Contemporary Mathematics
Volume 180, 1994

Two-level Additive Schwarz Preconditioners for Nonconforming Finite Elements

SUSANNE C. BRENNER

ABSTRACT. *Two-level additive Schwarz preconditioners are developed for the nonconforming finite element approximations of second order and fourth order elliptic boundary value problems. The condition numbers of the preconditioned systems are shown to be bounded independent of mesh sizes and the number of subdomains in the case of a generous overlap.*

1. Introduction

We generalize the Dryja-Widlund theory of two-level additive Schwarz preconditioners (cf. [**4**], [**5**] and [**9**]) to nonconforming finite element approximations of second and fourth order elliptic boundary value problems. Compared with conforming finite elements, the nonconforming finite elements have fewer degrees of freedom. The trade-off is that the communication between grids of different sizes is more complicated. The intergrid transfer operators must be constructed carefully. We show that under certain assumptions there is a uniform bound on the condition numbers of the preconditioned systems. Details and proofs of the results can be found in [**1**]. Recent works on domain decomposition methods for nonconforming finite elements can also be found in [**3**], [**6**] and [**2**].

2. The Preconditioner

The construction of the preconditioner is based on the idea of domain decomposition. Let Ω be a bounded polygonal domain in $\mathbb{R}^2$. Write $\Omega = \cup_{j=1}^{J}\Omega_j$ as a union of overlapping subdomains. $\mathcal{T}_H$ is a triangulation of Ω and $\mathcal{T}_h$ is a subdivision of $\mathcal{T}_H$ which is aligned with each $\partial\Omega_j$.

1991 *Mathematics Subject Classification.* Primary 65F10, 65N30, 65N55.

The author was supported in part by NSF Grant #DMS-92-09332.

The final detailed version of this paper has been submitted for publication elsewhere.

We assume that there exist nonnegative C^∞ functions $\theta_1, \theta_2, \ldots, \theta_J$ in $\mathbb{R}^2$ such that (G1) $\theta_j = 0$ on $\Omega\backslash\Omega_j$, (G2) $\sum_{j=1}^J \theta_j = 1$ on $\overline{\Omega}$, (G3) $\|\nabla\theta_j\|_{L^\infty} \leq \frac{C}{\delta}$ and $\|\nabla^2\theta_j\|_{L^\infty} \leq \frac{C}{\delta^2}$, and (G4) Each point in Ω can belong to at most N_c subdomains. (Here δ measures the size of the overlap and ∇^2 is the Hessian.) These are the *geometric assumptions.* Note that from here on the generic positive constant C is independent of h, H, δ, J and N_c.

Let V_h (resp., V_H) be the finite element space associated with $\mathcal{T}_h$ (resp., $\mathcal{T}_H$) and let $V_j = \{v \in V_h : v = 0 \text{ on } \Omega\backslash\Omega_j\}$. (We only treat homogeneous Dirichlet boundary conditions.) The existence of the partition of unity implies that $V_h = \sum_{j=1}^J V_j$. Let $a_h(\cdot,\cdot)$ (resp., $a_H(\cdot,\cdot)$) be a positive-definite symmetric bilinear form on V_h (resp,. V_H).

We will describe the preconditioner in operator notation (cf. [**8**]). We assume that $(\cdot,\cdot)_h$ and $(\cdot,\cdot)_H$ are inner products on V_h and V_H respectively. The operators $A_h : V_h \longrightarrow V_h$, $A_j : V_j \longrightarrow V_j$, $A_H : V_H \longrightarrow V_H$, $Q_j : V_h \longrightarrow V_j$ and $P_j : V_h \longrightarrow V_j$ are defined by

$$\begin{aligned} (A_h v, w)_h &= a_h(v,w), \qquad & (A_j v, w)_h &= a_h(v,w), \\ (A_H v, w)_H &= a_H(v,w), \qquad & (Q_j v, w)_h &= (v,w)_h, \\ a_h(P_j v, w) &= a_h(v,w). \end{aligned}$$

These operators are related by the equation $A_j P_j = Q_j A_h$.

Let $I_H^h : V_H \longrightarrow V_h$ be the coarse-to-fine intergrid transfer operator (the construction of I_H^h for some concrete applications are given in Section 5). The fine-to-coarse intergrid transfer operator $I_h^H : V_h \longrightarrow V_H$ is defined by $(I_h^H v, w)_H = (v, I_H^h w)_h$, and the operator $P_h^H : V_h \longrightarrow V_H$ is defined by $a_H(P_h^H v, w) = a_h(v, I_H^h w)$. These operators are related by the equation $A_H P_h^H = I_h^H A_h$.

Furthermore, we assume that we have approximate solvers $R_H \sim A_H^{-1}$ on the coarse grid and $R_j \sim A_j^{-1}$ on each subregion, and that these operators are symmetric positive-definite with respect to the inner products $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_h$.

