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Some Two-grid Finite Element Methods

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ABSTRACT. This paper is concerned with a class of finite element discretization techniques based on two finite element subspaces, one on a coarse grid and one on a fine grid. On the fine space, only one symmetric positive definite equation needs to be solved for a nonsymmetric or indefinite linear equation, and only one linear equation needs to be solved for a nonlinear equation. It is shown that the coarse grid can be extremely coarse to still achieve the optimal approximation on the fine space for these algorithms. A special nonlinear Galerkin method based on two-grid finite elements is also discussed for time-dependent problem.

1. Introduction. A class of finite element discretization techniques based on two different grids has been developed by the author recently. This type of algorithms was explored in [4, 7] for solving nonsymmetric and indefinite linear finite element algebraic systems and the idea was applied in [6] on the discretization level for nonsymmetric and indefinite linear problems and especially for nonlinear problems. In [8], an innovative technique was devised based on a further coarse grid correction to improve the performance of the algorithm without introducing much extra work. A new *nonlinear Galerkin method* was proposed in a recent work [2] by Marion and Xu based on two-grid finite element discretizations. This paper is to give a brief summary on these algorithms and some remarks on their applications with multigrid and domain decomposition methods.

Given a bounded domain $\Omega \subset \mathbb{R}^d$ $(1 \leq d \leq 3)$. We assume that Ω is convex with piecewise smooth boundary and that $\mathcal{W}_p^m$ is the standard Sobolev space defined on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p = \sum_{|\alpha|\leq m} \|D^\alpha v\|_{\mathcal{L}^p(\Omega)}^p$. For $p=2$, we denote $\mathcal{H}^m = \mathcal{W}_2^m$ and $\mathcal{H}_0^1 = \{v \in \mathcal{H}^1 : v|_{\partial\Omega} = 0\}$, where $v|_{\partial\Omega} = 0$ is in the sense of *trace*, $\|\cdot\|_m = \|\cdot\|_{m,2}$ and $\|\cdot\| = \|\cdot\|_{0,2}$.

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Throughout this paper, we shall use the letters C and c (with or without subscripts) to denote a generic positive constant which may stand for different values at its different appearances. When it is not important to keep track of these constants, we shall conceal the letter C or c into the notation $\lesssim$ or $\gtrsim$. Here $x \lesssim y$ means $x \le Cy$ and $x \gtrsim y$ means $x \ge cy$.

We assume that Ω is partitioned by two quasi-uniform triangulations T_H and T_h with two different mesh sizes H and h $(H > h)$. We assume that V_H and V_h are two subspaces of $\mathcal{H}_0^1$ that consist of piecewise linear functions with respect to T_H and T_h respectively. We shall call V_H to be a coarse space and V_h a fine space.

Assume that $u_h \in V_h$ is the standard finite element approximation to the exact solution of certain second order partial differential equation. In general, the best possible error estimate is

$$\|u - u_h\|_1 \lesssim h.$$

A two-grid finite element method is to produce an approximation $u^h \in V_h$ based on both V_H and V_h so that the following type of error estimates hold

$$\|u_h - u^h\|_1 \lesssim H^m$$

where $m \ge 2$. As a result, it suffices to take $H = O(h^{1/m})$ to achieve the optimal approximation. The point is that more difficult equations are solved only on V_H and easier equations are solved on V_h. As $\dim V_H \ll \dim V_h$, the efficiency of these algorithms is then evident.

2. Two-grid methods for linear problems. In this section, we shall discuss two-grid finite element discretizations for nonsymmetric and/or indefinite linear partial differential equations. Our motivation is based on the fact that a symmetric positive definite (SPD in short) system is in general much easier to solve (e.g. conjugate gradient like methods can be applied effectively) than a non-SPD system. The detailed analysis of these algorithms can be found in [6].

