Two-level Additive Schwarz Preconditioners for Nonconforming Finite Elements

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Abstract. Two-level additive Schwarz preconditioners are developed for the nonconforming finite element approximations of second order and fourth order elliptic boundary value problems. The condition numbers of the preconditioned systems are shown to be bounded independent of mesh sizes and the number of subdomains in the case of a generous overlap.

1. Introduction

We generalize the Dryja-Widlund theory of two-level additive Schwarz preconditioners (cf. [4], [5] and [9]) to nonconforming finite element approximations of second and fourth order elliptic boundary value problems. Compared with conforming finite elements, the nonconforming finite elements have fewer degrees of freedom. The trade-off is that the communication between grids of different sizes is more complicated. The intergrid transfer operators must be constructed carefully. We show that under certain assumptions there is a uniform bound on the condition numbers of the preconditioned systems. Details and proofs of the results can be found in [1]. Recent works on domain decomposition methods for nonconforming finite elements can also be found in [3], [6] and [2].

2. The Preconditioner

The construction of the preconditioner is based on the idea of domain decomposition. Let $\Omega$ be a bounded polygonal domain in $\mathbb{R}^2$. Write $\Omega = \bigcup_{j=1}^{J} \Omega_j$ as a union of overlapping subdomains. $T_H$ is a triangulation of $\Omega$ and $T_h$ is a subdivision of $T_H$ which is aligned with each $\partial \Omega_j$.

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We assume that there exist nonnegative $C^\infty$ functions $\theta_1, \theta_2, \ldots, \theta_J$ in $\mathbb{R}^2$ such that (G1) $\theta_j = 0$ on $\Omega \setminus \Omega_j$, (G2) $\sum_{j=1}^J \theta_j = 1$ on $\overline{\Omega}$, (G3) $\|\nabla \theta_j\|_{L^\infty} \leq \frac{C}{\delta}$ and $\|\nabla^2 \theta_j\|_{L^\infty} \leq \frac{C}{\delta^2}$, and (G4) Each point in $\Omega$ can belong to at most $N_c$ subdomains. (Here $\delta$ measures the size of the overlap and $\nabla^2$ is the Hessian.) These are the geometric assumptions. Note that from here on the generic positive constant $C$ is independent of $h$, $H$, $\delta$, $J$, and $N_c$.

Let $V_h$ (resp., $V_H$) be the finite element space associated with $T_h$ (resp., $T_H$) and let $V_j = \{ v \in V_h : v = 0 \text{ on } \Omega \setminus \Omega_j \}$. (We only treat homogeneous Dirichlet boundary conditions.) The existence of the partition of unity implies that $V_h = \sum_{j=1}^J V_j$. Let $a_h(\cdot, \cdot)$ (resp., $a_H(\cdot, \cdot)$) be a positive-definite symmetric bilinear form on $V_h$ (resp., $V_H$).

We will describe the preconditioner in operator notation (cf. [8]). We assume that $(\cdot, \cdot)_h$ and $(\cdot, \cdot)_H$ are inner products on $V_h$ and $V_H$ respectively. The operators $A_h : V_h \rightarrow V_h$, $A_J : V_J \rightarrow V_J$, $A_H : V_H \rightarrow V_H$, $Q_j : V_h \rightarrow V_J$ and $P_j : V_h \rightarrow V_J$ are defined by

$$
(A_h v, w)_h = a_h(v, w), \quad (A_J v, w)_h = a_h(v, w), \\
(A_H v, w)_H = a_H(v, w), \quad (Q_j v, w)_h = (v, w)_h, \\
a_h(P_j v, w) = a_h(v, w).
$$

These operators are related by the equation $A_J P_j = Q_j A_h$.

Let $I_H^h : V_H \rightarrow V_h$ be the coarse-to-fine intergrid transfer operator (the construction of $I_H^H$ for some concrete applications are given in Section 5). The fine-to-coarse intergrid transfer operator $I_h^H : V_h \rightarrow V_H$ is defined by $(I_h^H v, w)_H = (v, I_h^H w)_h$, and the operator $P_h^H : V_h \rightarrow V_H$ is defined by $a_H(P_h^H v, w) = a_h(v, I_h^H w)$. These operators are related by the equation $A_H P_h^H = I_h^H A_h$.