The preconditioner is defined by

$$B := I_H^h R_H I_h^H + \sum_{j=1}^J R_j Q_j.$$

The discretized problem is $A_h u_h = f_h$, and the preconditioned system is $BA_h u = Bf_h$. Using the relationships among the operators, we can write the operator for the preconditioned system as:

$$BA_h = I_H^h R_H I_h^H A_h + \sum_{j=1}^J R_j Q_j A_h = I_H^h R_H A_H P_h^H + \sum_{j=1}^J R_j A_j P_j.$$

It is easy to see that BA_h is symmetric positive-definite with respect to $a_h(\cdot,\cdot)$.

If we use exact solvers, i.e., if $R_H = A_H^{-1}$ and $R_j = A_j^{-1}$, then

$$BA_h = I_H^h P_h^H + \sum_{j=1}^{J} P_j = I_H^h A_H^{-1} I_h^H A_h + \sum_{j=1}^{J} P_j,$$

which shows that our preconditioner is a variant of the Dryja-Widlund preconditioner (cf. [**6**]).

3. Examples

EXAMPLE 1. Dirichlet problem for the Laplace equation:

$$-\Delta u = f \text{ in } \Omega, \qquad u|_{\partial\Omega} = 0.$$

We use the bilinear form $a_h(v_1, v_2) = \sum_{T\in\mathcal{T}_h} \int_T \nabla v_1 \cdot \nabla v_2 \, dx \quad \forall v_1, v_2 \in V_h$, where V_h is the nonconforming P1 finite element space $\{v \in L^2(\Omega) : v \in \mathcal{P}_1(T) \quad \forall T \in \mathcal{T}_h, v$ is continuous at the midpoints of interelement boundaries, and v vanishes at the midpoints along $\partial\Omega\}$. On the coarser grid we have two choices: nonconforming P1 finite elements and conforming P1 finite elements. Our theory is applicable to both choices.

EXAMPLE 2. Dirichlet problem for the biharmonic equation:

$$\Delta^2 u = f \text{ in } \Omega, \qquad u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

We use the bilinear form $a_h(v_1, v_2) = \sum_{T\in\mathcal{T}_h} \int_T \sum_{i,j=1}^{2} \frac{\partial^2 v_1}{\partial x_i \partial x_j} \frac{\partial^2 v_2}{\partial x_i \partial x_j} \, dx \quad \forall v_1, v_2 \in V_h$, where V_h is the Morley finite element space $\{v \in L^2(\Omega) : v \in \mathcal{P}_2(T) \quad \forall T \in \mathcal{T}_h, v$ is continuous at the vertices and vanishes at the vertices along $\partial\Omega$, $\frac{\partial v}{\partial n}$ is continuous at the midpoints of interelement boundaries and vanishes at the midpoints along $\partial\Omega\}$. Here we must use nonconforming finite element spaces on both grids since the Morley finite element space does not contain a conforming subspace.

4. An Abstract Theory

In what follows, k is a parameter which takes the value 1 (resp., 2) for second (resp., fourth) order problems. We use the nonconforming semi-norms

$$|v|_{H^m(\mathcal{T}_h)} := \left(\sum_{T\in\mathcal{T}_h} |v|^2_{H^m(T)}\right)^{1/2} \quad \text{and} \quad |v|_{H^m(\mathcal{T}_{h,j})} := \left(\sum_{\substack{T\subset\Omega_j \\ T\in\mathcal{T}_h}} |v|^2_{H^m(T)}\right)^{1/2}.$$

We make two assumptions on the variational forms:

(V1) $\sqrt{a_h(v,v)} \sim |v|_{H^k(\mathcal{T}_h)}, \quad \sqrt{a_H(v,v)} \sim |v|_{H^k(\mathcal{T}_H)}$

(V2) $a_h(v,w) \le C\,|v|_{H^k(\mathcal{T}_{h,j})}\,|w|_{H^k(\mathcal{T}_h)} \quad \forall v \in V_h, w \in V_j.$

We need two properties on the coarse-to-fine intergrid transfer operators:

$$|I_H^h v|_{H^k(\mathcal{T}_h)} \le C\,|v|_{H^k(\mathcal{T}_H)} \quad \forall\, v \in V_H \quad\text{and} \tag{I1}$$

$$|I_H^h v - v|_{H^\ell(\mathcal{T}_h)} \le C\,H^{k-\ell}\,|v|_{H^k(\mathcal{T}_H)} \quad \forall\, v \in V_H, \tag{I2}$$

$0 \le \ell \le k-1$.

LEMMA. *Under assumptions* (V1)–(V2), (I1) and (G4),

$$\lambda_{\max}(BA_h) \le C\,\omega_1\,N_c,$$

where $\omega_1 := \max(\rho(R_H A_H), \rho(R_1 A_1), \dots, \rho(R_J A_J))$.

