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Preconditioning Via Asymptotically-defined Domain Decomposition

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ABSTRACT. Asymptotic analysis is used to derive preconditioners based on operator splitting and domain decomposition for the numerical solution of the advection-diffusion equation. Specifically, asymptotics is used to identify subdomains in which the solution is dominated by a certain operator, and this information is used to construct an effective preconditioner. We analyze the one-dimensional case in a function space setting and present numerical results for both one and two dimensions.

1. Introduction

In this paper, we construct and apply a preconditioning technique for the matrices arising in the numerical solution of differential equations. The preconditioning strategy is based on asymptotic analysis and uses the interaction of solution phenomena with the differential operator to create an approximate inverse that depends on both the operator and on the right hand side. This preconditioning differs from the usual approach that considers only the discrete or differential operator, and not how that operator interacts with the solution. Our preconditioning strategy is based on decomposing the computational domain in such a way that each subdomain

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isolates a fundamentally different type of physical behavior. We then associate a simple "partial" operator with each subdomain, the choice being made according to the dominant physics within that subdomain. To identify these partial operators, we use a uniform asymptotic expansion. The partial operators are then discretized and combined to obtain a preconditioner for use in an iterative method.

Our goal is to approximate the solution of a singular-perturbation problem governed by a differential operator where the singular behavior of the solution is governed by a small (but non-zero) parameter. Toward this end, we discretize the operator so that exact solution to the discrete problem has an error no larger than some specified tolerance. We then obtain an approximate solution to the discrete problem to within the same tolerance. We use asymptotic analysis to define a *preconditioner* for accelerating convergence of an iterative method that is applied to the original problem. We remark that the idea of using a uniform asymptotic expansion to precondition the problem was first proposed by Chin, Hedstrom, and Howes [9]. This idea has been applied to canonical problems in fluid flows [10, 12, 6] and isentropic gas dynamics [4].

2. Model Problem

Our target application is the numerical simulation of fluid flow. In certain high-speed regimes, shock layers or boundary layers can arise, and the flow is governed by the compressible viscous gas dynamics equation,

$$\nabla \cdot F(\mathbf{u}) - \nabla \cdot G(\nabla \times \mathbf{u}) = 0,$$

with appropriate boundary conditions. The components of the vector $\mathbf{u}$ represent density, velocity, and energy, F is the flux, and G is the stress. In this paper we consider problems in which the "ratio" of $\nabla \cdot F(\mathbf{u})$ to $\nabla \cdot G(\nabla \times \mathbf{u})$ changes throughout the computational domain. For our purposes, it is sufficient to study a canonical problem whose solution is smooth except for the presence of a boundary layer. We will develop the technique for the linear one-dimensional singularly-perturbed advection-diffusion equation, and then demonstrate the effectiveness of the preconditioner on examples in one and two dimensions. Specifically, we will consider

$$(L_c - \epsilon L_d)[u] = f \tag{1}$$

on the domain $\Omega = [0, 1]$, with homogeneous Dirichlet boundary conditions

$$u(0) = 0, \quad u(1) = 0. \tag{2}$$

Here the operators $L_c = \nabla$ and $L_d = \Delta$ represent the effects of advection and diffusion, respectively. We are interested in solutions to this problem when $1 \gg \epsilon > 0$ is small, which gives rise to a boundary layer. The forcing function $f(x)$ may depend on ϵ, but we assume that f and its derivatives of all orders are bounded independently of ϵ. The analysis of this simple problem will illustrate the fundamental ideas.

3. Quantitative Analysis

The solution to (1-2) may be decomposed into two components, $u = u_{null} + u_f$. One component, u_f, has derivatives bounded independent of ϵ for the entire domain, and the other component, u_{null}, has derivatives bounded independently of ϵ except for a region that we can locate. Component u_f satisfies the non-homogeneous equation

$$(L_c - \epsilon L_d)[u_f] = f \tag{3}$$

with boundary conditions that yield a regular perturbation problem. The singular nature of the solution is captured in u_{null}, which satisfies

$$(L_c - \epsilon L_d)[u_{null}] = 0.$$

Thus, u_{null} is in the null space of the operator, and it is chosen so that the sum $u_{null} + u_f$ satisfies (2). In general, u_{null} is nonzero. More specifically, let

