Two-Grid Methods for Mixed Finite Element Approximations of Nonlinear Parabolic Equations

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ABSTRACT. Mixed finite element approximation of nonlinear parabolic equations is discussed. The equation considered is a prototype of a model that arises in flow through porous media. A two-grid approximation scheme is developed and analyzed for implicit time discretizations. In this approach, the full nonlinear system is solved on a "coarse" grid of size H. The nonlinearities are expanded about the coarse grid solution, and the resulting linear but nonsymmetric system is solved on a "fine" grid of size h. Error estimates are derived which demonstrate that the error is $\mathcal{O}(h^{k+1} + H^{2(k+1)-d/2} + \triangle t)$, where k is the degree of the approximating space for the primary variable and d is spatial dimension, with $k \geq 1$ for $d \geq 2$. For the RT_0 space ($k = 0$) on rectangular domains, we present a modified scheme for treating the coarse grid problem. Here we show that the error is $\mathcal{O}(h + H^{3-d/2} + \triangle t)$. The above estimates are useful for determining an appropriate H for the coarse grid problem.

1. Introduction

In this paper, we consider the mixed finite element approximation of ρ satisfying

$$(1) \qquad \frac{\partial \rho}{\partial t} - \nabla \cdot (K(\rho)\nabla\rho) = f(x,t), \quad \text{on } \Omega \times (0,T],$$

with initial condition

$$(2) \qquad \rho(x,0) = \rho^0(x), \quad \text{on } \Omega,$$

and boundary condition

$$(3) \qquad (K(\rho)\nabla\rho) \cdot \nu = 0, \text{ on } \partial\Omega \times (0,T],$$

where $\Omega \subset IR^d$, $d \leq 3$, is a bounded, convex domain with C^1 boundary $\partial\Omega$, ν is the unit exterior normal to $\partial\Omega$, and K is a symmetric positive definite tensor and $K : \Omega \times IR \to IR^{d \times d}$. We further assume that there exist positive constants K_* and K^* such that for $z \in IR^d$

$$(4) \qquad K_* \parallel z \parallel^2 \leq z^t K(x,s)z \leq K^* \parallel z \parallel^2, \text{ for } x \in \Omega \text{ and } s \in IR,$$

and that each element of K is twice continuously differentiable with derivatives up to second order bounded above by K^*.

Equation (1) is a simplification of a nonlinear parabolic equation (Richard's Equation) which arises in the modeling of two-phase flow in porous media, where the two phases are air and water [3]. The major difference between Richards' Equation and (1) is in the time derivative term, where ρ is replaced by a nonlinear function $\theta(\rho)$. The function $\theta'(\rho)$ may be zero and this complicates the analysis; however, this difficulty may be handled by techniques given in [1]. For brevity, we do not consider this generalization here.

Let $L^p(\Omega)$ for $p \geq 1$ denote the standard Banach space defined on Ω, with norm $\parallel \cdot \parallel_p$. We shall denote by $W^{m,p}(\Omega)$ the standard Sobolev space (m partial derivatives in L^p) with norm $\parallel \cdot \parallel_{m,p}$. The $W^{m,p}$ norm for vectors will be understood to be for each component. When $p = 2$ we omit the subscript on the norms.

Let $(\cdot,\cdot)$ denote the $L^2(\Omega)$ inner product, scalar and vector. Let $V = H(div;\Omega) = \{v \in (L^2(\Omega))^d : \nabla \cdot v \in L^2(\Omega)\}$, $\tilde{V} = V \cap \{v \cdot \nu = 0\}$, and $W = L^2(\Omega)$.

As in [2] we define the mixed weak form of (1) as $(\rho, \Gamma, \Psi) \in W \times V \times \tilde{V}$ satisfying

$$(5) \qquad (\frac{\partial\rho}{\partial t}, w) + (\nabla \cdot \Psi, w) = (f, w), \quad w \in W,$$

$$(6) \qquad (\Gamma, v) - (\rho, \nabla \cdot v) = 0, \quad v \in \tilde{V},$$

$$(7) \qquad (\Psi, v) = (K(\rho)\Gamma, v), \quad v \in V.$$

We consider finite-dimensional subspaces W_h^k, $\tilde{V}_h^k$, and V_h^k of W, $\tilde{V}$, and V, respectively (they may be Raviart-Thomas-Nedelec spaces of index k RT_k [9, 10] or Brezzi-Douglas-Marini spaces of index k [4], for instance) associated with a quasi-uniform family of polygonal decompositions of Ω by triangles (tetrahedra) or bricks with diameter h. For simplicity in the discussion below, we will drop the superscript k.