Let α, β, γ (with the ranges in $\mathbb{R}^{2\times 2}, \mathbb{R}^2$ and $\mathbb{R}^2$ respectively) be smooth functions on $\overline{\Omega}$ satisfying (for some positive constant α_0)

$$\xi^T \alpha(x)\xi \ge \alpha_0|\xi|^2 \quad \forall\ x \in \Omega, \xi \in \mathbb{R}^2.$$

Consider the following boundary value problem

$$\widehat{\mathcal{L}}\, v \equiv -\mathrm{div}(\alpha(x)\nabla v) + \beta(x)\cdot\nabla v + \gamma(x)v = f(x), \quad x \in \Omega, \quad u|_{\partial\Omega} = 0.$$

Our basic assumption is that $\widehat{\mathcal{L}}$ is nonsingular (a simple sufficient condition is $\gamma(x) \ge 0$.)

We define two bilinear forms, for $u, v \in \mathcal{H}_0^1$, as follows

$$A(u,v) = \int_\Omega \alpha(x)\nabla u\cdot\nabla v\, dx, \quad \widehat{A}(u,v) = A(u,v) + \int_\Omega ((\beta(x)\cdot\nabla u)v + \gamma(x)uv)\, dx.$$

The standard finite element approximation of (2.1) is to find $u_h \in V_h$ so that

$$\hat{A}(u_h, \chi) = (f, \chi) \quad \forall\ \chi \in V_h.$$

It is well-know that the above equation is uniquely solvable if h is sufficiently small and

$$\|u - u_h\| + h\|u - u_h\|_1 \lesssim h^2\|u\|_2.$$

The idea in the algorithms presented below is to reduce a non-SPD problem into a SPD problem by solving a non-SPD problem on a much smaller space.

Denoting the bilinear form of the lower order terms of the operator $\hat{\mathcal{L}}$ by

$$N(v, \chi) = (\hat{A} - A)(v, \chi) = (\beta \cdot \nabla v, \chi) + (\gamma v, \chi),$$

our first two-grid algorithm is

Algorithm 2.1.
1. $u_H \in V_H,\quad \widehat{A}(u_H, \varphi) = (f, \varphi) \quad \forall\ \varphi \in V_H$;
2. $u^h \in V_h,\quad A(u^h, \chi) + N(u_H, \chi) = (f, \chi) \quad \forall\ \chi \in V_h$.

We note that the linear system in step 2 of the above algorithm is SPD. The error estimates for the above algorithm are

$$\|u_h - u^h\|_1 \lesssim H^2\|u\|_2.$$

Algorithm 2.1 can be applied in a successive fashion.

Algorithm 2.2. *Let* $u_h^0 = 0$*; assume that* $u_h^k \in V_h$ *has been obtained,* $u_h^{k+1} \in V_h$ *is defined as follows*
1. $e_H \in V_H,\quad \widehat{A}(e_H + u_h^k, \varphi) = (f, \varphi) \quad \forall\ \varphi \in V_H$;
2. $u^h \in V_h,\quad A(u_h^{k+1}, \chi) + N(u_h^k + e_H, \chi) = (f, \chi) \quad \forall\ \chi \in V_h$.

The error estimate of the above algorithm is

$$\|u_h - u_h^k\|_1 \lesssim H^{k+1}\|u\|_2, \quad k \geq 1.$$

The SPD system in the second step of Algorithm 2.2 need not be solved exactly. The resultant algorithms correspond to those in [4,7].

3. Nonlinear problems. In this section, we shall present some two-grid methods for nonlinear equations. The detailed analysis of these algorithms can be found in [6].

3.1 Model problem and finite element discretization. We consider the following second order quasi-linear elliptic problem:

$$\mathcal{L}(u) \equiv -\mathrm{div}(F(x, u, \nabla u)) + g(x, u, \nabla u) = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0. \tag{3.1}$$

We assume that $F(x, y, z) : \overline{\Omega}\times\mathbb{R}^1\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $g(x, y, z) : \overline{\Omega}\times\mathbb{R}^1\times\mathbb{R}^2 \longrightarrow \mathbb{R}^1$ are smooth functions and that (3.1) has a solution $u \in \mathcal{H}_0^1 \cap \mathcal{H}^{2+\varepsilon}$ (for some $\varepsilon > 0$).