To obtain a lower bound for the eigenvalues of BA_h we need a connection operator $K_h^H : V_h \longrightarrow V_H$ which satisfies the following properties.

$$|K_h^H v|_{H^k(\mathcal{T}_H)} \le C\,|v|_{H^k(\mathcal{T}_h)} \quad \forall\, v \in V_h \tag{K1}$$

$$|K_h^H v - v|_{H^\ell(\mathcal{T}_H)} \le C\,H^{k-\ell}\,|v|_{H^k(\mathcal{T}_h)} \quad \forall\, v \in V_h, \tag{K2}$$

$0 \le \ell \le k-1$.

We also assume that the nodal interpolation opertor Π_h satisfies

$$|\Pi_h(\lambda v)|_{H^k(T)} \le C\,|\lambda v|_{H^k(T)} \quad\text{and} \tag{P1}$$

$$\|\Pi_h(gv)\|_{L^2(T)} \le C\Big(\|g\|_{L^\infty(T)} + (k-1)\,h\,\|\nabla g\|_{L^\infty(T)}\Big)\|v\|_{L^2(T)} \tag{P2}$$

for all $T \in \mathcal{T}_h, v \in \mathcal{P}(T)$ (the space of shape functions), $\lambda \in \mathcal{P}_{k-1}(T), g \in C^\infty(\overline{T})$.

LEMMA. *Under the assumptions* (P1)–(P2), (V1), (K1)–(K2), (I1)–(I2), (G1)–(G4), *given any* $v \in V_h$, *there exist* $v_0 \in V_H$, $v_j \in V_j$ $(1 \le j \le J)$ *such that*

$$v = I_H^h v_0 + \sum_{j=1}^J v_j \quad\text{and}$$

$$a_H(v_0, v_0) + \sum_{j=1}^J a_h(v_j, v_j) \le C\,N_c\left(1 + \left(\frac{H}{\delta}\right)^{2k}\right) a_h(v, v).$$

It is well-known (cf. [8]) that such a lemma implies that we have the following lower bound for the eigenvalues of BA_h.

LEMMA.

$$\lambda_{min}(BA_h) \ge C\,\frac{\omega_0}{N_c\left(1 + (\frac{H}{\delta})^{2k}\right)},$$

where $\omega_0 = \min(\lambda_{min}(R_H A_H), \lambda_{min}(R_1 A_1), \dots, \lambda_{min}(R_J A_J))$

Putting everything together, we obtain the bound for the condition number.

THEOREM. *Under the assumptions* (G), (V), (I), (K), and (P), we have

$$\frac{\lambda_{\max}(BA_h)}{\lambda_{\min}(BA_h)} \le C\,\frac{\omega_1}{\omega_0}\,N_c^2\left(1 + \left(\frac{H}{\delta}\right)^{2k}\right).$$

COROLLARY.
$$\frac{\lambda_{max}(BA_h)}{\lambda_{min}(BA_h)} \le CN_c^2$$
if $\omega_1 \le C_1, \quad \omega_0 \ge C_2 > 0, \quad \frac{H}{\delta} \le C_3.$

In the case of a small overlap, the factor $[1 + (H/\delta)]^4$ can be reduced to $[1 + (H/\delta)]^3$ by adopting the arguments in [**7**] (cf. [**2**]).

5. The Operators I_H^h and K_h^H

We use related nested conforming spaces W_H and W_h in the constructions of the intergrid transfer and connection operators. We define I_H^h and K_h^H by the following commutative diagrams, where i is the natural injection.

$$\begin{array}{ccc} V_h & \xleftarrow{F_h} & W_h \\ {\scriptstyle I_H^h}\uparrow & & \uparrow{\scriptstyle i} \\ V_H & \xrightarrow[E_H]{} & W_H \end{array} \qquad\qquad \begin{array}{ccc} V_h & \xrightarrow{E_h} & W_h \\ {\scriptstyle K_h^H}\downarrow & & \downarrow{\scriptstyle Q_h^H} \\ V_H & \xleftarrow[F_H]{} & W_H \end{array}$$

Below we describe W_H, W_h, E_h, E_H , F_h, F_H and Q_h^H for the two examples in Section 3.

