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Domain Decomposition Methods for Monotone Nonlinear Elliptic Problems

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ABSTRACT. In this paper, we study several overlapping domain decomposition based iterative algorithms for the numerical solution of some nonlinear strongly elliptic equations discretized by the finite element methods. In particular, we consider additive Schwarz algorithms used together with the classical inexact Newton methods. We show that the algorithms converge and the convergence rates are independent of the finite element mesh parameter, as well as the number of subdomains used in the domain decomposition.

1. Introduction

Schwarz type overlapping domain decomposition methods have been studied extensively in the past few years for linear elliptic finite element problems, see e. g., [**2, 4, 5, 11, 9**]. In this paper, we extend some of the theory and methods to the class of nonlinear strongly elliptic finite element problems. The first study of the classical Schwarz alternating method for nonlinear elliptic equations appeared in the paper of P. L. Lions [**14**], in which the class of continuous monotonic elliptic problems was investigated. There are basically two approaches that a domain decomposition method can be used to solve a nonlinear problem. The first approach is to locally linearize the nonlinear equation via a Newton-like algorithm and then to solve the resulting linearized problems at each nonlinear iteration by a domain decomposition method. The second approach is to use domain decomposition, such as the Schwarz alternating method, directly on the

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nonlinear problems. In this case, a number of smaller nonlinear problems need to be solved per domain decomposition iteration. In this paper, we focus on the first approach. We show under certain assumptions that the mesh parameters independent convergence can be obtained. Certain related multilevel approaches can be found in [**1, 16**].

Let $\Omega \subset R^d (d = 2, 3)$ be a polygonal domain with boundary $\partial\Omega$ and $a(u, v) = (\nabla u, \nabla v)_{L^2(\Omega)}$. Here $u, v \in V_h$ and V_h is the usual triangular finite element subspace of $H_0^1(\Omega)$(inner product $a(\cdot,\cdot)$ and norm $\|\cdot\|_a = a(\cdot,\cdot)^{1/2}$) consisting of continuous piecewise linear functions. Following the Dryja-Widlund construction of the overlapping decomposition of V_h (cf. [**11**]), the triangulation of Ω is introduced as follows. The region is first divided into nonoverlapping substructures Ω_i, $i = 1, \cdots, N$, whose union forms a coarse subdivision of Ω. Then all the substructures Ω_i, which have diameter of order H, are divided into elements of size h. The assumption, common in finite element theory, that all elements are shape regular is adopted. To obtain an overlapping decomposition of the domain, we extend each subregion Ω_i to a larger region $\Omega_i^{'}$, i.e., $\Omega_i \subset \Omega_i^{'}$. We assume that the overlap is uniform and $V_i \subset V_h$ is the usual finite element space over $\Omega_i^{'}$. Let $V_0 \subset V_h$ be a triangular finite element subspace defined on the coarse grid. It is clear that $\Omega = \bigcup_i \Omega_i^{'}$ and $V_h = V_0 + \cdots + V_N$.

Base on the decomposition of V_h discussed above, we introduce and analyze some algorithms for the finite element solution of the following quasilinear elliptic problem with Dirichlet boundary condition:

$$\mathcal{L}u = \sum_{i=1}^{d} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) + a_0(x, u, \nabla u) = f(x).$$

The corresponding variational problem reads as following: Find $u^* \in V_h$, such that

$$b(u^*, v) = (f, v) \quad \forall v \in V_h, \tag{1}$$

where

$$b(u, v) = \int_\Omega \left(\sum_{i=1}^{d} a_i(x, u, \nabla u) \frac{\partial v}{\partial x_i} + a_0(x, u, \nabla u) v \right) dx.$$

The existence and uniqueness of the continuous problem are understood under certain assumptions, see e.g., Ladyzhenskaya and Ural'Tseva [**13**]. Let $a_i(x, p_0, p_1, p_2) = a_i(x, u, u_x, u_y)$, $p = (p_0, p_1, p_2)$ and $p^{'} = (p_1, p_2)$. The basic assumptions are, for some positive constants c and C,

(A1) $a_i \in C^1(\Omega \times R^3)$;

(A2) $\max\left\{ |a_i|, \left|\frac{\partial a_i}{\partial x_j}\right|, \left|\frac{\partial a_i}{\partial p_k}\right| \right\} \le C$, for $i, k = 0, \cdots, d$, and $j = 1, \cdots, d$;

(A3) the operator is strongly elliptic; i.e.,

$$\sum_{i,j=0}^{d} \frac{\partial a_i(x, p)}{\partial p_j} \xi_i \xi_j \ge c \sum_{i=0}^{d} \xi_i^2.$$

As a direct consequence of assumptions (A1-3), we can prove the following lemmas, which will be used extensively in the convergence analysis in the subsequent sections of this paper.

