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Multilevel Methods for Elliptic Problems with Discontinuous Coefficients in Three Dimensions

MAKSYMILIAN DRYJA

ABSTRACT. Multilevel Schwarz methods are developed for a conforming approximation of second order elliptic problems. We focus on problems in three dimensions and with possibly large jumps in the coefficients across the interface separating the subregions. We establish a condition number estimate for the iterative operator which is independent of the coefficients and grows at most as the square of the number of levels. We also characterize a class of distributions of the coefficients, called quasi-monotone, for which the weighted L^2-projection is stable and for which we can use the standard piecewise linear function to construct a coarse space. In this case, we obtain optimal methods.

1. Introduction

In this paper, we discuss methods known as BPX algorithms (cf. Bramble, Pasciak and Xu [**1**] and Xu [**9**]) or multilevel Schwarz methods with one dimensional subspaces; see Zhang [**10**], and Dryja and Widlund [**5**]. It is well known that these methods are optimal when the coefficients are regular. A challenging problem is to extend these methods to problems which have very highly discontinuous coefficients. In [**5**], the BPX method was modified and applied to a Schur complement systems. In that case the condition number of the preconditioned system is bounded by $C_1\,(1+\log{(H/h)})^2$, where H and h are the parameters of the coarse and fine triangulations, respectively. In this paper, we obtain the same estimate for multilevel additive methods with several exotic coarse spaces; see Widlund [**8**]. For multiplicative versions such as V-cycle multigrid, we obtain rates of convergence bounded from above by $1-C_2\,(1+\log H/h)^{-2}$; see further Sarkis [**6**], and Dryja, Sarkis, and

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Widlund [2]. In this paper, all constants C_i are independent of the variables appearing in the inequalities and the parameters related to meshes, spaces and, especially, the weight ρ.

This brief paper represents joint work with Marcus Sarkis and Olof Widlund and all proofs and details can be found in [2].

2. Differential and Finite Element Model Problems

We consider the following selfadjoint second order problem:
Find $u \in H_0^1(\Omega)$, such that

$$a(u,v) = f(v) \; \forall \; v \in \; H_0^1(\Omega), \tag{1}$$

where

$$a(u,v) = \int_\Omega \rho(x) \, \nabla u \cdot \nabla v \, dx \text{ and } f(v) = \int_\Omega fv \, dx \text{ for } \; f \in \; L^2(\Omega).$$

Let Ω be a bounded Lipschitz region in $\Re^3$ with a diameter of order 1. A triangulation of Ω is introduced by dividing the region into nonoverlapping simplices $\{\Omega_i\}_{i=1}^N$, with diameters of order H, which are called substructures or subdomains. This partitioning induces a coarse triangulation associated with the parameter H.

We assume that $\rho(x) > 0$ is constant in each substructure with possibly large jumps occurring only across substructure boundaries. Therefore, $\rho(x) = \rho_i = \text{const}$ in each substructure Ω_i. The analysis of the methods introduced here can easily be extended to the case when $\rho(x)$ varies moderately in each subregion.

We define a sequence of nested triangulations $\{\mathcal{T}^k\}_{k=0}^{\ell}$ as follows. We start with the coarse triangulation $\mathcal{T}^0 = \{\Omega_i\}_{i=1}^N$ and let $h_0 = H$. A triangulation $\mathcal{T}^k$ on level k is obtained by subdividing each individual element in $\mathcal{T}^{k-1}$ into several elements. The assumptions on the regularity of the refinements are standard; see Zhang [10]. For each level of triangulation, we define a finite element space $V_0^k(\Omega)$ as the space of continuous piecewise linear functions associated with the triangulation $\mathcal{T}^k$ and which vanish on $\partial\Omega$, the boundary of Ω. We denote $V_0^h(\Omega) = V_0^\ell(\Omega)$. The discrete problem associated with (1) is given by:

Find $u \in V_0^h(\Omega)$, such that

$$a(u,v) = f(v) \;\; \forall \; v \in \; V_0^h(\Omega). \tag{2}$$

The bilinear form $a(u,v)$ is directly related to a weighted Sobolev space $H_\rho^1(\Omega)$ defined by the seminorm

$$|u|^2_{H_\rho^1(\Omega)} = a(u,u).$$

We also define a weighted L^2 norm by:

$$\|u\|^2_{L_\rho^2(\Omega)} := \int_\Omega \rho(x) \; |u(x)|^2 \, dx \text{ for } u \; \in L^2(\Omega). \tag{3}$$

3. Multilevel Additive Schwarz Methods

The multilevel methods that we consider are based on the MDS-multilevel diagonal scaling introduced by Zhang [**10**], enriched with a coarse space V_{-1} as in Dryja and Widlund [**4**], and Dryja, Smith, and Widlund [**3**].

