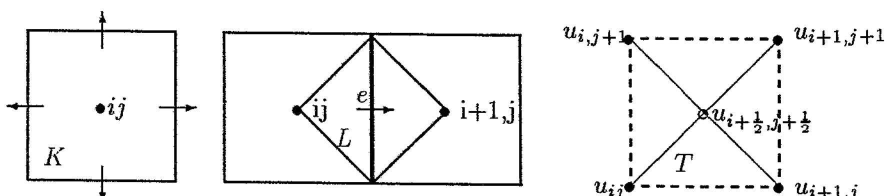


fig. 1 fig. 2 fig. 3

$$u_{i+\frac{1}{2},j+\frac{1}{2}} \stackrel{def}{=} \frac{1}{4}(u_{i,j} + u_{i+1,j} + u_{i+1,j+1} + u_{i,j+1})$$

We remark that $dim(W_{1h}) = dim(W_{2h})$ and $dim(M_{1h}) = dim(M_{2h})$. With these choices of spaces, the assumptions of Theorem (1) and Theorem (2) hold.

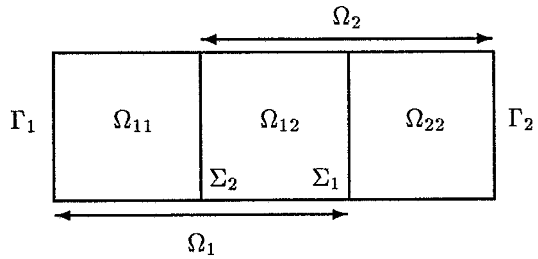
4. Application to domain decomposition methods

We will see here, how we can adapt domain decomposition methods to this new mixed formulation.

In the sequel, for the sake of simplicity, we shall assume that Ω is a rectangle.

4.1. Overlapping methods. Let Ω be an open rectangular domain of $\mathbf{R}^2$, whose boundary will be denoted by $\partial\Omega$, and partitionned into two subdomains Ω_1 and Ω_2 which satisfy:

$$\Omega_1 \cap \Omega_2 = \Omega_{12} \neq \emptyset \quad , \quad \bar{\Omega}_1 \cup \bar{\Omega}_2 = \bar{\Omega}$$



We note

$\Omega_{11} = \Omega/\Omega_2$, $\Omega_{22} = \Omega/\Omega_1$,
$\Sigma_1 = \partial\Omega_1 \cap \partial\Omega_{22}$,
$\Sigma_2 = \partial\Omega_2 \cap \partial\Omega_{11}$,
$\Gamma_1 = \partial\Omega_1 \cap \partial\Omega$,
$\Gamma_2 = \partial\Omega_2 \cap \partial\Omega$.

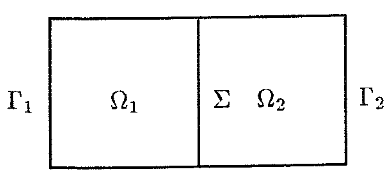
Let $W_{1h}(\Omega_i)$, $W_{2h}(\Omega_i)$, $M_{1h}(\Omega_i)$ and $M_{2h}(\Omega_i)$ be the spaces introduced in Section 3 but defined locally on Ω_i (i=1,2).
We solve the problem on Ω by a Schwarz type algorithm which can here be written as:

$$(\mathcal{P}_h^{2n+1}) \begin{cases} (p_h^{2n+1}, u_h^{2n+1}) \in W_{1h}(\Omega_1) \times M_{2h}(\Omega_1) & \textit{solution of} \\ \int_{\Omega_1} \frac{1}{K} p_h^{2n+1}.q_h \, dx - \int_{\Omega_1} grad(u_h^{2n+1}).q_h \, dx = 0 & \forall \ q_h \in W_{2h}(\Omega_1) \\ \int_{\Omega_1} v_h div p_h^{2n+1} \, dx = -\int_{\Omega_1} f v_h \, dx & \forall \ v_h \in M_{1h}(\Omega_1) \\ p_h^{2n+1}.\nu = p_h^{2n}.\nu & on \ \Sigma_1 \end{cases}$$

$$(\mathcal{P}_h^{2n+2}) \begin{cases} (p_h^{2n+2}, u_h^{2n+2}) \in W_{1h}(\Omega_1) \times M_{2h}(\Omega_2) & \textit{solution of} \\ \int_{\Omega_2} \frac{1}{K} p_h^{2n+2}.q_h \, dx - \int_{\Omega_2} grad(u_h^{2n+2}).q_h \, dx = 0 & \forall \ q_h \in W_{2h}(\Omega_2) \\ \int_{\Omega_2} v_h div p_h^{2n+2} \, dx = -\int_{\Omega_2} f v_h \, dx & \forall \ v_h \in M_{1h}(\Omega_2) \\ p_h^{2n+2}.\nu = p_h^{2n+1}.\nu & on \ \Sigma_2 \end{cases}$$