$$u_f(x) = F(x) + \sum_{n=1}^{\infty} \epsilon^n (f^{(n-1)}(x) - f^{(n-1)}(0)) \tag{4}$$

where $F(x) = \int_0^x f(y)dy$. This is a regular asymptotic expansion for u_f that satisfies the differential Eqn. (3). For example, using expansion (4) for u_f, we find

$$u_{null}(x) = K_0 e^{\frac{x-1}{\epsilon}} + K_1 + O(\epsilon) \tag{5}$$

where the constants K_0 and K_1 are determined by applying the boundary conditions

$$K_0 e^{-\frac{1}{\epsilon}} + K_1 = 0 \quad \text{and} \quad K_0 + K_1 = -F(1).$$

4. Qualitative Analysis

In this section, the qualitative behavior of the solution is obtained by deriving a uniform asymptotic expansion. The expansion is not new [7], but is repeated here as motivation for our preconditioner.

Consider Eqn. (1), and let

$$u_{\text{as}}^{\text{outer}}(t, x, \epsilon) = U_0(t, x) + \epsilon U_1(t, x) + \epsilon^2 U_2(t, x) + \cdots \tag{6}$$

be the regular asymptotic expansion of u in the (outer) region of smooth behavior $\Omega_{outer} = [0, \tau]$, for some τ yet to be determined. Since ϵ is small, the dominant behavior is given by U_0, which satisfies

$$L_c[U] = 0. \tag{7}$$

In other words, the operator in (1) is advection dominated in this subdomain. Next let

$$u_{as}^{inner}(t,x,\epsilon) = \hat{u}_0(t,x) + \epsilon\hat{u}_1(t,x) + \epsilon^2\hat{u}_2(t,x) + \cdots \tag{8}$$

be the asymptotic expansion for the solution in the (inner) boundary layer region $\Omega_{inner} = (\tau, 1]$. Here the behavior is given by $\hat{u}_0$, which satisfies

$$(L_c - \epsilon L_d)[\hat{u}] = 0. \tag{9}$$

Thus, in the boundary layer region, the solution is neither advection nor diffusion dominated. By applying appropriate conditions at the interface between the subdomains (i.e., $U_0(\tau) = \hat{u}_0(\tau)$), the first term in the uniform asymptotic expansion for the solution to our model problem is

$$u(x,y) \approx u_0^{uniform} = \begin{cases} U_0(x), & x \in \Omega_{outer} \\ \hat{u}_0(x), & x \in \Omega_{inner} \end{cases} . \tag{10}$$

We discuss how to choose τ in §6.

5. Preconditioning

In this section, we describe how asymptotics can be used to define a preconditioner for our problem. The goal is to find an operator M such that the problem

$$M^{-1}(L_c - \epsilon L_d)[u] = M^{-1}f \tag{11}$$

is easier to solve than the original problem. It would be natural to choose the preconditioner

$$M = (A_{outer})\chi_{[0,\tau)} + (A_{inner})\chi_{[\tau,1]}. \tag{12}$$

based on the portions of the operator that are dominant in the various subdomains. Here, χ_I denotes the characteristic function of the interval I. Next, we define each of the operators on the right-hand-side of (12).

Preconditioner in the Outer Region. In the outer region, the action of L_c dominates the differential operator, hence we define $V = A_{outer}[v]$ to be the solution of

$$L_c[V] = v, \quad \text{subject to} \quad V(0) = 0.$$

This preconditioner is applied to the equation $(L_c - \epsilon L_d)[v] = f$ for $x \in \Omega_{outer}$ to obtain

$$v(x) - \epsilon v'(x) - (v(0) - \epsilon v'(0)) = F(x).$$

Preconditioner in the Boundary Layer Region. In the boundary layer, neither the action of L_c nor the action of ϵL_d dominates the differential operator, and so the full operator needs to be used in this (smaller) region. The boundary condition at $x = \tau$ is provided by the preconditioner A_{outer}. To summarize, the preconditioner for the boundary layer region $A_{inner}[v]$ is defined as the solution V of

$$(L_c - \epsilon L_d)[V] = v$$

$$\text{subject to} \quad V(\tau) = A_{outer}[v' - \epsilon v''](\tau) \quad \text{and} \quad V(1) = 0.$$

Note that A_{outer} must be applied before A_{inner}. We apply this preconditioner to both sides of the equation $(L_c - \epsilon L_d)[v] = f$ for $x \in \Omega_{inner}$ to obtain $v(x) - \mu(x) = \hat{F}(x)$, where $\hat{F} = A_{inner}[f]$, and μ is an element of the null space of $(L_c - \epsilon L_d)$ that satisfies the boundary conditions imposed by our preconditioner, namely, $v(1)+\mu(1) = 0$ and $v(\tau)+\mu(\tau) = A_{outer}[v'-\epsilon v''](\tau)$.

