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Domain-oriented multilevel methods

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ABSTRACT. For the discretization of elliptic linear PDE's, instead of the usual nodal basis, we use a generating system that contains the nodal basis functions of the finest and all coarser levels. The Galerkin approach now results in an enlarged semidefinite linear system to be solved. Traditional iterative methods for that system turn out to be equivalent to modern multilevel methods for the fine grid system. Besides level-oriented iterative methods that lead to multilevel algorithms, other orderings of the unknowns of the enlarged system can be considered as well. A domain-wise block Gauss-Seidel iteration for the enlarged system results in a certain domain decomposition method with convergence rates independent of the mesh width of the fine grid. Furthermore, this approach directly leads to a $O(1)$-preconditioner for the Schur complement that arises in conventional domain decomposition methods.

1. The Generating System

Consider a partial differential equation with linear, symmetric and elliptic operator $Lu = f$ in Ω, with Dirichlet boundary conditions and associated weak formulation $a(u,v) = f(v), \forall v \in V$. For the discretization on some grid Ω_k with uniform mesh width $h_k = 2^{-k}$ usually a basis $B_k = \{\phi_i^{(k)}, i = 1,..,n_k\}$ with nodal basis functions $\phi_i^{(k)}$ is used, that span the corresponding space $V_k = \text{span}\{\phi_i^{(k)}, i = 1,..,n_k\}$. Here, n_k denotes the number of interior grid points and thus the dimension of V_k. Any function $u \in V_k$ can be denoted by

$$u = \sum_{i=1}^{n_k} u_{k,i} \cdot \phi_i^{(k)}$$

with corresponding coefficient vector $u_k^B = (u_{k,i})_{i=1,..,n_k}$ of nodal values. Now, the Galerkin approach leads to the linear system

$$L_k^B u_k^B = f_k^B \tag{1}$$

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with the vector u_k^B of unknowns.

In the context of multilevel methods for the iterative solution of (1), a sequence $\Omega_1, \Omega_2, ..., \Omega_2$ of grids, with associated sequence $B_1, B_2, ..., B_k$ of nodal bases and corresponding spaces $V_1, V_2, ..., V_k$ with dimensions $n_1, n_2, ..., n_k$ is employed. Inspired by that, we now will use directly the generating system

$$E_k = B_1 \cup B_2 \cup ... \cup B_k = \bigcup_{l=1}^{k} B_k$$

for the representation of functions in V_k and for the discretization process. Compare also [**2, 4**]. This corresponds to the level-wise splitting $V_k = \sum_{l=1}^{k} V_l$ of the underlying discretization space V_k. Since E_k is only a generating system and not a basis, the representation of any function $u \in V_k$ by

$$u = \sum_{l=1}^{k} \sum_{i=1}^{n_l} u_{i,l} \cdot \phi_i^{(l)}$$

with the enlarged vector $u_k^E = (u_1^{B^T}, u_2^{B^T}, ..., u_k^{B^T})^T$ is not unique any more.

For the simple 1D case, Figure 1 shows the functions contained in E_3 and one example for a multilevel representation of a function.

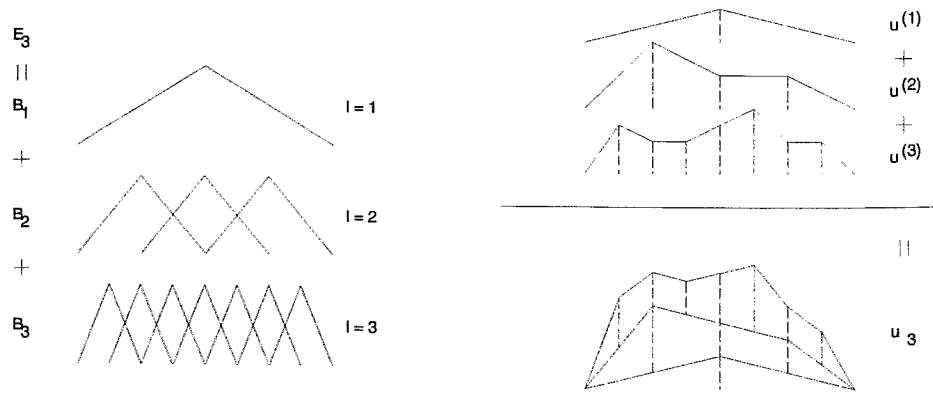


FIGURE 1. The generating system E_3 in 1D (left) and the multilevel representation of u_3 by E_3 (right).