The P1 Nonconforming Finite Element. We take W_h to be the conforming P2 finite element space $\{w \in C(\overline{\Omega}) : w|_T \in \mathcal{P}_2(T) \quad \forall T \in \mathcal{T}_h$ and $w = 0$ on $\partial\Omega\}$. The space W_H is defined similarly. Note that the nodal variables of the nonconforming P1 space are also nodal variables of the conforming P2 space. The operator $E_h : V_h \longrightarrow W_h$ is defined by

$$\begin{cases} (E_h v)(m) = v(m) \\ (E_h v)(p) = \text{average of}\, v_i(p) \end{cases}$$

where $v_i = v|_{T_i}$ and $T_i \in \mathcal{T}_h$ contains p as a vertex, and the operator $F_h : W_h \longrightarrow V_h$ is defined by

$$(F_h w)(m) = w(m).$$

The operators $E_H : V_H \longrightarrow W_H$ and $F_H : W_H \longrightarrow V_H$ are defined similarly. The operator $Q_h^H : W_h \longrightarrow W_H$ is the L^2-orthogonal projection.

The Morley Finite Element. We take $\tilde{W}_h$ to be the P5 Argyris finite element space $\{w \in C^1(\overline{\Omega}) : w|_T \in \mathcal{P}_5(T) \quad \forall T \in \mathcal{T}_h, D^\alpha w$ is continuous at the vertices for $|\alpha| = 2$ and $w = \frac{\partial w}{\partial n} = 0$ on $\partial\Omega\}$, which is contained in the larger space $W_h := \{w \in C^1(\overline{\Omega}) : w|_T \in \mathcal{P}_5(T) \quad \forall T \in \mathcal{T}_h,\ w = \frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega\}$. The spaces $\tilde{W}_H$ and W_H are defined similarly. The operator $E_h : V_h \longrightarrow \tilde{W}_h \subseteq W_h$ is defined by

$$\begin{cases} (E_h v)(p) = v(p) \\ (\partial^\alpha E_h v)(p) = \text{average of}\, (\partial^\alpha v_i)(p), |\alpha| = 1 \\ (\partial^\alpha E_h v)(p) = 0, |\alpha| = 2 \\ \left(\frac{\partial}{\partial n} E_h v\right)(m) = \frac{\partial v}{\partial n}(m) \end{cases}$$

where $v_i = v|_{T_i}$ and T_i contains p as a vertex, and the operator $F_h : W_h \longrightarrow V_h$ is defined by

$$\begin{cases} (F_h w)(p) = w(p) \\ \left(\frac{\partial}{\partial n}(F_h w)\right)(m) = \frac{\partial w}{\partial n}(m) \end{cases} .$$

Again, $Q_h^H : W_h \longrightarrow W_H$ is the L^2-orthogonal projection.

The estimates (I1)–(I2) and (K1)–(K2) are obtained from the corresponding estimates of E_h, E_H, F_h, F_H and Q_h^H.

6. Stationary Stokes Equations

Our theory can also be applied to the stationary Stokes equations

$$\begin{aligned} -\Delta \underset{\sim}{u} + \underset{\sim}{\operatorname{grad}}\, p &= \underset{\sim}{f} \quad \text{in } \Omega \\ \operatorname{div} \underset{\sim}{u} &= 0 \quad \text{in } \Omega \\ \underset{\sim}{u} &= \underset{\sim}{0} \quad \text{on } \partial\Omega \end{aligned}$$

using the divergence-free P1 nonconforming finite element. The main difficulty is the divergence-free constraint. It can be circumvented by the connection between the divergence-free P1 nonconforming finite element space and the Morley finite element space (cf. [**1**]).

References

1. S.C. Brenner, *Two-level additive Schwarz preconditioners for nonconforming finite element methods*, preprint.
2. ———, *A two-level additive Schwarz preconditioner for nonconforming plate elements*, preprint.
3. L.C. Cowsar, *Domain decomposition methods for nonconforming finite elements spaces of Lagrange-type*, Technical Report TR93-11, Rice University (1993).
4. M. Dryja and O.B. Widlund, *An additive variant of the Schwarz alternating method in the case of many subregions*, Technical Report 339, Department of Computer Science, Courant Institute (1987).
5. ———, *Some domain decomposition algorithms for elliptic problems*, Technical Report 438, Department of Computer Science, Courant Institute (1989).
6. M. Sarkis, *Two-level Schwarz methods for nonconforming finite elements and discontinuous coefficients*, Technical Report 629, Department of Computer Science, Courant Institute (1993).
7. O.B. Widlund, *Some Schwarz methods for symmetric and nonsymmetric elliptic problems*, Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations (D.E. Keyes, et.al., eds.), SIAM, Philadelphia, 1991, pp. 19–36.
8. J. Xu, *Iterative methods by space decomposition and subspace correction*, SIAM Review **34** (1992), 581–613.
9. X. Zhang, *Studies in Domain Decomposition: Multi-level Methods and the Biharmonic Dirichlet Problem*, Dissertation, (Technical Report 584, Department of Computer Science) Courant Institute (1991).

Department of Mathematics and Computer Science, Clarkson University, Potsdam, NY 13699

Current address: Department of Mathematics, University of South Carolina, Columbia, SC 29208

E-mail address: brenner@math.scarolina.edu