Preconditioned Problem. After applying the preconditioner on each subdomain, Eqn. (11) becomes

$$A_{outer}[u]\chi[0,\tau] + A_{inner}[u]\chi[\tau,1] = G$$

where $G(x) = [F(x)]\,\chi[0,\tau] + \left[\hat{F}(x)\right]\chi[\tau,1]$. Since the family of functions $J_1 e^{\frac{x-1}{\epsilon}} + J_2$ is in the null space of $(L_c - \epsilon L_d)$, the above equation may be written

$$\begin{aligned} &u(x) + \epsilon\Bigg([u'(0) - u'(x)]\,\chi[0,\tau] \\ (13) \qquad &+ \left[(u'(0) - u'(\tau)) \left(\frac{e^{\frac{x-1}{\epsilon}} - 1}{e^{\frac{\tau-1}{\epsilon}} - 1} \right) \right] \chi[\tau,1] \Bigg) = G(x). \end{aligned}$$

Using the homogeneous Dirichlet boundary conditions, we write our preconditioned problem more generally as: Find a constant α and a function $u(x)$ such that

$$[u(x) - \epsilon u'(x) + \epsilon\alpha]\,\chi[0,\tau] + [u(x) + \mu(x)]\,\chi[\tau,1] = G(x)$$

where u satisfies the boundary conditions $u(0) = 0$ and $u(1) = 0$. Notice that this is a first-order equation for u with two boundary conditions. The parameter α reflects the use of (the inverse of) a first-order operator to precondition a second-order problem.

6. Analysis of the Iterative Method

In this section, we analyze the convergence of the Richardson and GMRES iterative methods in a function space setting. This analysis makes clear the dependence of the L_2 convergence properties on bounds for high derivatives of the solution in a way that an analysis of a discretized problem could not. We emphasize that this analysis has not been extended to the discrete case, but we present supporting numerical evidence in the next two sections.

Let $A = L_c - \epsilon L_d$ and apply homogeneous Dirichlet boundary conditions on $[0,1]$. We will solve $Au = f$ via the GMRES method [11] using M as a left preconditioner. Let the functions $\{v_k(x)\}$ be the GMRES iterates. Denote the error by $e_k(x) = u(x) - v_k(x)$. In the L^2 norm we have [11, 8]

$$\|e_k\|_2 = \|P_{opt}(M^{-1}A)[e_0]\|_2, \tag{14}$$

where P_{opt} is a k^{th} degree *residual* polynomial, that is, $P_{opt}(0) = 1$. GMRES chooses the k^{th} degree polynomial P_{opt} that is optimum in the *residual error* norm. That is, P_{opt} is a residual polynomial that minimizes the L_2 norm of the residual error,

$$||r_k||_2 = ||P_{opt}(M^{-1}A)[r_0]||_2 \le ||P(M^{-1}A)[r_0]||_2 \tag{15}$$

over all k^{th} degree residual polynomials. To demonstrate the convergence of the GMRES iterates, we may choose any k^{th} degree residual polynomial, and any initial iterate that makes the left side of (15) easy to estimate. Toward that end, we choose the polynomial $P(z) = (1-z)^k$, which corresponds to a stationary Richardson iteration. In addition, we assume that the initial iterate is zero, that is, $v_0(x) = 0$. This implies that the initial error is $e_0 = u$, which allows us to exploit the smoothness properties of u. We emphasize that this choice of initial iterate is crucial to the analysis that follows. Of course, in practice, one may wish to use a different v_0, but one should ensure that it is sufficiently smooth.