2. The Semidefinite System

Now, we use the generating system E_k directly in the Galerkin discretization process. Then, we obtain the enlarged linear system

$$L_k^E u_k^E = f_k^E \tag{2}$$

with semidefinite matrix L_k^E where for $i_1 = 1, .., n_{l_1}$, $i_2 = 1, .., n_{l_2}$ and $l_1, l_2 = 1, .., k$

$$(L_k^E)_{i_1,i_2,l_1,l_2} = a(\phi_{i_1}^{(l_1)}, \phi_{i_2}^{(l_2)}) \quad \text{and} \quad (f_k^E)_{i_2,l_2} = f(\phi_{i_2}^{(l_2)}).$$

This linear system is of size $n_k^E = \sum_{l=1}^{k} n_l$, which is in 1D about 2 times, in 2D about 4/3 times and in 3D about 8/7 times larger than n_k, i.e. the size of (1).

Assuming a level-oriented ordering of the unknowns, we obtain the following structure for L_k^E (here for the simple example of $k = 3$):

$$L_k^E = \begin{pmatrix} R_3^1 L_3^B P_1^3 & R_3^1 L_3^B P_2^3 & R_3^1 L_3^B \\ R_3^2 L_3^B P_1^3 & R_3^2 L_3^B P_2^3 & R_3^2 L_3^B \\ L_3^B P_1^3 & L_3^B P_2^3 & L_3^B \end{pmatrix} = \begin{pmatrix} R_3^1 \\ R_3^2 \\ I_3 \end{pmatrix} \cdot L_3^B \cdot \begin{pmatrix} P_1^3 & P_2^3 & I_3 \end{pmatrix}$$

where $R_i^j = P_j^{iT}$ and P_j^i denotes the interpolation/prolongation from V_j to V_i, $j < i$, and I_i denotes the identity in V_i. Note that $P_j^i = \prod_{q=1}^{i-j} P_{i-q}^{i-q+1}$. Thus, we see that with help of the matrix

$$S_k = \begin{pmatrix} P_1^k & P_2^k & \cdots & P_{k-1}^k & I_k \end{pmatrix}$$

our enlarged system (2) can be written as

$$S_k^T L_k^B S_k u_k^E = S_k^T f_k^B.$$

Now, we see that the discrete Galerkin operators $L_l^B = R_k^l L_k^B P_l^k, l = 1, .., k$, i.e. the stiffness matrices of every level of discretization, are contained as diagonal blocks. The couplings between different levels are contained in the outer diagonal blocks.

Note that our enlarged system is consistent, i.e. $\text{rank}(L_k^E) = \text{rank}(L_k^E, f_k^E)$, and therefore solvable. There exist many different solutions due to the semidefiniteness of L_k^E. Since the unique solution u_k^B of (1) can be obtained from *any* solution u_k^E of (2) by $u_k^B = S_k u_k^E$, the idea is now to produce some u_k^E for (2) by a traditional iterative method and to apply S_k. This will be studied in the following sections.

3. Level-Oriented Methods

In the previous example we employed a level-wise ordering of the unknowns u_k^E that resulted in a level-block partitioning of the matrix L_k^E and the system (2) and was associated with the splitting $V_k = \sum_{l=1}^{k} V_l = \sum_{l=1}^{k} \sum_{i=1}^{n_l} V_{l,x_i}$, where $V_{l,x_i} = \text{span}\{\phi_i^{(l)}\}$.

It can be seen easily that traditional iterative methods for (2) are equivalent to modern multilevel methods for (1), c.f. [**2, 4**]. Note that this has also been shown in [**10**] (in a slightly different but equivalent language). For instance, the simple Jacobi-preconditioner for (2) is equivalent to the BPX-preconditioner [**1**] for (1). The BPX-preconditioner can be written as $BPX_k = S_k D_k^{E^{-1}} S_k^T$, where $D_k^E = \text{diag}(L_k^E)$. Now, if we define the generalized condition number κ of a positive semidefinite matrix to be the quotient of the largest and non-vanishing smallest eigenvalue, we obtain directly

$$\kappa(BPX_k L_k^B) = \kappa(S_k D_k^{E^{-1}} S_k^T L_k^B) = \kappa(D_k^{E^{-1}} S_k^T L_k^B S_k) = \kappa(D_k^{E^{-1}} L_k^E)$$

and since $\kappa(BPX_k L_k^B) = O(1)$ (see [**6, 7, 10, 13**]) we have

$$\kappa(D_k^{E^{-1}} L_k^E) = O(1). \tag{3}$$

Thus, the Jacobi-preconditioned CG-method for (2) converges to some solution within a number of iterations that is independent of k.