For any $w \in \mathcal{W}^1_\infty$, we denote

$$a(w) = D_z F(x, w, \nabla w) \in \mathbb{R}^{2\times 2}, \quad b(w) = D_y F(x, w, \nabla w) \in \mathbb{R}^2,$$
$$c(w) = D_z g(x, w, \nabla w) \in \mathbb{R}^2, \quad d(w) = D_y g(x, w, \nabla w) \in \mathbb{R}^1.$$

The linearized operator $\mathcal{L}$ at w (namely the Fréchet derivative of $\mathcal{L}$ at w) is then given by

$$\mathcal{L}_w v = -\mathrm{div}(a(w)\nabla v + b(w)v) + c(w)\nabla v + d(w)v.$$

Our basic assumptions are, first of all, $\mathcal{L}_u$ is nonsingular and for the solution u of (3.1)

$$\xi^T a(u)\xi \geq \alpha_0|\xi|^2 \quad \forall\ \xi \in \mathbb{R}^2, \quad x \in \overline{\Omega}.$$

For convenience of exposition, we introduce two parameters δ_1 and δ_2 as follows.

$$\delta_2 = \begin{cases} 0 & \text{if } D_z^2 F(x,y,z) \equiv 0, D_z^2 g(x,y,z) \equiv 0, \\ 1 & \text{otherwise;} \end{cases}$$

and

$$\delta_1 = \begin{cases} 0 & \text{if } \delta_2 = 0,\ D_y D_z F(x,y,z) \equiv 0,\ D_y D_z g(x,y,z) \equiv 0, \\ 1 & \text{otherwise.} \end{cases}$$

If $\delta_2 = 0$ and $\delta_1 = 1$, (3.1) is mildly nonlinear for which

$$\mathcal{L}(u) = -\mathrm{div}(\alpha(x,u)\nabla u + \beta(x,u)) + \gamma(x,u)\cdot\nabla u + g(x,u);$$

if $\delta_1 = \delta_2 = 0$, (3.1) is semilinear for which

$$\mathcal{L}(u) = -\mathrm{div}(\alpha(x)\nabla u + \beta(x,u)) + g(x,u). \tag{3.2}$$

The classic finite element approximation of (3.1) is to find $u_h \in V_h$ such that

$$A(u_h, \chi) = 0 \quad \forall\ \chi \in V_h, \tag{3.3}$$

where

$$A(v, \varphi) = (F(\cdot, v, \nabla v), \nabla\varphi) + (g(\cdot, v, \nabla v), \varphi).$$

It can be proven that (cf. [6] and the references cited therein) that, for sufficiently small h, the equation (3.3) has a (locally unique) solution u_h satisfying

$$\|u - u_h\|_{0,p} + h\|u - u_h\|_{1,p} \lesssim h^2 \quad \text{if} \quad 2 \leq p < \infty, u \in \mathcal{W}^2_p,$$
$$\|u - u_h\|_{0,\infty} \lesssim h^2|\log h| \quad \text{and} \quad \|u - u_h\|_{1,\infty} \lesssim h \quad \text{if} \quad u \in \mathcal{W}^2_\infty.$$

The above estimates are important in the analysis for the algorithms described in this section (cf. [6]).

3.2. Simple two-grid methods. The technique presented here is similar to that for algorithms for non-SPD linear problems in section 2. We first define, for $u, v, \chi \in \mathcal{W}^1_\infty$

$$\tilde{A}(u; v, \chi) = (\alpha(\cdot, u)\nabla v + \beta(\cdot, u), \nabla\chi) + (\gamma(\cdot, u)\cdot\nabla v + g(\cdot, u), \chi).$$

Our first algorithm is a nonlinear extension of Algorithm 2.1.

Algorithm 3.1.

1. $u_H \in V_H, \quad A(u_H, \varphi) = 0 \quad \forall\ \varphi \in V_H;$

2. $u^h \in V_h, \quad \tilde{A}(u_H; u^h, \chi) = 0 \quad \forall\ \chi \in V_h.$

The error estimates of the above algorithm are

$$\|u_h - u^h\|_1 \lesssim H^2 \quad \text{if} \quad u \in \mathcal{H}^2,$$
$$\|u_h - u^h\|_{1,\infty} \lesssim H^2 |\log h| \quad \text{if} \quad u \in \mathcal{W}^2_\infty.$$

The following algorithm is obtained by combining Algorithm 2.1 with Algorithm 3.1 and it reduces a nonlinear problem to a SPD linear problem and a nonlinear system of smaller size.

