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A Domain Decomposition Method for Bellman Equations

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ABSTRACT. We apply a domain decomposition technique with subdomains without overlapping to construct an approximation scheme for Bellman equations in $\mathbb{R}^n$. The algorithm is presented for a 2-domain decomposition where the original problem is split into two problems with state constraints plus a linking condition on the interface. We establish the convergence to the viscosity solution and show the results of a numerical experiment.

1. Introduction.

We deal with the numerical solution of the Bellman equation related to the infinite horizon problem with state constraints in an open bounded convex subset Ω of $\mathbb{R}^n$, namely

$$(B) \qquad \lambda u(x) + \max_{a\in A}\left[-b(x,a)\cdot \nabla u(x) - f(x,a)\right] = 0 \quad , \qquad x \in \Omega,$$

where λ is a positive real parameter and A is a compact subset of $\mathbb{R}^m$ representing the set of admissible controls. It is known (see Soner [8], Capuzzo Dolcetta-Lions [3]) that, under rather general assumptions, the value function of the problem is the unique constrained viscosity solution of (B). We mention that the numerical solution of (B) gives complete information about the control problem since it provides approximate optimal controls in feedback form and the corresponding approximate optimal trajectories. However, this solution requires to solve a partial differential equation in Ω and this can be unaffordable when the number of state variables is large (f.e. in many economic problems $\Omega \subset \mathbb{R}^n$ and $n >> 10$). This is the main obstacle that has limited the application of the dynamic programming approach to the solution of real problems. The application of a

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domain decomposition strategy seems to be an answer to this problem since it permits a huge problem (in Ω) to be split into a number of problems (in Ω_r) of managable size. This strategy can be directly implemented on parallel machines enlarging the possibilities of the dynamic programming approach.

We refer to [6] for a first step in this direction, where we studied a splitting algorithm for (B) based on a domain decomposition *with overlapping* between the sets Ω_r of the decomposition, $\Omega = \bigcup_r \Omega_r$, $r = 1, \ldots, d$. Here we extend our result to the situation in which we do *not* have overlapping using a variable step technique. For simplicity we present our algorithm in the case of a 2-domains decomposition but the extension to an m-domains decomposition requires only technical adaptations.

Our approach is based on recent results in the numerical approximation of the infinite horizon problem with state constraints. We refer to [2] for an a priori estimate of the fully discrete scheme with fixed time-step and to the references therein for other numerical methods for Hamilton–Jacobi–Bellman equations. It is important to notice that the basic ideas of the method presented here are general enough to be applied to other first order Hamilton–Jacobi–Bellman equations. Finally, we should also mention the work [9] on the numerical solution of the Bellman equation related to an exit time problem for diffusion processes (i.e. for second order elliptic problems) wherein a different algorithm is considered.

2. The infinite horizon problem with state constraints

Let Ω be an open bounded convex subset of $\mathbb{R}^n$ with regular boundary ($\nu(x)$ being its outward normal at $x \in \partial\Omega$). We will make the following assumptions on $b : \mathbb{R}^n \times A \to \mathbb{R}^n$ and $f : \mathbb{R}^n \times A \to \mathbb{R}$:

$$(A1) \qquad b \text{ and } f \text{ are continuous in } \overline{\Omega} \times A$$

$$(A2) \qquad |g(x_1, a) - g(x_2, a)| \leq L_g |x_1 - x_2| \qquad \text{for } g = b, f \text{ and } a \in A$$

Soner [8] has extended the notion of "viscosity solution" in order to characterize the value function for the constrained problem. To this end, he also proved that the value function is continuous provided there exists a positive constant c such that

$$(A3) \qquad \forall x \in \partial\Omega \qquad \exists a \in A \text{ such that } < b(x, a), \nu(x) > \leq -c < 0,$$