Let $N > 0$ and $\triangle t = T/N$, $t^n = n\triangle t$, and $\varphi^n = \varphi(\cdot, t^n)$. In our analysis we shall use the following norms: for X a normed vector space defined on Ω,

$$||\varphi||_{l^2((0,T);X)} \equiv (\sum_{n=1}^{N} \triangle t ||\varphi^n||_X^2)^{\frac{1}{2}},$$

$$||\varphi||_{l^\infty((0,T);X)} \equiv \max_{1 \leq n \leq N} ||\varphi^n||_X,$$

and

$$||\varphi||_{L^2((0,T);X)} \equiv (\int_0^T ||\varphi^n(\cdot, t)||_X^2 dt)^{\frac{1}{2}}.$$

The discrete time mixed finite element approximation to (5)-(7) is defined as follows: Given $(\rho_h^0, \Gamma_h^0, \Psi_h^0) \in W_h \times V_h \times \tilde{V}_h$, for $n = 1, \ldots, N$, let $(\rho_h^n, \Gamma_h^n, \Psi_h^n) \in W_h \times V_h \times \tilde{V}_h$, satisfy

$$(8) \qquad (\frac{\rho_h^n - \rho_h^{n-1}}{\triangle t}, w_h) + (\nabla \cdot \Psi_h^n, w_h) = (f^n, w_h), \quad w_h \in W_h,$$

$$(9) \qquad\qquad\qquad (\Gamma_h^n, v_h) - (\rho_h^n, \nabla \cdot v_h) = 0, \quad v_h \in \tilde{V}_h,$$

$$(10) \qquad\qquad\qquad (\Psi_h^n, v_h) = (K(\rho_h^n)\Gamma_h^n, v_h), \quad v_h \in V_h.$$

This procedure is based on a modification of the standard mixed finite element procedure and was introduced for linear elliptic problems in [2]. For brevity, a proof of existence and uniqueness of the solution to the nonlinear algebraic system (8)-(10) is not given; the reader is referred to [7] since the proof closely follows the argument of Milner for standard mixed method approximations to nonlinear elliptic problems.

Instead of solving (8)–(10) each time step for a fine mesh h we shall also consider two two-level procedures, both of which involve solving the nonlinear problem only a coarse grid of diameter $H >> h$. This work has been motivated by the work of Xu [12] for Galerkin procedures applied to nonlinear elliptic equations. In the simplest scheme, the fine mesh discrete problem is linearized by evaluating $K(\rho)$ at the coarse mesh solution ρ_H. In the second scheme $K(\rho)$ is approximated by a first-order Taylor expansion about ρ_H, a correction by one Newton iteration on the fine space. We shall show that the simplest scheme yields no improvement in accuracy over the coarse grid approximation. The second scheme; however, yields substantial improvement and is a viable computational approach.

This paper is divided into five sections. Notation and approximation assumptions are given and error estimates are derived for (8)-(10) in Section 2. The two two-level schemes are defined and estimates derived for both in Section 3. A new procedure based on postprocessing the coarse grid solution and then applying a Newton correction is defined and analyzed in Section 4 for the space RT_0. In Section 5, we give conclusions and extensions.

2. Notation and Approximation Results

We shall employ several projection operators.

Denote by P the L^2 projection operator. More precisely, let $P_{W_h} : L^2(\Omega) \to W_h$ and $P_{V_h} : (L^2(\Omega))^d \to V_h$, where for $g \in L^2(\Omega)$,

$$(11) \qquad (P_{W_h}g, w_h) = (g, w_h), \quad w_h \in W_h,$$

and for $q \in (L^2(\Omega))^d$,

$$(12) \qquad (P_{V_h}q, v_h) = (q, v_h), \quad v_h \in V_h.$$

For convenience we shall set $\hat{g}_h = Pg$, where P is understood to be P_{W_h} or P_{V_h} depending on whether g is a scalar or vector quantity.

We shall use the well known Π_h projection for mixed finite element approximation spaces. We shall assume that there exists a projection operator $\Pi_h : (H^1(\Omega))^d \to V_h$ such that

$$(13) \qquad (\nabla \cdot \Pi_h g, w_h) = (\nabla \cdot g, w_h), \quad w_h \in W_h.$$

We assume that

$$\nabla \cdot \Pi_h = P_{W_h}, \quad \nabla \cdot : V \overset{\text{onto}}{\to} W_h.$$

We also assume that the following approximation properties hold.

Approximation Properties A:
- There exists a positive constant Q independent of h such that

$$(14) \qquad \|u - \Pi_h u\| \leq Q\|u\|_r h^r, \quad 1 \leq r \leq k+1,$$

$$(15) \qquad \|p - \hat{p}_h\| \leq Q\|p\|_r h^r, \quad 0 \leq r \leq k+1,$$

and

$$(16) \qquad \|p - \hat{p}_h\|_\infty \leq Q\|p\|_\infty h^r, \quad 0 \leq r \leq k+1.$$

- Given $\Upsilon \in (W^{1,\infty}(\Omega))^d$ there exists a $\Upsilon_h \in V_h$ such that

$$(17) \qquad \|\Upsilon - \Upsilon_h\| \leq Qh\|\Upsilon\|_{1,\infty}.$$

Finally, we assume the following inverse property on V_h holds, namely, for $v_h \in V_h$,

$$(18) \qquad \|v_h\|_\infty \leq \|v_h\|h^{-d/2}.$$

These assumptions are known to hold for the Raviart-Thomas-Nedelec and Brezzi-Douglas-Marini spaces mentioned above.