Furthermore, we assume that we have approximate solvers $R_H \sim A_H^{-1}$ on the coarse grid and $R_j \sim A_J^{-1}$ on each subregion, and that these operators are symmetric positive-definite with respect to the inner products $(\cdot, \cdot)_H$ and $(\cdot, \cdot)_h$.

The preconditioner is defined by

$$
B := I_h^H R_H I_h^H + \sum_{j=1}^J R_j Q_j.
$$

The discretized problem is $A_h u_h = f_h$, and the preconditioned system is $B A_h u = B f_h$. Using the relationships among the operators, we can write the operator for the preconditioned system as:

$$
B A_h = I_h^H R_H I_h^H A_h + \sum_{j=1}^J R_j Q_j A_h = I_h^H R_H A_H P_h^H + \sum_{j=1}^J R_j A_J P_j.
$$

It is easy to see that $B A_h$ is symmetric positive-definite with respect to $a_h(\cdot, \cdot)$. 
TWO-LEVEL PRECONDITIONERS

If we use exact solvers, i.e., if \( R_H = A_H^{-1} \) and \( R_j = A_j^{-1} \), then

\[
BA_h = I_H^h P_H^h + \sum_{j=1}^J P_j = I_H^h A_H^{-1} I_H^h A_h + \sum_{j=1}^J P_j,
\]

which shows that our preconditioner is a variant of the Dryja-Widlund preconditioner (cf. [6]).

3. Examples

EXAMPLE 1. Dirichlet problem for the Laplace equation:

\[-\Delta u = f \text{ in } \Omega, \quad u|_{\partial \Omega} = 0.\]

We use the bilinear form \( a_h(v_1, v_2) = \sum_{T \in T_h} \int_T \nabla v_1 \cdot \nabla v_2 \, dx \quad \forall v_1, v_2 \in V_h, \)

where \( V_h \) is the nonconforming \( P1 \) finite element space \( \{ v \in L^2(\Omega) : v \in \mathcal{P}_1(T) \quad \forall T \in T_h, v \text{ is continuous at the midpoints of interelement boundaries, and } v \text{ vanishes at the midpoints along } \partial \Omega \}. \) On the coarser grid we have two choices: nonconforming \( P1 \) finite elements and conforming \( P1 \) finite elements. Our theory is applicable to both choices.

EXAMPLE 2. Dirichlet problem for the biharmonic equation:

\[\Delta^2 u = f \text{ in } \Omega, \quad u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega.\]

We use the bilinear form \( a_h(v_1, v_2) = \sum_{T \in T_h} \int_T \sum_{i,j=1}^2 \frac{\partial^2 v_1}{\partial x_i \partial x_j} \frac{\partial^2 v_2}{\partial x_i \partial x_j} \, dx \quad \forall v_1, v_2 \in V_h, \)

where \( V_h \) is the Morley finite element space \( \{ v \in L^2(\Omega) : v \in \mathcal{P}_2(T) \quad \forall T \in T_h, v \text{ is continuous at the vertices and vanishes at the vertices along } \partial \Omega, \frac{\partial v}{\partial n} \text{ is continuous at the midpoints of interelement boundaries and vanishes at the midpoints along } \partial \Omega \}. \) Here we must use nonconforming finite element spaces on both grids since the Morley finite element space does not contain a conforming subspace.

4. An Abstract Theory

In what follows, \( k \) is a parameter which takes the value 1 (resp., 2) for second (resp., fourth) order problems. We use the nonconforming semi-norms

\[
|v|_{H^m(T_h)} := \left( \sum_{T \in T_h} |v|_{H^m(T)}^2 \right)^{1/2} \quad \text{and} \quad |v|_{H^m(T_h, j)} := \left( \sum_{T \in \tau_j} |v|_{H^m(T)}^2 \right)^{1/2}.
\]

We make two assumptions on the variational forms:

\[
\begin{align*}
(V_1) & \quad \sqrt{a_h(v, v)} \sim |v|_{H^k(T_h)}, \quad \sqrt{a_H(v, v)} \sim |v|_{H^k(T_h)} \\
(V_2) & \quad a_h(v, w) \leq C |v|_{H^k(T_h)} |w|_{H^k(T_h)} \quad \forall v \in V_h, w \in V_j.
\end{align*}
\]
We need two properties on the coarse-to-fine intergrid transfer operators:

(I1) \[ |I^H v|_{H^k(T_h)} \leq C |v|_{H^k(T_h)} \quad \forall v \in V_H \quad \text{and} \]

(I2) \[ |I^H v - v|_{H^k(T_h)} \leq C H^{k-\ell} |v|_{H^k(T_h)} \quad \forall v \in V_H, \]

0 \leq \ell \leq k - 1.