To show that a specified tolerance δ in the L_2 norm is obtained when $k = k_\delta$, we show that τ can be chosen as a function of ϵ and δ so that

$$||r_k||_2 \le ||(I - M^{-1}A)^k M^{-1}A[e_0]||_2 \le \delta. \tag{16}$$

Let v be a general function satisfying homogeneous Dirichlet boundary conditions, then

$$\begin{aligned}(I - M^{-1}A)^k[v] = \epsilon^k \Bigg(&\left[v^{(k)}(x) - v^{(k)}(0)\right] \chi[0,\tau] \\ &+ \left[\left(v^{(k)}(\tau) - v^{(k)}(0)\right)\left(\frac{e^{\frac{x-1}{\epsilon}} - 1}{e^{\frac{\tau-1}{\epsilon}} - 1}\right)\right] \chi[\tau,1]\Bigg).\end{aligned} \tag{17}$$

Thus, since $r_0 = M^{-1}A[u]$, the bound in the L_2 norm of the residual error at the k^{th} iterate is

$$||r_k||_2^2 \le \epsilon^{2k}C_k^2 + \epsilon^{2k+1}C_kC_{k+1} + \epsilon^{2k+1}C_{k+1}^2. \tag{18}$$

Here, C_k is an upper bound on the magnitude of $u^{(k)}$ for $x \in \Omega_{outer}$. Inequality (18) does not imply that the GMRES iterates converge in the limit as $k \to \infty$ because the sequence $\{C_k\}$ is unbounded (see below). However, given a δ, we show that τ can be chosen so that (16) holds for $k = k_\delta$.

To choose τ, we use the analysis in §3 to obtain

$$C_k = \max_{\Omega_{outer}} u^{(k)}(x) = \frac{K}{\epsilon^k} e^{\frac{\tau-1}{\epsilon}} + \gamma_k. \tag{19}$$

Here, $\gamma_k = \max_{\Omega_{outer}} f^{(k)}(x) + O(\epsilon)$ is a constant based on (4). The constant

$$K = K_0 + O(\epsilon) \tag{20}$$

is determined by (5). From Eqn. (18), we must satisfy $\epsilon^k \left(\Gamma + \frac{2K}{\epsilon^k} e^{\frac{\tau-1}{\epsilon}}\right)(1+ O(\epsilon)) \leq \delta$ where Γ is an upper bound on γ_k and γ_{k+1}. Assuming $\Gamma > \epsilon$, the desired tolerance on the residual error is satisfied for $k = k_\delta$ if $\epsilon^k \left(2\Gamma + \frac{2K}{\epsilon^k} e^{\frac{\tau-1}{\epsilon}}\right) \leq \delta$. Next, we will show that it is possible to choose τ so that the above relation holds. Suppose we wish to converge in k_δ iterations. Then the above relation is satisfied if $\tau < 1 + \epsilon \ln \left(\frac{\delta}{4K} - \frac{\epsilon^{k_\delta}\Gamma}{K}\right)$. We assume that k_δ is large enough so that $\frac{\delta}{4K} > \Gamma\epsilon^{k_\delta}$. (This provides a lower bound on the number of iterations given δ and ϵ). We choose τ so that $\tau < 1 + \epsilon \ln \left(\frac{\delta}{8K}\right)$. This expression is the basis for the heuristic $\tau = 1 + c_\tau \epsilon \ln(\delta)$, for choosing τ, where the constant c_τ is near unity.

Error Analysis. In general, a small residual error $||r_k||_2$ does not guarantee a small error $||e_k||_2$. We can establish a bound on the error based on the residual. This bound follows from an examination of the solution to $(M^{-1}A)[e_k] = r_k$, where $||r_k||_2 \leq \delta$. For $x \in \Omega_{outer}$, it has been shown [5] that $e_k = r_k + O(\epsilon)$. In addition, for $x \in \Omega_{inner}$ we have that $e_k = r_k + O(\epsilon) + O(e^{\frac{\tau-1}{\epsilon}})$. Thus, the norm of the error is related to the norm of the residual by $||e_k||_2 = ||r_k||_2 + O(\epsilon^{1/2})$. A tighter bound on the norm of the error might be possible; however, with simple analysis, we have shown that a small residual norm implies a small error norm, assuming a sufficiently smooth initial error.