Theorem 3 : *Let (p_h, u_h) be the solution of the discrete problem (4). The sequences (p_h^{2n})$_n$ and (p_h^{2n+1})$_n$ converge respectively to $p_h/_{\Omega_2}$ and $p_h/_{\Omega_1}$ in $H(div,\Omega_2)$ and $H(div,\Omega_1)$. The sequences $(u_h^{2n})_n$ and $(u_h^{2n+1})_n$ converge respectively to $u_h/_{\Omega_2}$ and $u_h/_{\Omega_1}$ in $H^1(\Omega_2)$ and $H^1(\Omega_1)$. Moreover the convergences are geometric.*

4.2. Nonoverlapping method. Let Ω be an open rectangular domain of $\mathbf{R}^2$ partitionned into two subdomains Ω_1 and Ω_2 which satisfy:



$\Omega_1 \cap \Omega_2 = \emptyset$ *and* $\bar{\Omega}_1 \cup \bar{\Omega}_2 = \bar{\Omega}$.
We note $\Sigma = \partial\Omega_1 \cap \partial\Omega_2$,
$\Gamma_1 = \partial\Omega_1 \cap \partial\Omega$, $\Gamma_2 = \partial\Omega_2 \cap \partial\Omega$.

Let $W_{1h}(\Omega_i)$, $W_{2h}(\Omega_i)$, $M_{1h}(\Omega_i)$ and $M_{2h}(\Omega_i)$ be the spaces introduced in Sec-

tion 3 but defined locally on Ω_i (i=1,2). Let $\Phi_{1h} = \{v_h/_\Sigma \ ; \ v_h \in M_{2h}(\Omega_1)\}$. We use here an iterative algorithm with interface relaxation (cf Marini and Quarteroni [2]). The procedure can be written as:
Let $\lambda_{1h}^0 \in \Phi_{1h}$,

$$(\mathcal{P}_{1h}^{2n+1}) \left\{ \begin{array}{ll} (p_{1h}^{2n+1}, u_{1h}^{2n+1}) \in W_{1h}(\Omega_1) \times M_{2h}(\Omega_1) & solution \quad of \\ \int_{\Omega_1} \frac{1}{K} p_{1h}^{2n+1}.q_1 \, dx \ - \ \int_{\Omega_1} gradu_{1h}^{2n+1}.q_1 \, dx \ = \ 0 & \forall \ q_1 \in W_{2h}(\Omega_1) \\ \int_{\Omega_1} v_1 divp_{1h}^{2n+1} \, dx \ = \ -\int_{\Omega_1} f_1 v_1 \, dx & \forall \ v_1 \in M_{1h}(\Omega_1) \\ u_{1h}^{2n+1} \ = \ \lambda_{1h}^{2n} & on \quad \Sigma \end{array} \right.$$

$$(\mathcal{P}_{2h}^{2n+2}) \left\{ \begin{array}{ll} (p_{2h}^{2n+2}, u_{2h}^{2n+2}) \in W_{1h}(\Omega_2) \times M_{2h}(\Omega_2) & solution \quad of \\ \int_{\Omega_2} \frac{1}{K} p_{2h}^{2n+2}.q_2 \, dx \ - \ \int_{\Omega_2} gradu_{2h}^{2n+2}.q_2 \, dx \ = \ 0 & \forall \ q_2 \in W_{2h}(\Omega_2) \\ \int_{\Omega_2} v_2 divp_{2h}^{2n+2} \, dx \ = \ -\int_{\Omega_2} f_2 v_2 \, dx & \forall \ v_2 \in M_{1h}(\Omega_2) \\ (p_{2h}^{2n+2}.\nu) \ = \ (p_{1h}^{2n+1}.\nu) & on \quad \Sigma \end{array} \right.$$

$$\lambda_{1h}^{2n+2} \ = \ \theta(u_{2h/\Sigma}^{2n+2}) \ + \ (1-\theta)\lambda_{1h}^{2n}$$

We conclude this paper with the following convergence theorem.

Theorem 4 : *There exists a positive constant $\theta^\star$ such that, for all θ in $]0, \theta^\star[$, the sequence $(u_{1h}^{2n+1}, u_{2h}^{2n+2}, p_{1h}^{2n+1}, p_{2h}^{2n+2})_n$ converges geometrically in $H^1(\Omega_1) \times H^1(\Omega_2) \times H(div, \Omega_1) \times H(div, \Omega_2)$ to the solution of the discrete problem on Ω.*

Numerical results are presented in Trujillo [5].

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