**Lemma.** Under assumptions (V1)–(V2), (I1) and (G4),

\[ \lambda_{\max}(BA_h) \leq C \omega_1 N_c, \]

where \( \omega_1 := \max(\rho(R_H A_H), \rho(R_J A_J), \ldots, \rho(R_J A_J)) \).

To obtain a lower bound for the eigenvalues of \( BA_h \) we need a connection operator \( K^H_h : V_h \rightarrow V_H \) which satisfies the following properties.

(K1) \[ |K^H_h v|_{H^k(T_h)} \leq C |v|_{H^k(T_h)} \quad \forall v \in V_h \]

(K2) \[ |K^H_h v - v|_{H^k(T_h)} \leq C H^{k-\ell} |v|_{H^k(T_h)} \quad \forall v \in V_h, \]

0 \leq \ell \leq k - 1.

We also assume that the nodal interpolation operator \( \Pi_h \) satisfies

(P1) \[ ||\Pi_h(\lambda v)|_{H^k(T)} \leq C |\lambda v|_{H^k(T)} \quad \text{and} \]

(P2) \[ ||\Pi_h(gv)||_{L^p(T)} \leq C \left( ||g||_{L^\infty(T)} + (k-1) h \|\nabla g\|_{L^\infty(T)} \right) ||v||_{L^p(T)} \]

for all \( T \in T_h, v \in P(T) \) (the space of shape functions), \( \lambda \in P_{k-1}(T), g \in C^\infty(T) \).

**Lemma.** Under the assumptions (P1)–(P2), (V1), (K1)–(K2), (I1)–(I2), (G1)–(G4), given any \( v \in V_h \), there exist \( v_0 \in V_H, v_j \in V_j \) (1 \leq j \leq J) such that

\[ v = I^H_h v_0 + \sum_{j=1}^J v_j \quad \text{and} \]

\[ a_H(v_0, v_0) + \sum_{j=1}^J a_h(v_j, v_j) \leq C N_c \left( 1 + \left( \frac{H}{\delta} \right)^{2k} \right) a_h(v, v). \]

It is well-known (cf. [8]) that such a lemma implies that we have the following lower bound for the eigenvalues of \( BA_h \).

**Lemma.**

\[ \lambda_{\min}(BA_h) \geq C \frac{\omega_0}{N_c (1 + \left( \frac{H}{\delta} \right)^{2k})}, \]

where \( \omega_0 = \min(\lambda_{\min}(R_H A_H), \lambda_{\min}(R_J A_J), \ldots, \lambda_{\min}(R_J A_J)) \).

Putting everything together, we obtain the bound for the condition number.

**Theorem.** Under the assumptions (G), (V), (I), (K), and (P), we have

\[ \frac{\lambda_{\max}(BA_h)}{\lambda_{\min}(BA_h)} \leq C \frac{\omega_1}{\omega_0} N_c^2 \left( 1 + \left( \frac{H}{\delta} \right)^{2k} \right). \]
COROLLARY.

\[
\frac{\lambda_{\text{max}}(BA_h)}{\lambda_{\text{min}}(BA_h)} \leq C N_c^2
\]

if \(\omega_1 \leq C_1, \quad \omega_0 \geq C_2 > 0, \quad \frac{H}{\delta} \leq C_3.\)

In the case of a small overlap, the factor \([1 + (H/\delta)]^4\) can be reduced to \([1 + (H/\delta)]^3\) by adopting the arguments in [7] (cf. [2]).