Let $\mathcal{N}^k$ be the set of nodal points associated with the space V_0^k. Let ϕ_j^k be a standard nodal basis function of V_0^k, and let $V_j^k = \text{span}\{\phi_j^k\}$. We decompose V_0^h as

$$V_0^h = V_{-1}^X + \sum_{k=0}^{\ell} V_0^k = V_{-1}^X + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{N}^k} V_j^k.$$

We note that this decomposition is not a direct sum and that $\dim(V_j^k) = 1$. Four different coarse spaces V_{-1}^X and associated bilinear forms $b_{-1}^X(u,v) : V_{-1}^X \times V_{-1}^X \to \Re$, $X = F, E, NN$, and W are considered; see next section.

We introduce operators $P_j^k : V_0^h \to V_j^k$, by

$$a(P_j^k u, v) = a(u,v) \;\; \forall\, v \in V_j^k,$$

and an operator $T_{-1}^X : V^h \to V_{-1}^X$, by

$$b_{-1}^X(T_{-1}^X u, v) = a(u,v) \;\; \forall\, v \in V_{-1}^X.$$

Let

$$T^X = T_{-1}^X + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{N}^k} P_j^k. \tag{4}$$

We now replace (2) by

$$T^X u = g, \;\; g = T_{-1}^X u + \sum_{k=0}^{\ell} \sum_{j \in \mathcal{N}^k} P_j^k u. \tag{5}$$

The equation (5) is typically solved by a conjugate gradient method. In order to estimate its rate of convergence, we need to obtain upper and lower bounds for the spectrum of T^X.

THEOREM 1. *For $u \in V_0^h(\Omega)$, we have*

$$C_3\,(1 + \log(H/h)^{-2} a(u,u) \le a(T^X u, u) \le C_4\, a(u,u).$$

4. Coarse Spaces and Bilinear Forms

Let $\mathcal{F}_{ij}$ represent the open face which is shared by two substructures Ω_i and Ω_j. Let $\mathcal{W}_i$ denote the wire basket of the subdomain Ω_i, i.e. the union of the closures of the edges of $\partial\Omega_i$. We define the *wire basket* by $\mathcal{W} = \cup \mathcal{W}_i \backslash \partial\Omega$. The sets of nodes on $\mathcal{F}_{ij}$, $\mathcal{W}$, and $\mathcal{W}_i$ are denoted by $\mathcal{F}_{ij,h}$, $\mathcal{W}_h$, and $\mathcal{W}_{i,h}$.

• **A face and wire basket based coarse space.** The first coarse space is denoted by V_{-1}^F, and is based on the wire basket $\mathcal{W}_h$ and the average over each face $\mathcal{F}_{ij,h}$. This space can conveniently be defined as the range of an interpolation operator $I_h^F : V_0^h \to V_{-1}^F$, defined by

$$I_h^F u(x)_{|\bar{\Omega}_i} = \sum_{x_p \in \mathcal{W}_{i,h}} u(x_p)\varphi_p(x) + \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} \bar{u}_{\mathcal{F}_{ij}} \theta_{\mathcal{F}_{ij}}(x).$$

Here, $\varphi_p(x)$ is the discrete harmonic function into Ω_i which equals 1 at x_p and vanishes elsewhere on $\partial\Omega_{i,h}$. $\bar{u}_{\mathcal{F}_{ij}}$ is the average value of u on $\mathcal{F}_{ij,h}$, and $\theta_{\mathcal{F}_{ij}}(x)$ the discrete harmonic function in Ω_i which equals 1 on $\mathcal{F}_{ij,h}$ and is zero on $\partial\Omega_{i,h} \backslash \mathcal{F}_{ij,h}$.

We define the bilinear form by

$$\begin{aligned} b_{-1}^F(u,u) &= \sum_i \rho_i \{ \sum_{x_p \in \mathcal{W}_{i,h}} h\,(u(x_p) - \bar{u}_i)^2 \\ &\quad + H(1 + \log H/h) \sum_{\mathcal{F}_{ij} \subset \partial\Omega_i} (\bar{u}_{\mathcal{F}_{ij}} - \bar{u}_i)^2 \}, \end{aligned}$$

where $\bar{u}_i$ is the average of the discrete values of u over $\partial\Omega_{i,h}$.