(see also [3] for further theoretical results on constrained problems). In [5] and [2] the following fully discrete scheme has been studied

$$(B_h^k) \ u(x_i) = \inf_{a \in A_h(x_i)} \{(1 - \lambda h) u(x_i + hb(x_i, a)) + hf(x_i, a)\}, \qquad i = 1, \ldots, N,$$

where x_i, is a node of a regular triangulation of Ω, h is a positive parameter to be interpreted as a (fixed) time-step and for any $i \in I \equiv \{1, \ldots, N\}$ the set

$A_h(x_i)$ is defined as

(1) $$A_h(x_i) = \{a \in A : \; x_i + hb(x_i, a) \in \Omega\}, \quad i \in I.$$

Notice that, due to (A1) and to the boundary condition (A3), there exists a $\overline{h} > 0$ such that for any $h \leq \overline{h}$

(2) $$A_h(x_i) \neq \emptyset, \quad i \in I.$$

Working in the space of piecewise linear finite elements, (B_h^k) is reduced to a finite dimensional fixed point problem which admits a unique solution V^* by the contraction mapping theorem.

3. A convergence result for the domain decomposition method

In this section we modify the previous approach by considering a variable time step η in order to deal with a decomposition *without overlapping.*

Let Ω be partitioned into two open subdomains Ω_1 and Ω_2, such that $\Omega = \Omega_1 \cup \Omega_2$, and let Γ be the interface, i.e. $\Gamma \equiv \partial\Omega_1 \cap \partial\Omega_2$. Given the above decomposition and a positive parameter h, we define the variable step $\eta : \overline{\Omega} \times A \to \mathbb{R}_+$ as follows

$$\eta(x,a) \equiv \begin{cases} \eta^r(x,a) & (x,a) \in \overline{\Omega}_r \setminus \Gamma \times A_h(x), \\ h & (x,a) \in \Gamma \times A_h(x) \end{cases}$$

where

$$\eta^r(x,a) \equiv \min\{\inf[t \in \mathbb{R}_+ : x + tb(x,a) \in \mathbb{R}^n \setminus \overline{\Omega}_r], h\}, \quad r = 1, 2.$$

We define the following operator $D_h : L^\infty(\Omega) \to L^\infty(\Omega)$,

$$[D_h u](x) \equiv \begin{cases} \inf\limits_{a \in A_h(x)} \{\beta(x,a)u(x + \eta(x,a)b(x,a)) + \eta(x,a)f(x,a)\} & x \in \overline{\Omega} \setminus \Gamma, \\ \inf\limits_{a \in A_h(x)} \{(1 - \lambda h)u(x + hb(x,a)) + hf(x,a)\} & x \in \Gamma \end{cases}$$

where $\beta(x,a) \equiv (1 - \lambda\eta(x,a))$. In order to establish the convergence of the domain decomposition algorithm, we study the properties of D_h.

THEOREM 3.1. *If (A1), (A3) are verified and $h \in (0, \frac{1}{\lambda}]$, then there exists a unique solution $u_h \in L^\infty(\Omega)$ of*

(3) $$u(x) = D_h u(x) \qquad x \in \overline{\Omega}$$

PROOF. *Uniqueness.* Let us suppose that there exist two solutions $u, v \in L^\infty(\Omega)$ of (3). Let $x \in \Gamma$, then there exists $a \in A_h(x)$ such that

$$u(x) - v(x) \leq (1 - \lambda h)[u(x + hb(x,a)) - v(x + hb(x,a))]$$

and, by symmetry, we get

(4) $$|u(x) - v(x)| \leq (1 - \lambda h)\|u - v\|_{L^\infty(\Omega)}$$

For $x \in \overline{\Omega} \setminus \Gamma$ we have

$$u(x) - v(x) \le \beta(x,a)[u(x + \eta(x,a)b(x,a)) - v(x + \eta(x,a)b(x,a))].$$

If $\eta(x,a) = h$ we conclude as in (4). When $\eta(x,a) < h$ we have $x + \eta(x,a)b(x,a) \in \Gamma$ so that by (4), we obtain