7. One-Dimensional Implementation and Demonstration

We demonstrate the convergence properties by approximating the solution of the advection-diffusion problem (1-2) with $f(x) = -1$ so that the error is no larger than a tolerance TOL. The error tolerance applies to both the discretization and to the iterative process.

The discretization of the advection operator is a second-order strictly upwind scheme, and the diffusion operator is approximated using a second-

Table 1: Experiments varying TOL ($\epsilon = .01$, $c_\tau = 1.0$)

TOL	Δx_{outer}	Number of grid points	Number of GMRES steps
.1	.33	138	1
.01	.11	144	1
.001	.032	166	2

Table 2: Experiments varying ϵ ($TOL = .025$, $c_\tau = 1.0$, $\Delta x_{outer} = .17$)

ϵ	Number of grid points	Number of GMRES steps
.1	26	1
.01	141	1
.005	270	1

order centered scheme. Based on known behavior of the solution [5], the boundary layer is in a neighborhood of $x = 1$ of size $O(\epsilon \ln(\epsilon))$. The nonuniform grid is chosen to resolve this boundary layer. In the outer region, we use $\Delta x_{outer} \approx TOL^{1/2}$. This grid spacing is halved for successive mesh points in the interface region between the boundary layer and the outer region. The boundary layer region uses $\Delta x_{inner} \approx \epsilon \Delta x_{outer}$. This will provide a uniformly second-order accurate method. At the interface we use the second-order upwind method for the advection operator for all points $x_i < \tau$, and the discretization of the full operator for all points $x_i > \tau$ (here, we assume τ is between two points in our grid.)

The experiments were performed in Matlab. The GMRES iteration was halted when $||r_k||_2/||r_0||_2 < TOL$. These experiments suggest that the conclusions from our analysis of the continuous problem also apply to the discretized problem (e.g., implementation in a finite-dimensional space).

Table 3: Experiments varying c_τ

c_τ	Number of outer-region points	Number of GMRES steps
$1.2 - 1.0$	30	2
.8	31	2
.6	32	3
.4	57	27

The results summarized in Table 1 show that the method has slow growth in the number of iterations as TOL is varied. The experiments reported in Table 2 show that the number of iterations is (nearly) independent of ϵ.

However, total work per iteration increases. Next, in Table 3, only c_τ is varied and the discretization fixed. With $\epsilon = .01$ and $TOL = .001$, there are 166 points in the grid. The number of iterations is sensitive to c_τ, as seen in the dramatic increase in the number of GMRES steps for $c_\tau = 0.4$. Other experiments indicated that there is a strong interaction between the choice of c_τ and the location of the interface between the *outer* and *inner* subdomains. That is, τ should be located outside or barely inside the region that is refined. When τ is well inside the refined region, the number of GMRES iterations increases dramatically, regardless of whether τ is inside the boundary layer or not.

8. Two-Dimensional Results

Discretizations. Consider a tensor-product mesh $P_{i,j} = (x_i, y_j)$, where $i = 0, \ldots, N_x$ and $j = 0, \ldots, N_y$. Also let $u_{i,j}$ represent the numerical approximation to the solution at $P_{i,j}$, that is, $u_{i,j} \approx u(x_i, y_j)$. The points need not be uniformly distributed. In practice, the coarse mesh would be used in the "smooth" subdomain Ω_c, and a fine mesh in the "boundary layer" subdomain Ω_d. We use a first order upwind approximation for the first order spatial derivatives, and centered differences are used for the second order spatial derivatives. Consequently, the discretization of (1) can be formulated as the system of linear equations,

$$Au = f \tag{21}$$

where $A = A_c - \epsilon A_d$, with A_c, A_d, and f being the discretizations of L_c, L_d, and the boundary conditions, respectively.