In this paper C shall denote a generic constant. For convenience we will also assume that the solution is smooth and that the maximum index of convergence $k+1$ is attained.

We rewrite (5)-(7) with $t = t^n$. Using the definition of $\hat{\rho}$, the Π_h projection, and the assumption that $\nabla \cdot V_h \subset W_h$, we deduce that

$$(19) \, (\frac{\hat{\rho}_h^n - \hat{\rho}_h^{n-1}}{\triangle t}, w_h) + (\nabla \cdot \Pi_h \Psi^n, w_h) = (f^n, w_h) + (\epsilon^n, w_h), \quad w_h \in W_h,$$

$$(20) \qquad (\hat{\Gamma}_h^n, v_h) - (\hat{\rho}_h^n, \nabla \cdot v_h) = 0, \quad v_h \in \tilde{V}_h,$$

and

$$(21) \qquad (\Pi_h \Psi^n, v_h) = (\Pi_h \Psi^n - \Psi^n, v_h) + (K(\rho^n)\Gamma^n, v_h), \quad v_h \in V_h,$$

where ϵ^n is a time truncation error of order $\triangle t$.

We now derive an error estimate for the algorithm (8)-(10).

Set $\mu^n = \hat{\rho}_h^n - \rho_h^n$, $\zeta^n = \hat{\Gamma}_h^n - \Gamma_h^n$, and $\chi^n = \Pi_h \Psi^n - \Psi_h^n$. Subtracting (8) from (19), (9) from (20), and (10) from (21) and in the resulting equations using the test functions $w_h = \mu^n$, $v_h = \chi^n$, and $v_h = \zeta^n$ in the first, second, and third equations, respectively, we obtain the error equations

$$(22) \qquad (\frac{\mu^n - \mu^{n-1}}{\triangle t}, \mu^n) + (\nabla \cdot \chi^n, \mu^n) = (\epsilon^n, \mu^n),$$

$$(23) \qquad (\zeta^n, \chi^n) = (\mu^n, \nabla \cdot \chi^n),$$

and

$$
\begin{aligned}
(\chi^n, \zeta^n) &= (\Pi_h \Psi^n - \Psi^n, \zeta^n) + (K(\rho_h^n)\zeta^n, \zeta^n) \\
&\quad -((K(\rho_h^n) - K(\rho^n))\Gamma^n, \zeta^n) \\
&\quad +(K(\rho_h^n)(\Gamma^n - \hat{\Gamma}_h^n), \zeta^n).
\end{aligned}
$$

(24)

Combining (22) - (24), applying smoothness and boundedness assumptions on $K(\rho)$, and Hölder's inequality we obtain

$$
\frac{1}{2\triangle t}[\|\mu^n\|^2 - \|\mu^{n-1}\|^2] + \|K(\rho_h^n)^{1/2}\zeta^n\|^2
$$

$$
\leq (\frac{\mu^n - \mu^{n-1}}{\triangle t}, \mu^n) + \|K(\rho_h^n)^{1/2}\zeta^n\|^2
$$

$$
\leq \frac{1}{2}\|\mu^n\|^2 + \frac{1}{2}\|\epsilon^n\|^2 + (\Pi_h\Psi^n - \Psi^n, \zeta^n)
$$

$$
+((K(\rho_h^n) - K(\rho^n))\Gamma^n, \zeta^n) + (K(\rho_h^n)(\Gamma^n - \hat{\Gamma}_h^n), \zeta^n)
$$

(25)
$$
\leq \frac{1}{2}\|\mu^n\|^2 + \frac{1}{2}\|\epsilon^n\|^2 + \delta\|\zeta^n\|^2
$$

$$
+ C\left[\|\Pi_h\Psi^n - \Psi^n\|^2 + \|\hat{\Gamma}_h^n - \Gamma^n\|^2 + (\|\rho^n - \hat{\rho}_h^n\| \, \|\Gamma^n\|_\infty)^2\right],
$$

where $\delta \leq K_*$. Multiplying (26) by $\triangle t$ and summing on $n, n = 1, 2, ..., N$ and applying Gronwall's Lemma, we see that

$$
\|\mu^N\|^2 + \sum_{n=1}^{N} \triangle t\|K(\rho_h^n)^{1/2}\zeta^n\|^2
$$

(26)
$$
\leq C\sum_{n=1}^{N}\triangle t\left[\|\Pi_h\Psi^n - \Psi^n\|^2 + \|\hat{\Gamma}_h^n - \Gamma^n\|^2 + (\|\rho^n - \hat{\rho}_h^n\| \, \|\Gamma^n\|_\infty)^2\right]
$$

$$
+\|\mu^0\|^2 + C\triangle t^2.
$$

Using (27), Approximation Properties A, and the triangle inequality we deduce the following theroem:

THEOREM 1. *Let* $V_h = V_h^k$, $\tilde{V}_h = \tilde{V}_h^k$, *and* $W_h = W_h^k$ *and define the triplet* $(\rho_h^n, \Gamma_h^n, \Psi_h^n) \in W_h^k \times V_h^k \times \tilde{V}_h^k$ *by (8) - (10), for* $n \geq 1$. *Assume that the Approximation Properties A hold. Take* $\rho_h^0 = \hat{\rho}_h(\cdot, 0)$. *Then there exists a positive constant* C, *independent of* h *such that*

(27) $\|\rho_h^N - \rho^N\| + (\sum_{n=1}^{N}\triangle t\|K(\rho_h^n)^{1/2}(\Gamma_h^n - \Gamma^n)\|^2)^{\frac{1}{2}} \leq C(h^{k+1} + \triangle t).$

3. Two-Level Schemes

We first consider a scheme based on a correction by one Newton iteration on the fine space. More precisely, we solve (8)–(10) on a coarse mesh with $h = H$

and then solve the following linear system for $(\tilde{\rho}_h^n, \tilde{\Gamma}_h^n, \tilde{\Psi}_h^n) \in W_h \times V_h \times \tilde{V}_h$, for $n \geq 1$ and $h \ll H$:

$$(28) \qquad (\frac{\tilde{\rho}_h^n - \tilde{\rho}_h^{n-1}}{\triangle t}, w_h) + (\nabla \cdot \tilde{\Psi}_h^n, w_h) = (f^n, w_h), \quad w_h \in W_h,$$

$$(29) \qquad (\tilde{\Gamma}_h^n, v_h) = (\tilde{\rho}_h^n, \nabla \cdot v_h), \quad v_h \in \tilde{V}_h,$$

$$(30) \quad (\tilde{\Psi}_h^n, v_h) = (K(\rho_H^n)\tilde{\Gamma}_h^n, v_h) + (K'(\rho_H^n)\Gamma_H^n(\tilde{\rho}_h^n - \rho_H^n), v_h), \quad v_h \in V_h.$$

Equation (30) is motivated by the Taylor expansion

$$K(\rho^n)\Gamma^n = K(\rho_H^n)\Gamma^n + K'(\rho_H^n)\Gamma^n(\rho^n - \rho_H^N) + \frac{K''(\alpha^n)}{2}\Gamma^n(\rho^n - \rho_H^n)^2,$$

for some α^n between ρ^n and ρ_H^n.

We now derive an estimate for this two-level scheme. Set $\xi^n = \hat{\rho}_h^n - \tilde{\rho}_h^n$, $v^n = \hat{\Gamma}_h^n - \tilde{\Gamma}_h^n$, and $\theta^n = \Pi_h\Psi^n - \tilde{\Psi}_h^n$.

Subtracting (28) from (19), (29) from (20), and (30) from (21), letting $w_h = \xi^n$, $v_h = \theta^n$, and $v_h = v^n$, in the first, second, and third equations respectively, we obtain

$$(31) \qquad (\frac{\xi^n - \xi^{n-1}}{\triangle t}, \xi^n) + (\nabla \cdot \theta^n, \xi^n) = (\epsilon^n, \xi^n)$$

$$(32) \qquad (v^n, \theta^n) = (\xi^n, \nabla \cdot \theta^n),$$

and

$$
\begin{aligned}
(33) \quad (\theta^n, v^n) = {} & (\Pi_h\Psi^n - \Psi^n, v^n) - (K(\rho^n)(\hat{\Gamma}_h^n - \Gamma^n), v^n) \\
& + (K(\rho_H^n)v^n, v^n) + ((\xi^n + (\rho^n - \hat{\rho}_h^n))K'(\rho_H^n)\Gamma_H^n, v^n) \\
& + ((K'(\rho_H^n)(\rho^n - \rho_H^n)(\hat{\Gamma}_h^n - \Gamma_H^n), v^n) \\
& + \frac{1}{2}((\rho^n - \rho_H^n)^2 K''(\alpha^n)\hat{\Gamma}_h^n, v^n).
\end{aligned}
$$