$$u(x) - v(x) \le \beta(x,a)(1 - \lambda h)\|u - v\|_{L^\infty(\Omega)}.$$

In conclusion, (4) is verified for any $x \in \overline{\Omega}$.
Existence. Let $\|f\|_\infty \le M_f$, $u_0 \equiv -M_f/\lambda$ and $u^0 \equiv M_f/\lambda$. It is simple to check that u_0 and u^0 are respectively a sub-solution and a super-solution of (3). Let $\mathcal{K}$ be the convex, closed subset of $L^\infty(\Omega)$ defined by $\mathcal{K} \equiv \{u \in L^\infty(\Omega) : u_0 \le u \le u^0\}$. The operator D_h is compact and $D_h(\mathcal{K}) \subset \mathcal{K}$, so Schauder's fixed point theorem (see e.g. [7]) implies that there exists a solution of (3). □

Remark. In order to give a constructive method to compute the fixed point u_h, we use the following strategy. We define $\eta_\varepsilon(x,a) = \eta(x,a) \vee \varepsilon$, for $\varepsilon > 0$ and introduce a new operator $D_{h\varepsilon}$ which is obtained replacing $\eta(x,a)$ with $\eta_\varepsilon(x,a)$ in the definition of D_h. $D_{h\varepsilon}$ is a contraction map in $L^\infty(\Omega)$ and we will denote by $u_{h\varepsilon}$ the corresponding fixed point. Then we can prove that

$$\|u_h - u_{h\varepsilon}\|_{L^\infty} \le \lambda(\varepsilon \vee \eta - \varepsilon)\|u_h\|_{L^\infty} + (1 - \lambda(\varepsilon \vee \eta))\|u_h - u_{h\varepsilon}\|_{L^\infty}$$

which implies

$$\|u_h - u_{h\varepsilon}\|_{L^\infty} \le (1 - \frac{\eta}{\varepsilon \vee \eta})\|u_h\|_{L^\infty}.$$

Therefore the sequence of fixed points $u_{h\varepsilon}$ converges to u_h as $\varepsilon \longrightarrow 0^+$.

THEOREM 3.2. *Let (A1) and (A3) be verified. Then $\{u_h\}$ converges uniformly in $\overline{\Omega}$ to the unique constrained viscosity solution u of (B).*

PROOF. Let us define

$$\overline{u} = \limsup_{\substack{h \to 0^+ \\ y \to x}} u_h(y), \qquad \underline{u} = \liminf_{\substack{h \to 0^+ \\ y \to x}} u_h(y)$$

Then $\underline{u} \le \overline{u}$. If we prove that $\underline{u}$ is a viscosity sub-solution in Ω of (B) and $\overline{u}$ is a viscosity super-solution in $\overline{\Omega}$, by the comparison theorem in [1] we can conclude that $u = \overline{u} = \underline{u}$ in $\overline{\Omega}$ and $u_h \longrightarrow u$ uniformly . The proof in $\overline{\Omega} \setminus \Gamma$ can be obtained by a straightforward modification of Theorem 2.7 in [2]. Therefore we limit ourselves to the case $x \in \Gamma$.

Given $\phi \in C^\infty(\overline{\Omega})$, let $x_0 \in \Gamma \cap \Omega$ be a maximum point for $\overline{u} - \phi$. Then, by definition of $\overline{u}$, there exist two sequences $\{h_n\}$ and $\{y_n\}$ such that $h_n \longrightarrow 0^+$, $y_n \longrightarrow x$

$$\overline{u}(x_0) = \lim_{n \to +\infty} u_{h_n}(y_n)$$

and y_n is a maximum point for $u_{h_n} - \phi$. Since $x_0 \in \Omega$, $A_{h_n}(y_n) = A$, for h sufficiently small. Since y_n is a maximum point for $u_{h_n} - \phi$, for any fixed $a \in A$, we have

$$\lambda\eta(y_n,a)u_{h_n}(y_n) - \beta(y_n,a)\left[u_{h_n}(y_n + \eta(y_n,a)b(y_n,a)) - u_{h_n}(y_n)\right] + \\ - \eta(y_n,a)f(y_n,a) \leq \lambda\eta(y_n,a)u_{h_n}(y_n) - \beta(y_n,a)\left[\phi(y_n + \eta(y_n,a)b(y_n,a)) + \\ -\phi(y_n)\right] + \eta(y_n,a)f(y_n,a)$$