LEMMA 1. *The functional* $b(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to R$ *satisfies the strong monotonicity condition, i.e., there exists a constant* $c > 0$*, such that for any* $u, v \in H_0^1(\Omega)$,

$$b(u, u - v) - b(v, u - v) \geq c\|u - v\|_a^2 \tag{2}$$

or, equivalently, for any $v, z \in H_0^1(\Omega)$,

$$b(v + z, z) - b(v, z) \geq c\|z\|_a^2.$$

LEMMA 2. *The functional* $b(\cdot,\cdot)$ *is uniformly bounded in the sense that there exists a constant* $C > 0$*, such that*

$$|b(u, w) - b(v, w)| \leq C\|u - v\|_a\|w\|_a, \tag{3}$$

for any $u, v, w \in H_0^1(\Omega)$.

Let $V_h = span\{\phi_1, \cdots, \phi_n\}$ and the finite element solution $u^* = \sum_{i=1}^n u_i\phi_i$. Define

$$b_i(u_1, \cdots, u_n) = b\left(\sum_{j=1}^n u_j\phi_j, \phi_i\right), \qquad f_i = (f, \phi_i)$$

$B = (b_1, \cdots, b_n)^T$ and $\hat{f} = (f_1, \cdots, f_n)^T$. The rest of the paper is devoted to the solution of the following nonlinear algebraic equation

$$G(u) = B(u) - \hat{f} = 0. \tag{4}$$

Here and in the remainder of the paper, we use u (or v, w, z) to denote either a function in V_h or its corresponding vector representation in terms of the basis functions, i.e., $u = (u_1, ..., u_n)^T \in R^n$ and $u = \sum_{i=1}^n u_i\phi_i \in V_h$. We consider the well-known Newton-like method [7].

2. A Simple Poisson-Schwarz-Newton Method

In this section, we discuss a simple algorithm that combines the Schwarz preconditioning technique with a Newton's method. The preconditioner is defined by using the Poisson operator(i.e., using $a(\cdot,\cdot)$), which generally has nothing to do with the nonlinear problem to be solved. We show that with a properly chosen relaxation parameter λ the algorithm converges at an optimal rate, which is independent of the mesh parameters. The involvement of the parameter λ makes the algorithm not very practical, but nevertheless, it provides some theoretical insight to the preconditioning process.

For each subspace V_i, let us define an operator $Q_i : V_h \to V_i$, by

$$a\left(Q_i(u), v\right) = b(u, v), \quad \forall u \in V_h, \; v \in V_i.$$

$Q_i(u)$ can also be understood in the matrix form $Q_i(u) = R_i^T A_i^{-1} R_i B(u)$, where A_i is the subdomain discretization of $a(\cdot,\cdot)$ and $R_i : V_h \to V_i$ is a restriction operator, [**3**]. To define the additive Schwarz method, let us define

$$Q = Q_0 + Q_1 + \cdots + Q_N.$$

We note that the operators Q_i and Q are not linear in general. We shall show that the following nonlinear equation

$$\tilde{G}(u) \equiv Q(u) - \tilde{g} = 0 \tag{5}$$

has a unique solution, and is equivalent to equation (4), i.e., they have the same solution. Here the right-hand vector $\tilde{g} = \sum_{i=0}^{N} \tilde{g}_i$, and $\tilde{g}_i = Q_i u^*$. These $\tilde{g}_i$ can be pre-computed without knowing the exact solution u^*, as illustrated in [**11**]. Let us define

$$M^{-1} = \sum_{i=0}^{N} R_i^T A_i^{-1} R_i \text{ and } M = \left(\sum_{i=0}^{N} R_i^T A_i^{-1} R_i \right)^{-1}.$$

From the additive Schwarz theory of Dryja and Widlund [**11**], we understand that M is symmetric and positive definite and the norm generated by M ($\|\cdot\|_M$) is equivalent to the norm $\|\cdot\|_a$.

ALGORITHM 1 (ADDITIVE-SCHWARZ-RICHARDSON). *For a properly chosen parameter λ, iterate for $k = 0, 1, \cdots$ until convergence*

$$u^{k+1} = u^k + \lambda s^k,$$

where $s^k = -\tilde{G}(u^k) = -M^{-1}G(u^k)$.

We note that the algorithm can also be written as $u^{k+1} = u^k - \lambda\left(Q(u^k) - \tilde{g}\right)$. The following technical lemma plays a key role in our optimal convergence theory.

LEMMA 3. *There exists two constants δ_0 and δ_1, such that*

$$(Q(u+z) - Q(u), z)_M \geq \delta_0 \|z\|_M^2, \quad \forall u, z \in V_h, \tag{6}$$

and

$$\|Q(u+z) - Q(u)\|_M^2 \leq \delta_1 \|z\|_M^2, \quad \forall u, z \in V_h. \tag{7}$$

The optimal convergence of the Algorithm 1 is stated in the main theorem of this section.