Combining (31)–(34) and using Hölder's inequality, we deduce that

$$
\begin{aligned}
(34) \quad & \frac{1}{2\triangle t}[\|\xi^n\|^2 - \|\xi^{n-1}\|^2] + \|K(\rho_H^n)^{1/2}v^n\|^2 \\
& \leq |(\xi^n, \epsilon^n)| + \delta\|v^n\|^2 \\
& \quad + C\|v^n\|\,\|\xi^n\|(\|\Gamma^n\|_\infty + \|(\Gamma_H^n - \Gamma^n)\|_\infty) \\
& \quad + C\left[\|\Pi_h\Psi^n - \Psi^n\|^2 + \|\hat{\Gamma}_h^n - \Gamma^n\|^2 \right. \\
& \qquad\quad + \|(\rho^n - \hat{\rho}_h^n)\Gamma_H^n\|^2 \\
& \qquad\quad + \|(\rho^n - \rho_H^n)(\hat{\Gamma}_h^n - \Gamma_H^n)\|^2 \\
& \qquad\quad \left. + \|(\rho^n - \rho_H^n)^2\hat{\Gamma}_h^n\|^2 \right],
\end{aligned}
$$

where $\delta \leq K_*/2$, and $C = C(K^*, K_*)$.

We now choose $\hat{N}$ to be the index corresponding to the maximum of $\|\xi^n\|$ for $n = 1, \ldots, N$. Multiplying (35) by $\triangle t$ and summing from $n = 1$ to $M, M \leq N$, using Approximation Properties A and Theorem 1, we see that

$$\|\xi^M\|^2 - \|\xi^0\|^2 + \sum_{n=1}^{M} \triangle t \|K(\rho_H^n)^{1/2} v^n\|^2$$

$$(35) \quad \leq C\triangle t^2 \|\rho_{tt}\|_{L^2((0,T),L^2(\Omega))}^2 + \delta \sum_{n=1}^{M} \triangle t \|v^n\|^2 + T^* + T^{**}$$

$$+ C \left[\sum_{n=1}^{M} \triangle t \|\xi^n\|^2 + \|\Psi\|_{l^2((0,T),H^{k+1})}^2 h^{2(k+1)} + \|\Gamma\|_{l^2((0,T),H^{k+1})}^2 h^{2(k+1)} \right],$$

where T^* and T^{**} are defined as follows.

First, by Theorem 1 and the inverse assumption (18)

$$T^* = K^* \|\xi^{\hat{N}}\| \left(\sum_{n=1}^{M} \triangle t \|v^n\|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{M} \triangle t \|(\Gamma_H^n - \hat{\Gamma}_H^n)\|_{\infty}^2 \right)^{\frac{1}{2}}$$

$$(36) \quad \leq K^* \|\xi^{\hat{N}}\| \left(\sum_{n=1}^{M} \triangle t \|v^n\|^2 \right)^{\frac{1}{2}} (H^{-d/2}(H^{k+1} + \triangle t)).$$

For $d \geq 2$, $k \geq 1$, H and $\triangle t$ can be chosen sufficiently small so that

$$(37) \quad T^* \leq \frac{1}{2} \|\xi^{\hat{N}}\|^2 + \frac{K_*}{8} \sum_{n=1}^{N} \triangle t \|v^n\|^2.$$

Moreover,

$$T^{**} = C^* \sum_{n=1}^{M} (\|(\rho^n - \hat{\rho}_h^n)\Gamma_H^n\|^2 + \|(\rho^n - \rho_H^n)(\hat{\Gamma}_h^n - \Gamma_H^n)\|^2$$

$$+ \|(\rho^n - \rho_H^n)^2 \hat{\Gamma}_h^n\|^2) \triangle t$$

$$\leq C^* \sum_{n=1}^{M} \triangle t (\|\rho^n - \hat{\rho}_h^n\|^2 \|\Gamma^n\|_{\infty}^2$$

$$+ \|\rho^n - \hat{\rho}_h^n\|_{\infty}^2 \|\Gamma_H^n - \Gamma^n\|^2$$

$$+ (\|\hat{\rho}_H^N - \rho_H^N\|_{\infty} + \|\rho^n - \hat{\rho}_H^n\|_{\infty})^2 \|\hat{\Gamma}_h^n - \Gamma_H^n\|^2$$

$$+ \|\rho^n - \rho_H^n\|_{\infty}^2 \|\rho^n - \rho_H^n\|^2 \|\hat{\Gamma}_h^n\|_{\infty}^2).$$

Thus, from Theorem 1 and approximation properties,

$$T^{**} \leq C^* \sum_{n=1}^{M} \triangle t [h^{2(k+1)} + (H^{-d/2}(H^{k+1} + \triangle t) + H^{k+1})^2 (H^{k+1} + \triangle t)^2$$

$$(38) \quad + (H^{-d/2}(H^{k+1} + \triangle t))^2 (H^{k+1} + \triangle t)^2].$$

Here $C*$ depends on $\|\rho\|_{l^2((0,T);W^{(k+1),\infty})}$, $\|\Gamma\|_{l^2((0,T);W^{1,\infty})}$, $\|\Gamma\|_{l^2((0,T);H^{(k+1)})}$, K^* and K_*.