By definition, $\eta(y_n,a) \longrightarrow 0^+$ for $n \to +\infty$ and $\eta(x,a) > 0$ for any $(x,a) \in \overline{\Omega} \times A_h(x)$, so dividing the above inequality by $\eta(y_n,a)$ and passing to the limit for n tending to $+\infty$ we obtain

$$\lambda\overline{u}(x_0) - b(x,a) \cdot \nabla\phi(x_0) - f(x_0,a) \leq 0.$$

Since a is arbitrary, this implies that $\overline{u}$ is a subsolution of (B) at x_0.
Let $x_0 \in \Gamma \cap \overline{\Omega}$ be a minimum point for $\underline{u} - \phi$. Then we can define a sequence $\{y_n\}$ of minimum points for $u_{h_n} - \phi$ such that

$$\underline{u}(x_0) = \lim_{n\to+\infty} u_{h_n}(y_n).$$

Let $\overline{a}_n$ be a control such that the infimum in (D_{h_n}) is obtained. Since $\{\overline{a}_n\}$ is contained in the compact set A, there exists a subsequence, still denoted by $\overline{a}_n$ and $\overline{a} \in A$ such that $\lim_{n\to+\infty} \overline{a}_n = \overline{a}$. Then a straightforward computation gives

$$0 \leq \lambda\eta(y_n,\overline{a}_n)u_{h_n}(y_n) - \beta(y_n,\overline{a}_n)\left[\phi(y_n + \eta(y_n,\overline{a}_n)b(y_n,\overline{a}_n)) - \phi(y_n)\right] + \\ - \eta(y_n,\overline{a}_n)f(y_n,\overline{a}_n)$$

Now, dividing the above inequality by $\eta(y_n,\overline{a}_n)$, we get for $n \to +\infty$

$$0 \leq \lambda\underline{u}(x_0) - b(x,\overline{a}) \cdot \nabla\phi(x_0) - f(x_0,\overline{a})$$

which implies that $\underline{u}$ is a supersolution of (B) at x_0. □

In order to define the numerical algorithm we introduce the following notations,

(5) $$A_h^r(x) \equiv \{a \in A : x + \eta(x,a)b(x,a) \in \overline{\Omega}_r\}, \quad x \in \overline{\Omega}_r\ , \quad r = 1,2.$$

The definition of $\eta(x,a)$ implies that

(6) $$A_h(x) = A_h^r(x) \qquad \forall x \in \overline{\Omega}_r \setminus \Gamma \quad r = 1,2$$

(7) $$A_h(x) = A_h^1(x) \cup A_h^2(x) \qquad \forall x \in \Gamma$$

We shall always assume that the triangulation of Ω is such that

$(A4)$ no simplex crosses Γ.

We will divide the nodes x_i, $i \in I \equiv \{1,\dots,N\}$, into three classes depending on the region to which they belong, defining

$$I_0 = \{i : x_i \in \Gamma\} \text{ and } I_r = \{i : x_i \in \overline{\Omega}_r \backslash \Gamma\}\ ,\ r = 1,2.$$

Let N_r, $r = 1, 2$, be the number of nodes in $\overline{\Omega}_r$. We define the "discrete" restriction operator

$$(8) \qquad R_r : \mathbb{R}^N \to \mathbb{R}^{N_r}, \qquad R_r(U) = \{U_i\}_{i \in I_r \cup I_0}, \quad r = 1, 2.$$

Since the numerical solution of (3) on the triangulation requires to compute the value of u in points which are not nodes, we use a linear interpolation defining

$$u(x_i + \eta(x_i, a)b(x_i, a)) \equiv \sum_{j=1}^{N} \lambda_{ij}(a)u(x_j),$$

where $\lambda_{ij}(a)$ are the baricentric coordinates of the point $x_i + \eta(x_i, a)b(x_i, a)$ with respect to the vertices of the simplex which contains it. The decomposition of Ω corresponds to split the $(N \times N)$–matrix $\Lambda(a)$ into two submatrices $\Lambda^{(1)}(a) = \{\lambda_{ij}^{(1)}(a)\}_{i,j \in I_1 \cup I_0}$ and $\Lambda^{(2)}(a) = \{\lambda_{ij}^{(2)}(a)\}_{i,j \in I_2 \cup I_0}$.