In this section, we introduce and compare several preconditioning matrices, including two based on asymptotics and domain decomposition. The description of the preconditioning matrices is most easily accomplished if we introduce a block partitioning of A (as in [1]). Let $u = [u_c^T \; u_d^T]^T$, where u_c are the unknowns corresponding to the smooth subdomain Ω_c and u_d are the unknowns corresponding to the boundary layer subdomain Ω_d. Moreover, let the unknowns within a subdomain be ordered according to the direction of the convection [2, 3]. Since we are using a strictly upwind differencing scheme for the convection terms, this means A_c is lower triangular. Given this ordering, we can partition the matrices as follows:

$$A_c = \begin{bmatrix} A_{11} & \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad A_d = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \tag{22}$$

where A_{11} and A_{22} are lower tridiagonal (we have assumed a and b are positive), D_{11} and D_{22} are pentadiagonal, and A_{21}, D_{21}, and D_{12} contain one nonzero sub- or super-diagonal. The off-diagonal blocks A_{21}, D_{21}, and D_{12} represent the portions of the difference stencils that couple the two subdomains, and the diagonal blocks contain the portions of the difference stencils that couple points entirely within one subdomain or the other. That is, we solve the equivalent linear system $M^{-1}Au = M^{-1}f$ where M is chosen to approximate A in some sense and is easy to invert.

We present several preconditioners derived from asymptotic analysis and compare them to "standard" preconditioners. Our baseline "preconditioner" is the identity matrix (i.e., no preconditioning), $M_{id} = I$. Since the diagonal of A is constant in our test problems, diagonal preconditioning has no effect on GMRES convergence. We also consider the Gauss-Seidel (lower triangular part of A) preconditioner,

$$M_{gs} = \begin{bmatrix} A_{11} - \epsilon\tilde{D}_{11} & 0 \\ A_{21} - \epsilon D_{21} & A_{22} - \epsilon\tilde{D}_{22} \end{bmatrix}, \tag{23}$$

where $\tilde{D}_{ii}$ is the lower triangular part of D_{ii}. The next preconditioner is the matrix representation of the discrete convection operator,

$$M_{co} = A_c = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}. \tag{24}$$

This preconditioner should be effective in the smooth subdomain, but it neglects the importance of the diffusion term in the boundary layer subdomain. To compensate for this, we might consider the block diagonal preconditioner

$$M_{bd} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} - \epsilon D_{22} \end{bmatrix}. \tag{25}$$

This is our first asymptotics-motivated domain decomposition preconditioner. It is equivalent to solving the convection equation (7) in Ω_c and the full equation (9) in Ω_d, and ignoring the coupling of unknowns across the subdomain interface. Thus, this preconditioner corresponds to solving the original problem with certain nonphysical conditions imposed at the interface. The advantage of this preconditioner is its inherent large-grain parallelism: the two subproblems can be solved independently. We rely on the iterative method to "glue" the two pieces together.

If we wish to obtain a more physically realistic preconditioner, we can include the coupling of the Ω_c and Ω_d unknowns. This yields the following block lower triangular matrix

$$M_{dd} = \begin{bmatrix} A_{11} & 0 \\ A_{21} - \epsilon D_{21} & A_{22} - \epsilon D_{22} \end{bmatrix}. \tag{26}$$

Finally, note that the physically motivated domain decomposition preconditioners M_{bd} and M_{dd} require the solution of the original equation (1),

Table 4: Condition Number

Grid Size	10 × 10		20 × 20		30 × 30	
	Actual	Estimate	Actual	Estimate	Actual	Estimate
$M = I$	33.	23.	110	74.	240	150
M_{gs}	8.0	4.7	29	16.	66	43.
M_{co}	4.8	2.3	12	3.8	23	4.9
M_{bd}	10.	4.5	33	8.9	81	15.
M_{dd}	8.9	2.4	35	4.1	97	6.1

Table 5: Number of iterations

Grid Size	10 × 10	20 × 20	30 × 30	40 × 40
$M = I$	48	90	112	133
M_{gs}	15	38	79	87
M_{co}	15	27	39	52
M_{bd}	19	35	58	89
M_{dd}	15	27	39	53

but on a smaller domain. Since the boundary layer region is a fraction of the total computational domain, this subproblem is small compared to the full problem.

The condition number of the upper Hessenberg matrix that is generated during the GMRES iterations is an estimate for the condition number of the preconditioned matrix, and is an indication of the *stiffness* of the problem. The estimates in Table 4 were obtained after the method had obtained a relative residual of 10^{-4}. By comparing the results presented in Table 4 with those in Table 5, we see that this estimate is a better predictor of the total number of iterations than the actual condition number. The results in Table 5 indicate that the most efficient preconditioner would be M_{co}; however, this might change if a nonuniform mesh were used.

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