Taking $M = \hat{N}$, and for $d \geq 2$ taking $k \geq 1$, noting that $\xi^0 = 0$, and applying Gronwall's Lemma to (36) we see that for $\triangle t$ and H sufficiently small,

$$\|\xi^{\hat{N}}\| \leq C(h^{k+1} + \triangle t + H^{2k+2-d/2}). \tag{39}$$

Combining (36) - (39), taking $M = N$, we deduce that

$$\|\xi^N\| + (\sum_{n=1}^{N} \triangle t\|K(\rho_H^n)^{1/2}v^n\|^2)^{\frac{1}{2}}$$
$$\leq C(C^*)(h^{k+1} + \triangle t + H^{2k+2-d/2}). \tag{40}$$

Applying the triangle inequality and approximation properties, we obtain the following theorem:

THEOREM 2. *Let* $V_h = V_h^k$, $\tilde{V}_h = \tilde{V}_h^k$, *and* $W_h = W_h^k$ *and define the one Newton correction triplet* $(\tilde{\rho}_h^n, \tilde{\Gamma}_h^n, \tilde{\Psi}_h^n), \in W_h^k \times V_h^k \times \tilde{V}_h^k$ *by (28) - (30). Assume that the Approximation Properties A hold, and take* $\tilde{\rho}_h^0 = \hat{\rho}_h(\cdot, 0)$. *Then there exists a positive constant* C^{**}, *independent of* h, *such that*

$$\|\tilde{\rho}_h^N - \rho^N\| + (\sum_{n=1}^{N} \triangle t\|K(\rho_H^n)^{1/2}(\tilde{\Gamma}_h^n - \Gamma^n)\|^2)^{\frac{1}{2}}$$
$$\leq C^{**}(h^{k+1} + \triangle t + H^{2k+2-d/2}). \tag{41}$$

C^{**} *depends on* $\|\rho\|_{l^2((0,T);W^{(k+1),\infty})}$, $\|\Gamma\|_{l^2((0,T);W^{1,\infty})}$, $\|\Gamma\|_{l^2((0,T);H^{(k+1)})}$, K^* *and* K_*.

We now consider the simple two level scheme: solve(8)–(10) on a coarse mesh with $h = H$ and then solve the following linear system for $(\bar{\rho}_h^n, \bar{\Gamma}_h^n, \bar{\Psi}_h^n) \in W_h \times V_h \times \tilde{V}_h$ for $n \geq 1$,

$$(\frac{\bar{\rho}_h^n - \bar{\rho}_h^{n-1}}{\triangle t}, w_h) + (\nabla \cdot \bar{\Psi}_h^n, w_h) = (f^n, w_h), \quad w_h \in W_h, \tag{42}$$

$$(\bar{\Gamma}_h^n, v_h) = (\bar{\rho}_h^n, \nabla \cdot v_h), \quad v_h \in \tilde{V}_h, \tag{43}$$

$$(\bar{\Psi}_h^n, v_h) = (K(\rho_H^n)\bar{\Gamma}_h^n, v_h) \quad v_h \in V_h. \tag{44}$$

We now derive an estimate for this two-level scheme. Set $\alpha^n = \hat{\rho}_h^n - \bar{\rho}_h^n$, $\beta^n = \hat{\Gamma}_h^n - \bar{\Gamma}_h^n$, and $\lambda^n = \Pi_h \Psi^n - \bar{\Psi}_h^n$. Subtracting (42) from (19), (43) from (20), and (44) from (21), letting $w_h = \alpha^n$, $v_h = \lambda^n$, and $v_h = \beta^n$, in the first, second, and third equations respectively, we obtain

$$(\frac{\alpha^n - \alpha^{n-1}}{\triangle t}, \alpha^n) + (\nabla \cdot \lambda^n, \alpha^n) = (\epsilon^n, \alpha^n) \tag{45}$$

$$(\beta^n, \lambda^n) = (\alpha^n, \nabla \cdot \lambda^n), \tag{46}$$

and

$$(\lambda^n, \beta^n) = (\Pi_h \Psi^n - \Psi^n, \beta^n) - (K(\rho^n)(\bar{\Gamma}_h^n - \Gamma^n), \beta^n)$$

(47) $$+ (K(\rho_H^n)\beta^n, \beta^n) + ((K(\rho^n) - K(\rho_H^n))\hat{\Gamma}_h^n, \beta^n).$$

Combining (45) - (47) we obtain

$$\|\alpha^N\|^2 + \sum_{n=1}^{N} \triangle t \|K(\rho_H^n)^{\frac{1}{2}}\beta^n\|^2$$

(48) $$\leq C(\|\rho_H^n - \rho^n\|^2 + \|\Gamma^n - \bar{\Gamma}_h^n\|^2 + \triangle t^2).$$

THEOREM 3. *The error bound for the simple two level scheme defined by (42) - (44) is given by*

$$\|\bar{\rho}^N - \rho^N\| + (\sum_{n=1}^{N} \triangle t \|K(\rho_H^n)^{\frac{1}{2}}(\bar{\Gamma}_h^n - \Gamma^n)\|)^{\frac{1}{2}}$$

(49) $$\leq C(H^{k+1} + \triangle t + h^{k+1}).$$

In the estimate (49) we observe that solving a linear problem on a fine grid with no Newton-type correction yields no asymptotic improvement over a coarse grid solution. This is different from the result obtained for Galerkin methods for nonlinear elliptic equations [12].