THEOREM 1. *If we choose $0 < \lambda < 2\delta_0/\delta_1$, where δ_0 and δ_1 are both defined in Lemma 3, then Algorithm 1 converges optimally in the sense that*

$$\|u^k - u^*\|_a \leq C\rho^k \|u^0 - u^*\|_a. \tag{8}$$

Here $\rho^2 = 1 - \lambda\delta_1(2\delta_0/\delta_1 - \lambda) < 1$ and C are independent of the mesh paramenters h and H. The optimal $\lambda_{opt} = \delta_0/\delta_1$ and $\rho_{opt}^2 = 1 - \delta_0^2/\delta_1$.

3. A Newton-Krylov-Schwarz Method (NKS)

In this section, we study an outer-inner iterative method for solving (1). Classical Newton is used as the outer iterative method, and a Schwarz preconditioned Krylov subspace method is used as the inner iterative method. We prove that under certain conditions that if the number of inner iterations is sufficiently large, then the outer iteration converges at a rate independent of the finite element mesh parameters, and the number of subdomains.

At each point $u \in V_h$, let us define

$$M_{AS}^{-1}(u) = \sum_{i=0}^{N} R_i^T L_i^{-1}(u) R_i,$$

as the additive Schwarz preconditioner corresponding to the Jacobi operator $L(u)$ of $B(u)$. Here $L_i^{-1}(u)$ is the inverse of $L(u)$ in the subspace V_i and $R_i : V_h \to V_i$ is the restriction operator. To solve for the kth Newton correction, we use n_k steps of a Schwarz-preconditioned Krylov subspace iterative method with initial guess $v^0 = 0$. Let F_k be the iteration operator, i.e., at the lth Krylov iteration, the error is given by

$$v^l - v = F_k(v^0 - v). \tag{9}$$

Or, equivalently, we have $v = (I - F_k)^{-1} v^l$. For the simplicity of presentation, we replace the Krylov iterative method by a simpler Richardson's method. The operator F_k has the form

$$F_k(u_k) = \left(I - \tau_k M_{AS}^{-1}(u_k) L(u_k)\right)^l,$$

where the τ_k are relaxation parameters. We assume that the operator F_k is bounded, i.e., there exists a constant $0 < \rho_k < 1$, such that

$$\|F_k\|_a \leq \rho_k. \tag{10}$$

The estimate (10) is satisfied for a number of Krylov space methods, such as GMRES [**15**]. In the rest this section, we study the convergence of the following NKS algorithm.

ALGORITHM 2 (NEWTON-KRYLOV-ADDITIVE-SCHWARZ ALGORITHM). *For any given $u_0 \in V_h$, iterate with $k = 0, 1, \ldots$ until convergence*

$$L(u_k)(I - F_k)^{-1}(u_{k+1} - u_k) = -B(u_k) + \hat{f}. \tag{11}$$

In practice, a damping parameter can usually be used in each outer iteration to accelerate the convergence of the Newton method. The parameters can be selected by using either a line search or a trust region approach, see e.g. [**7**]. Since we are interested mostly in theoretical aspects of the algorithm, the selection of parameters is omitted from its description.

Before giving the main result, we present a few auxiliary lemmas. Let $A = \{a(\phi_i, \phi_j)\}$, $i, j = 1, \cdots n$. We assume that $L(u)$ satisfies the Lipschitz condition, i.e.,

$$\|L(u) - L(v)\|_{A^{-1}} \leq \gamma \|u - v\|_A.$$

LEMMA 4. *There exist two constants γ_0 and γ_1, such that for any $v, w \in V_h$,*

$$a(L^{-1}(v)w, L^{-1}(v)w) \leq \gamma_0 a(A^{-1}w, A^{-1}w) \tag{12}$$

and

$$a(A^{-1}L(v)w, A^{-1}L(v)w) \leq \gamma_1 a(w, w). \tag{13}$$

LEMMA 5.

$$\|B(v + z) - B(v) - L(v)z\|_{A^{-1}} \leq C\|z\|_A^2 \tag{14}$$

LEMMA 6. *Let $e_k = u^* - u_k$, we then have*

$$\|e_{k+1}\|_a \leq C_0 \left(\|e_k\|_a^2 + \delta_k\right), \tag{15}$$

where $\delta_k \leq \rho_k(1 - \rho_k)^{-1}\|u_{k+1} - u_k\|_a$.

Based on this lemma, we prove that

THEOREM 2. *There exist constants c_1 and c_2, both sufficiently small, such that if $\|u^* - u_0\|_a \leq c_1$ and $\rho_k \leq c_2$, for all k, then*

$$\|u^* - u_k\|_a \leq \rho^k \|u^* - u_0\|_a.$$

Here $0 \leq \rho < 1$ is a constant independent of the mesh parameters. In addition, if $\rho_k \to 0$ in such a way that $\rho_k \leq \{C\|e_k\|_a, 1/(2C_0)\}$, then the convergence is quadratic, i.e.,

$$\|u^* - u_{k+1}\|_a \leq C\|u^* - u_k\|_a^2,$$

where C is independent of the mesh parameters.

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