- **A face, edge, and vertex based coarse space.** We can decrease the dimension of the coarse space given above and define another coarse space denoted by V_{-1}^E. Rather than using the values of all the nodes on the edges as degrees of freedom, only one degree of freedom per edge, an average value is used; see [**3**].
- **A Neumann-Neumann coarse space.** We can also consider the coarse space V_{-1}^{NN}; see [**4**]. This space is of minimal dimension with only one degree of freedom per substructure.
- **A wire basket based coarse space.** Finally, we consider a coarse space V_{-1}^W, due to Barry Smith see [**7**], or [**3**]. It is based only on the wire basket $\mathcal{W}_h$.

REMARK 1. *We can decrease the complexity of our algorithm by considering approximate discrete harmonic extension given by simple explicit formulas in* [**2**].

5. Quasi-Monotone Coefficients and an Optimal Algorithm

In this section, we indicate that if the coefficients ρ_i satisfy certain assumptions, the L_ρ^2-projection is stable and we can use the space of piecewise linear functions $V^H(\Omega)$ as a coarse space to obtain an *optimal* multilevel preconditioner.

Let $\{\mathcal{V}_m\}_{m=1}^M$ be the set of substructure vertices. We also include the vertices that are on $\partial\Omega$. Let Ω_{m_i}, $i = 1, \cdots, s(m)$, denote the substructures that have the vertex $\mathcal{V}_m$ in common, and let the ρ_{m_i} denote their coefficients. Let Ω_m' be the interior of the closure of the union of these substructures Ω_{m_i}, i.e. the interior of $\cup_{i=1}^{s(m)} \bar{\Omega}_{m_i}$. By using the fact that all substructures are simplices, we see that each Ω_{m_i} has a whole face in common with $\partial\Omega_m'$. Thus, the vertex $\mathcal{V}_m$ is the only internal cross point in $\bar{\Omega}_m'$.

DEFINITION 1. *For each Ω_m', we order its substructures such that*

$$\rho_{m_1} = \max_{i:1,\cdots,s(m)} \rho_{m_i}.$$

We say that a distribution of ρ_{m_i} is quasi-monotone on Ω'_m if for each $i = 1, \cdots, s(m)$, there exists a sequence $i_j,\ j = 1, \cdots, R$, with

$$\rho_{m_i} = \rho_{m_{i_R}} \leq \cdots \leq \rho_{m_{i_{j+1}}} \leq \rho_{m_{i_j}} \leq \cdots \leq \rho_{m_{i_1}} = \rho_{m_1}, \tag{6}$$

where the substructures $\Omega_{m_{i_j}}$ and $\Omega_{m_{i_{j+1}}}$ have a face in common. If the vertex $\mathcal{V}_m \in \partial\Omega$, then we additionally assume that $\partial\Omega_{m_1} \cap \partial\Omega$ contains a face for which $\mathcal{V}_m$ is a vertex.

A distribution ρ_i on Ω is quasi-monotone if it is quasi-monotone on each Ω'_m.

THEOREM 2. *For a quasi-monotone distribution of the coefficients on Ω, we have*

$$\|(I - Q^H_\rho)u\|_{L^2_\rho(\Omega)} \preceq H\,|u|_{H^1_\rho(\Omega)}\ \forall u \in V^h_0(\Omega). \tag{7}$$

Here, Q^H_ρ is the weighted L^2-projection from $V^h_0(\Omega)$ to $V^H_0(\Omega)$.

THEOREM 3. *Let $T^H = T^X$ be defined by (4) with $V^X_{-1} = V^H(\Omega)$ and $b_{-1}(\cdot,\cdot) = a(\cdot,\cdot)$. For a quasi-monotone distribution of the coefficients on Ω, we have*

$$C_5\, a(u,u) \leq a(T^H u, u) \leq C_6\, a(u,u)\ \forall u \in V^h_0(\Omega).$$

REMARK 2. *The analysis can be extended to problems with Neumann or mixed boundary conditions, and quasi-monotone coefficients. In this case, we also obtain an optimal method.*

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DEPARTMENT OF MATHEMATICS, WARSAW UNIVERSITY, BANACHA 2, 02-097 WARSAW, POLAND.
E-mail address: dryja@mimuw.edu.pl