4. Newton Correction for RT_0 Spaces

Our previous results did not treat the case $k = 0$ for $d \geq 2$. Here we assume Ω is a rectangular parallelepiped, and we restrict our attention to the RT_0 ($k = 0$) space defined on a tensor product grid. We also assume K is a diagonal matrix or the mesh is uniform.

In this case, the space W_h is the space of discontinuous piecewise constants defined on a tensor product partition. The ith component of the velocity field V_h is a continuous piecewise linear polynomial in the ith direction and discontinuous piecewise constants in the other directions.

The mixed finite element method for the RT_0 space with special numerical quadrature rules has been shown [11] to be equivalent to cell centered finite difference methods. The results obtained here are applicable to the latter methods.

A discrete time mixed finite element approximation for the RT_0 space is defined as follows: Given $(\rho_h^0, \Gamma_h^0, \Psi_h^0) \in W_h \times V_h \times \tilde{V}_h$, for $n = 1, \ldots, N$, let $(\rho_h^n, \Gamma_h^n, \Psi_h^n) \in W_h \times V_h \times \tilde{V}_h$, satisfy

(50) $$(\frac{\rho_h^n - \rho_h^{n-1}}{\triangle t}, w_h) + (\nabla \cdot \Psi_h^n, w_h) = (f^n, w_h), \quad w_h \in W_h,$$

(51) $$(\Gamma_h^n, v_h) - (\rho_h^n, \nabla \cdot v_h) = 0, \quad v_h \in \tilde{V}_h,$$

(52) $$(\Psi_h^n, v_h) = (K(\mathcal{P}(\rho_h^n))\Gamma_h^n, v_h), \quad v_h \in V_h.$$

Here $\mathcal{P} : W_h \to M_h$ is a "postprocessing" operator, where M_h denotes the space of continuous bilinears $(d = 2)$ or trilinears $(d = 3)$ defined on the same partition as the RT_0 space. For $d = 2$, $\mathcal{P}$ linearly interpolates the four adjacent cell centered values to the vertices of the tensor product grid. A similar procedure can be defined for $d = 3$. This postprocessing operator is motivated by well-known superconvergence results for the scalar variable ρ at the center of each grid block [5, 8].

In deriving an error estimate for this algorithm we proceed as in Section 2; however here we will employ superconvergence results for the RT_k spaces on rectangular elements. It is known for example that the Π_h and the P_{V_h} projections are super close, namely $O(h^{k+2})$ in L^2 [8, 6]. It is also known that two weighted L^2 projections are $O(h^{k+2})$ provided the weight functions are C^1. We shall use these results in the analysis given below.

Set $\mu^n = \check{\rho}_h^n - \rho_h^n$, $\zeta^n = \hat{\Gamma}_h^n - \Gamma_h^n$, and $\chi^n = \Pi_h \Psi^n - \Psi_h^n$. Subtracting (50) from (19), (51) from (20), and (52) from (21) and in the resulting equations using the test functions $w_h = \mu^n$, $v_h = \chi^n$, and $v_h = \zeta^n$ in the first, second, and third equations, respectively, we obtain the error equations

$$(53) \qquad (\frac{\mu^n - \mu^{n-1}}{\triangle t}, \mu^n) + (\nabla \cdot \chi^n, \mu^n) = (\epsilon^n, \mu^n),$$

$$(54) \qquad (\zeta^n, \chi^n) = (\mu^n, \nabla \cdot \chi^n),$$

and

$$
\begin{aligned}
(\chi^n, \zeta^n) \;=\; & (\Pi_h \Psi^n - \hat{\Psi}^n, \zeta^n) + (K(\mathcal{P}(\rho_h^n)\zeta^n, \zeta^n) \\
& - ((K(\mathcal{P}(\rho_h^n)) - K(\rho^n))\check{\Gamma}_h^n, \zeta^n) \\
(55) \qquad & + (K(\mathcal{P}(\rho_h^n))(\check{\Gamma}_h^n - \hat{\Gamma}_h^n), \zeta^n).
\end{aligned}
$$

Here we have defined $\check{\Gamma}_h^n$ by

$$(56) \qquad (K(\rho^n)(\check{\Gamma}_h^n - \Gamma^n), v_h) = 0, \quad v_h \in V_h.$$

Note we have also replaced Ψ^n on the right side of (55) by its L^2 projection.