We introduce the two discrete operators D_1 and D_2, $D_r : \mathbb{R}^{N_r} \to \mathbb{R}^{N_r}$, $r = 1, 2$, related to the subdomains Ω_1 and Ω_2

$$[D_r(U)]_i \equiv \min_{a \in A_h^r(x_i)} \left\{ \beta_i(a) \sum_{j \in I_r \cup I_0} \lambda_{ij}^{(r)}(a)U_j + \eta_i(a)F_i(a) \right\}, \quad i \in I_r \cup I_0,$$

where $\eta_i(a) = \eta(x_i, a)$, $\beta_i(a) = \beta(x_i, a)$ and $F_i(a) = f(x_i, a)$.

Finally, by D_1 and D_2 we define the operator $D : \mathbb{R}^N \to \mathbb{R}^N$ related to the domain Ω as follows:

$$[D(U)]_i = \begin{cases} [D_r(U^r)]_i & i \in I_r,\ r = 1, 2, \\ \min\{[D_1(U^1)]_i\ ,\ [D_2(U^2)]_i\} & i \in I_0 \end{cases}$$

where $U^r = R_r(U), r = 1, 2$.

Remark. The discrete map D corresponding to the operator D_h and the discrete map corresponding to the operator $D_{h\varepsilon}$ coincide for $\varepsilon = \min\{\eta(x_i, a), i \in I, a \in A_h(x_i)\}$. Therefore the convergence of the fully discrete operator is guaranteed by the contraction mapping theorem.

Numerical experiment.

Let us set $\Omega \equiv (-2, 2)^2$, $\Omega_1 \equiv (-2, 0) \times (-2, 2)$, $\Omega_2 \equiv (0, 2) \times (-2, 2)$

$$A \equiv [0, 1], \quad \lambda = 1, \quad b(x, y, a) = (ay, 0) \text{ and } f(x, y, a) = (|x| - 1)^2.$$

The exact solution is known. Numerical results in double precision FORTRAN where obtained on an IBM 3090 using only 1 CPU. The computation of the solution in Figure 1 required 136 iterations for a total of 85 seconds of CPU time giving an L^∞ error of 0.025.

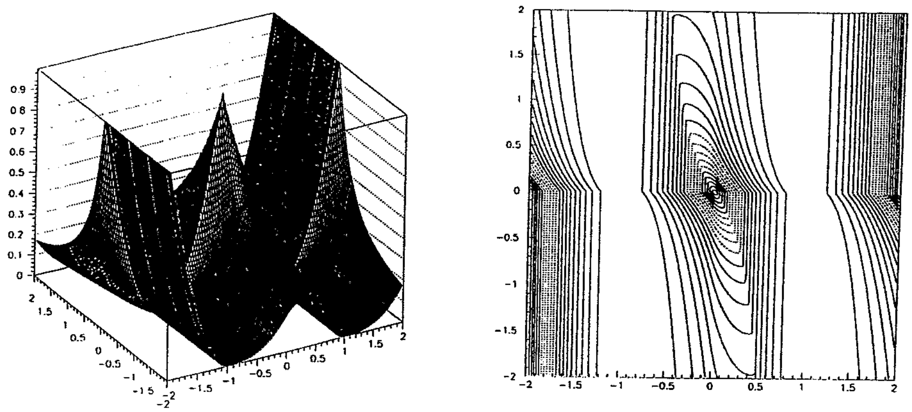

FIGURE 1. The approximate solution and its level curves ($h = 0.05$, $k = 0.025$).

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