5. The Operators \(I_h^H\) and \(K_h^H\)

We use related nested conforming spaces \(W_H\) and \(W_h\) in the constructions of the intergrid transfer and connection operators. We define \(I_h^H\) and \(K_h^H\) by the following commutative diagrams, where \(i\) is the natural injection.

\[
\begin{array}{ccc}
V_h & \xrightarrow{E_h} & W_h \\
\downarrow I_h^H & & \downarrow i \\
V_H & \xrightarrow{E_H} & W_H \\
\end{array}
\quad \quad
\begin{array}{ccc}
V_h & \xrightarrow{E_h} & W_h \\
\downarrow K_h^H & & \downarrow Q_H^H \\
V_H & \xrightarrow{F_H} & W_H \\
\end{array}
\]

Below we describe \(W_H, W_h, E_h, E_H, F_h, F_H\) and \(Q_H^H\) for the two examples in Section 3.

The P1 Nonconforming Finite Element. We take \(W_h\) to be the conforming P2 finite element space \(\{ w \in C^1(\Omega) : w|_T \in P_2(T) \quad \forall T \in T_h \text{ and } w = 0 \text{ on } \partial \Omega \}\). The space \(W_H\) is defined similarly. Note that the nodal variables of the nonconforming P1 space are also nodal variables of the conforming P2 space. The operator \(E_h : V_h \rightarrow W_h\) is defined by

\[
\begin{align*}
(E_h v)(m) &= v(m) \\
(E_h v)(p) &= \text{average of } v_i(p)
\end{align*}
\]

where \(v_i = v|_{T_i}\) and \(T_i \in T_h\) contains \(p\) as a vertex, and the operator \(F_h : W_h \rightarrow V_h\) is defined by

\[
(F_h w)(m) = w(m).
\]

The operators \(E_H : V_H \rightarrow W_H\) and \(F_H : W_H \rightarrow V_H\) are defined similarly. The operator \(Q_H^H : W_H \rightarrow W_H\) is the \(L^2\)-orthogonal projection.

The Morley Finite Element. We take \(\tilde{W}_h\) to be the P5 Argyris finite element space \(\{ w \in C^1(\Omega) : w|_T \in P_5(T) \quad \forall T \in T_h, D^\alpha w \text{ is continuous at the vertices for } |\alpha| = 2 \text{ and } w = \frac{\partial w}{\partial n} = 0 \text{ on } \partial \Omega \}\), which is contained in the larger space \(W_h := \{ w \in C^1(\Omega) : w|_T \in P_5(T) \quad \forall T \in T_h, w = \frac{\partial w}{\partial n} = 0 \text{ on } \partial \Omega \}\). The spaces \(\tilde{W}_H\) and \(W_H\) are defined similarly. The operator \(E_h : V_h \rightarrow \tilde{W}_h \subseteq W_h\) is defined by

\[
\begin{align*}
(E_h v)(p) &= v(p) \\
(\partial^\alpha E_h v)(p) &= \text{average of } (\partial^\alpha v_i)(p), |\alpha| = 1 \\
(\partial^\alpha E_h v)(p) &= 0, |\alpha| = 2 \\
\left(\frac{\partial}{\partial n} E_h v\right)(m) &= \frac{\partial v}{\partial n}(m)
\end{align*}
\]
where $v_i = v|_{T_i}$ and $T_i$ contains $p$ as a vertex, and the operator $F_h : W_h \rightarrow V_h$ is defined by

\[
\begin{cases}
(F_h w)(p) = w(p) \\
\left( \frac{\partial}{\partial n} (F_h w) \right)(m) = \frac{\partial w}{\partial n}(m)
\end{cases}
\]

Again, $Q_h^H : W_h \rightarrow W_H$ is the $L^2$-orthogonal projection.

The estimates (I1)--(I2) and (K1)--(K2) are obtained from the corresponding estimates of $E_h$, $E_H$, $F_h$, $F_H$ and $Q_h^H$.

6. Stationary Stokes Equations

Our theory can also be applied to the stationary Stokes equations

\[
-\Delta u + \nabla \cdot p = f \quad \text{in } \Omega
\]

\[
\nabla \cdot u = 0 \quad \text{in } \Omega
\]

\[
u = 0 \quad \text{on } \partial \Omega
\]

using the divergence-free $P1$ nonconforming finite element. The main difficulty is the divergence-free constraint. It can be circumvented by the connection between the divergence-free $P1$ nonconforming finite element space and the Morley finite element space (cf. [1]).

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