Define

$$A_s(u; v, \chi) = (\alpha(\cdot, u)\nabla v, \nabla \chi),$$
$$N(u; v, \chi) = (\beta(\cdot, u), \nabla \chi) + (\gamma(\cdot, u) \cdot \nabla v + g(\cdot, u), \chi).$$

Algorithm 3.2.

1. $u_H \in V_H, \quad A(u_H, \varphi) = 0 \quad \forall\ \varphi \in V_H;$

2. $u^h \in V_h, \quad A_s(u_H; u^h, \chi) + N(u_H; u_H, \chi) = 0 \quad \forall\ \chi \in V_h.$

Note that the system in step 2 of the above algorithm is SPD. The error estimates for the above algorithm are

$$\|u_h - u^h\|_1 \lesssim H^2 \quad \text{if} \quad u \in \mathcal{H}^2,$$
$$\|u_h - u^h\|_{1,\infty} \lesssim H^2 |\log h| \quad \text{if} \quad u \in \mathcal{W}^2_\infty.$$

3.3 Correction by one Newton's iteration on the fine space. The essence of these methods is like the well-known Newton's method for nonlinear systems.

Algorithm 3.3.

1. $u_H \in V_H, \quad A(u_H, \varphi) = 0 \quad \forall\ \varphi \in V_H;$

2. $u^h \in V_h, \quad A_{u_H}(u^h, \chi) = A_{u_H}(u_H, \chi) - A(u_H, \chi) \quad \forall\ \chi \in V_h.$

Here

$$A_w(v, \varphi) = \langle \mathcal{L}_w v, \phi \rangle = (a(w)\nabla v + b(w)v, \nabla \varphi) + (c(w)\nabla v + d(w)v, \varphi).$$

The error estimates for the above algorithm are

$$\|u_h - u^h\|_1 \lesssim H^4 + \delta_1 H^3 + \delta_2 H^2 \quad \text{if} \quad u \in \mathcal{W}^2_4,$$
$$\|u_h - u^h\|_{1,\infty} \lesssim (H^4 + \delta_1 H^3 + \delta_2 H^2)|\log h| \quad \text{if} \quad u \in \mathcal{W}^2_\infty.$$

(The above second estimate for $\delta_2 \neq 0$ may also be deduced from Rannacher [3]).

If the Algorithm 3.3 is applied to the semi-linear equation (3.2), then

(3.4)
$$\|u_h - u^h\|_1 \lesssim H^4 \quad \text{if} \quad u \in \mathcal{W}^2_4, \quad \|u_h - u^h\|_{1,\infty} \lesssim H^4 |\log h| \quad \text{if} \quad u \in \mathcal{W}^2_\infty.$$

3.4. Correction by two Newton's iterations on the fine space. The algorithms presented above can be greatly improved if one further Newton's iteration is carried out on V_h.

Corresponding to Algorithm 3.2, we have

Algorithm 3.4.
1. $u_H \in V_H$, $\quad A(u_H, \varphi) = 0 \quad \forall\ \varphi \in V_H$;
2. $u^h \in V_h$, $\quad \tilde{A}(u_H; u^h, \chi) = 0 \quad \forall\ \chi \in V_h$;
3. $u_h^* \in V_h$, $\quad A_{u^h}(u_h^*, \chi) = A_{u^h}(u^h, \chi) - A(u^h, \chi) \quad \forall\ \chi \in V_h$.

Corresponding to Algorithm 3.3, we have

Algorithm 3.5.
1. $u_H \in V_H$, $\quad A(u_H, \varphi) = 0 \quad \forall\ \varphi \in V_H$;
2. $u^h \in V_h$, $\quad A_{u_H}(u^h, \chi) = A_{u_H}(u_H, \chi) - A(u_H, \chi) \quad \forall\ \chi \in V_h$;
3. $u_h^* \in V_h$, $\quad A_{u^h}(u_h^*, \chi) = A_{u^h}(u^h, \chi) - A(u^h, \chi) \quad \forall\ \chi \in V_h$.