Combining (53) - (55), applying smoothness and boundedness assumptions on $K(\rho)$, superconvergence, and Holder's inequality we obtain

$$
\begin{aligned}
\frac{1}{2\triangle t} & [\|\mu^n\|^2 - \|\mu^{n-1}\|^2] + \|K(\mathcal{P}(\rho_h^n)^{1/2}\zeta^n\|^2 \\
& \leq (\frac{\mu^n - \mu^{n-1}}{\triangle t}, \mu^n) + \|K(\mathcal{P}(\rho_h^n)^{1/2}\zeta^n\|^2 \\
& \leq \frac{1}{2}\|\mu^n\|^2 + \frac{1}{2}\|\epsilon^n\|^2 + (\Pi_h \Psi^n - \hat{\Psi}_h^n, \zeta^n) \\
& \quad + ((K(\mathcal{P}(\rho_h^n)) - K(\rho^n))\check{\Gamma}_h^n, \zeta^n) + (K(\mathcal{P}(\rho_h^n))(\check{\Gamma}_h^n - \hat{\Gamma}_h^n), \zeta^n) \\
(57) \qquad & \leq C\|\mu^n\|^2 + \delta\|\zeta^n\|^2 + C\left[h^{2(k+2)} + \triangle t^2\right],
\end{aligned}
$$

where $\delta \leq K_*$. Here we have also used the fact that

$$\|\mathcal{P}(\hat{\rho}_h^n) - \rho^n\| \leq Ch^2.$$

Multiplying (57) by $\triangle t$ and summing on $n, n = 1, 2, ..., N$ and applying Gronwall's Lemma, we see that

$$(58) \quad \|\mu^N\|^2 + \sum_{n=1}^{N} \triangle t \|K(\rho_h^n)^{1/2}\zeta^n\|^2 \leq \|\mu^0\|^2 + C(\triangle t^2 + h^{2(k+2)}).$$

Using (58), Approximation Properties A, and the triangle inequality we deduce the following theroem:

THEOREM 4. *Let $V_h = V_h^0$, $\tilde{V}_h = \tilde{V}_h^0$, and $W_h = W_h^0$, that is, the RT_0 spaces defined on a tensor product grid, and define the triplet $(\rho_h^n, \Gamma_h^n, \Psi_h^n) \in W_h^k \times V_h^k \times \tilde{V}_h^k$ by (50) - (52). Take $\rho_h^0 = \hat{\rho}_h(\cdot, 0)$. Then there exists a positive constant C, independent of h such that*

$$(59) \quad \|\rho_h^N - \hat{\rho}_h^N\| + (\sum_{n=1}^{N} \triangle t \|K(\rho_h^n)^{1/2}(\hat{\Gamma}_h^n - \Gamma_h^n)\|^2)^{\frac{1}{2}} \leq C(h^2 + \triangle t).$$

We now proceed as in Section 3, and let ρ_H^n be defined by (50)-(52) with $h = H$. The proof is identical to that given above accept for the bounds on the terms T^* and T^{**}. These bounds can be modified by simply replacing H^{k+1} by H^{k+2}; i.e., since $k = 0$, we replace H by H^2.

We obtain the following theorem:

THEOREM 5. *Let $V_h = V_h^0$, $\tilde{V}_h = \tilde{V}_h^0$, and $W_h = W_h^0$ and define the one Newton correction triplet $(\tilde{\rho}_h^n, \tilde{\Gamma}_h^n, \tilde{\Psi}_h^n), \in W_h^0 \times V_h^0 \times \tilde{V}_h^0$ by (28) - (30). Assume that the Approximation Properties A holds, and take $\tilde{\rho}_h^0 = \hat{\rho}_h(\cdot, 0)$. Then there exists a positive constant C, independent of h such that*

$$\|\tilde{\rho}_h^N - \rho^N\| + (\sum_{n=1}^{N} \triangle t \|K(\rho_H^n)^{1/2}(\tilde{\Gamma}_h^n - \Gamma^n)\|^2)^{\frac{1}{2}}$$

$$(60) \quad\quad\quad \leq C(h + \triangle t + H^{3-d/2}).$$

5. Conclusions and extensions

We remark that we even though we only considered the case $k = 0$ in Section 4, we have established superconvergence for the RT_k spaces, $k \geq 0$, for the scheme (50)-(52). The two-level scheme with the Newton correction outlined in Section 3 could be extended to multiple levels with multiple corrections. We are currently investigating these possibilities. Computational results for the algorithms outlined here are also in progress.

We also remark that one may be able to improve the rate of convergence given in Theorem 5 for the lowest-order case by substituting $\mathcal{P}(\rho_H^n)$ and $\mathcal{P}(\tilde{\rho}_h^n)$ for ρ_H^n

and $\tilde{\rho}_h^n$. Preliminary theoretical results indicate that this may give superconvergence for $\tilde{\rho}_h^n$ at the centers of each cell of order $h^2 + H^{4-d/2} + \triangle t$.

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