For Algorithm 3.4,
$$\|u_h - u_h^*\|_{1,\infty} \lesssim H^4 |\log h|^2,$$
and for Algorithm 3.5
$$\|u_h - u_h^*\|_{1,\infty} \lesssim (H^8 + \delta_1 H^6 + \delta_2 H^4) |\log h|^2.$$

Again we notice that if Algorithm 3.5 is applied to the semi-linear equation (3.2), then
$$\|u_h - u^h\|_{1,\infty} \lesssim H^8 |\log h|^2, \quad \text{if} \quad u \in \mathcal{W}_\infty^2.$$

4. Further coarse grid correction. The section is to discuss a technique devised in [8] that makes a further refinement in the aforementioned process by solving one more linear equation on the *coarse* space. This additional correction step (which needs very little extra work) improves the accuracy of the algorithms in [6] up to one or two orders. The fact that a further coarse grid correction *after* the fine grid correction can actually improve the accuracy appears to be of great interests.

For clarity, we shall consider a simple semilinear equation
$$-\Delta u + f(x, u) = 0, \quad x \in \Omega, \qquad u|_{\partial\Omega} = 0.$$
Here the function f is sufficiently smooth. For simplicity, we shall drop the dependence of variable x in $f(x, u)$ in the following exposition. We assume that the above equation has at least one solution $u \in \mathcal{H}_0^1 \cap \mathcal{H}^2$ and the linearized operator $L_u \equiv -\Delta + f_u(u)$ is nonsingular.

The main algorithm of this section is

Algorithm 4.1. *Find* $u_h^* = u_H + e_h + e_H$ *such that*
1. $u_H \in V_H$, $\quad (\nabla u_H, \nabla \phi) + (f(u_H), \phi) = 0 \quad \forall\ \phi \in V_H$;
2. $e_h \in V_h$, $\quad a_{u_H}(e_h, \chi) = -(f(u_H), \chi) - (\nabla u_H, \nabla \chi) \quad \forall\ \chi \in V_h$;
3. $e_H \in V_H$, $\quad a_{u_H}(e_H, \phi) = -\frac{1}{2}(f_{uu}(u_H) e_h^2, \phi) \quad \forall\ \phi \in V_H$.

The new feature of the above algorithm lies in step 3 where a further coarse grid correction is performed. We notice that the linearized operator used in this step is based on the first coarse grid approximation u_H (instead of the more accurate $u_H + e_h$). Such a correction indeed improves the accuary of the approximation. In fact, if $u \in \mathcal{W}_4^2$, then

$$\|u_h - u_h^*\|_1 \lesssim H^5, \qquad \|u_h - u_h^*\| \lesssim H^6.$$

Compared with the estimate (3.4), we notice that a further coarse grid correction gives rise one order improvement for the $\mathcal{H}^1$ error (and two order improvement for $\mathcal{L}^2$ error).

For most practical purposes, the Algorithm 4.1 which involves only one Newton's iteration on the fine grid may be sufficient for applications. Nevertheless more dramatic result can be derived if one more Newton's iteration is performed on the fine grid.

Algorithm 4.2. *Find* $\tilde{u}_h = u_h^* + e^h$ *such that*
1. $u_h^* \in V_h$ *is obtained by Algorithm 4.1;*
2. $e^h \in V_h, \quad a_{u_h^*}(e^h, \chi) = -(f(u_h^*), \chi) - (\nabla u_h^*, \nabla \chi) \quad \forall\ \chi \in V_h.$

If $u \in W_4^2$, the error estimate for the above algorithm is

$$\|u_h - (u_h^* + e^h)\|_1 \lesssim H^{12} |\log h|^{\frac{1}{2}}.$$

It is also possible to make an additional coarse grid correction in Algorithm 4.2 to further improve the accuracy, but such an improvement may not be that important since the order of H is already so high. We would also like to remark that the technique presented in this section may be extended to the general equation (3.1).

5. Nonlinear Galerkin methods for evolution problems. In this section, we shall discuss two-grid methods for time dependent problems. In principal, most of the aforementioned algorithms for stationary problems can be carried over to time dependent problems. Instead of discussing such extensions shall now discuss another type of two-grid method — the nonlinear Galerkin method.

The nonlinear Galerkin method is a special class of numerical algorithms being developed for solving nonlinear time-dependent partial differential equation motivated by inertial manifold theory (see Marion and Temam [1] and the references cited therein). In a recent paper by Marion and Xu [2], we have proposed a new nonlinear Galerkin method based on two-grid finite element discretizations. We shall now briefly describe such a special two-grid method.

We consider the following model problem:

$$u_t - \Delta u + f(x, u) = 0$$

with the initial condition $u(x, 0) = u_0(x)$ and boundary condition $u|_{\partial\Omega} = 0$.

An example of the standard Galerkin method for the above equation is to find $u_h \in V_h$ such that

$$(u_{h,t}, \chi) + (\nabla u_h, \nabla \chi) + (f(u_h), \phi) = 0 \quad \forall\ \chi \in V_h, \quad \text{and} \quad u_h(0) = Q_h u_0,$$

where $Q_h : \mathcal{H}_0^1 \mapsto V_h$ is the ordinary $\mathcal{L}^2$ projection.

Based on an intermediate space $V_h^H = (I - Q_H)V_h$, the two-grid nonlinear Galerkin method proposed in [2] is as follows

Algorithm 5.1. *Find $u^h = u^H + e_h \in V_h$, where $u^H \in V_H$ and $e_h \in V_h^H$ satisfy the following coupled equations:*

$$(u_t^H, \phi) + (\nabla(u^H + e_h), \nabla\phi) + (f(u^H + e_h), \phi) = 0 \quad \forall \ \phi \in V_H,$$
$$(\nabla(u^H + e_h), \nabla\chi) + (f(u^H), \chi) = 0 \quad \forall \ \chi \in V_h^H,$$

with initial condition $u^H(\cdot, t_0) = (Q_H u_h)(\cdot, t_0)$ for some $t_0 > 0$.

The above systems may be decoupled by choosing appropriate time discretizations. We note that on the fine grid, we need to solve a fixed stationary problem and the nonlinearity and time-dependence are all treated on the coarse gird. A rather attractive feature is that the stability of an explicit scheme for this coupled system appear to only depend on the coarse mesh size in the usual fashion (e.g. $\Delta t \le cH^2$ for explicit Euler method).

The error estimate for the above nonlinear Galerkin method is ([2])

$$\|u_h - u^h\|_1 \lesssim H^3, \quad t \ge t_0. \tag{5.1}$$

As no Newton's type linearization is used in our algorithm, the above error estimate is quite remarkable.

Algorithm 5.1 is the first nonlinear Galerkin method based on finite element discretization that admits error estimates like (5.1).

6. Applications with multigrid and domain decomposition methods. These methods are among the most efficient algorithms for solving linear algebraic systems and they can be naturally applied to the linear systems on the fine space appearing in the algorithms discussed in this paper.

Most domain decomposition methods are, in certain sense, two-grid methods. The set of subdomains in a domain decomposition gives rise to a natural coarse grid and in fact it is well-known that certain solvers on such a coarse grid are necessary to avoid the deterioration of the efficiency as the number of subdomains increases. Hence the domain decomposition techniques fit perfectly well with our algorithms and the coarse grid plays two different important roles in such an application. Similar arguments also apply to multigrid methods. Suppose we have a multiple subspaces $V_0 \subset V_1 \subset \cdots \subset V_j \subset \mathcal{H}_0^1$. Naturally we can choose $V_H = V_0$ (and $V_h = V_j$, of course) in our two-grid algorithms.

Applications of multigrid and domain decomposition methods with our two-grid methods for nonlinear problems are satisfactory from both theoretical and practical point of views, since the systems on the fine grid are all linear and hence theories and numerical codes for linear problems can be adopted with little modification.

The linear systems on the fine space in Algorithms 2.1 and 2.2 are SPD and their solution methods are well developed, we refer to [5] for a summary of these methods. The linear systems from the fine space on the algorithms in section 3 are mostly not SPD and may be solved by combining the algorithms in section 2. As a result, a nonlinear system on the fine space can be decomposed into few SPD linear systems on the fine space together with some linear and nonlinear systems on the coarse space.

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