Furthermore, we can consider Gauss-Seidel-type methods for (2). They are equivalent to multigrid methods with a Gauss-Seidel smoother, c.f. [**2, 4**]. For example, the simple Gauss-Seidel iteration on (2) with level-wise ordering $l = 1, .., k$ corresponds to the multigrid (0,1)-V-cycle with one post-smoothing step by Gauss-Seidel. The symmetric Gauss-Seidel-iteration corresponds to the (1,1)-V-cycle. Here, an outer iteration switches from level to level and an inner iteration operates on the specific grids.

The convergence rate of the Gauss-Seidel iteration on (2) can be estimated by

$$\rho = \sqrt{1 - K_0/(1 + K_1)^2} = 1 - O(1).$$

Here, $K_0 := \lambda_{\min_{\neq 0}}(\tilde{L}_k^E) \geq c_0 > 0$ (c.f. (3)), where $\tilde{L}_k^E = {D_k^E}^{-1/2} L_k^E {D_k^E}^{-1/2}$ and c_0 is some constant that is independent of k.

Furthermore, $K_1 := \|\tilde{F}_k^E\|_2$ where $\tilde{F}_k^E$ is given as the lower triangle part of $\tilde{L}_k^E$. With help of the Cauchy-Schwarz inequality (see, e.g. [**4, 10, 12, 13**]), K_1 can be estimated from above by some constant independent of k. We get $K_1 \leq \lambda_{max}(|\tilde{L}_k^E|) \leq c_1 < \infty$, where $|\tilde{L}_k^E|$ denotes the matrix that is produced from L_k^E by taking the absolute value of each entry. Note that with $\lambda_{\max}(\tilde{L}_k^E) \leq \lambda_{\max}(|L_k^E|)$, c_1 is an upper bound for the largest eigenvalue of the Jacobi-preconditioned matrix, i.e. $\tilde{L}_k^E$, as well.

In addition, $K_1 \leq c_1$ holds for *all* possible Gauss-Seidel traversal orderings, c.f. [**4**]. Therefore, we obtain a k-independent convergence rate not only for the Gauss-Seidel method for (2) with some level-weise traversal ordering that corresponds to a multigrid method, but also for *any other* traversal ordering as well. This will be exploited in the sequel.

4. Domain-Oriented Methods

Now, we consider domain-oriented Gauss-Seidel iterations for (2). We assume a decomposition on Ω in J non-overlapping subdomains $\Omega_j, j = 1, .., J$, i.e. $\Omega = \bigcup_{j=1}^J \Omega^j$ with mutually disjoint interiors so that no grid point lies on an internal boundary and split the grid points $\Omega_l = \bigcup_{j=1}^J \Omega_l^j$ on each level $l = 1, .., k$ accordingly. Then, we group together the associated functions of E_k and the unknowns of u_k^E that belong to the same subdomain. The system (2) is partitioned analogously. This corresponds to the splitting

$$V_k = \sum_{j=1}^{J} \left(\sum_{l=1}^{k} \sum_{x \in \Omega_l^j} V_{l,x} \right)$$

where j runs over the domains, l runs over the levels and x runs over the respective grid points. Note that in comparison to the level-wise splitting of the previous section the order of summation is exchanged. The term in parenthesis corresponds

now to the block structuring of system (2). Figure 2 shows a 2D example with four subdomains and $k = 3$.

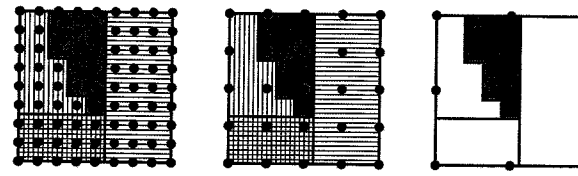


FIGURE 2. Domain-oriented regrouping of E_3 in 2D.

Now, we perform a block Gauss-Seidel iteration for the block partitioned system (2). Then, an outer iteration switches from subdomain to subdomain. If we treat the arising subdomain problems by one inner Gauss-Seidel iteration where within each block a level-wise ordering of the unknowns is applied, we obtain in a natural way a *local* multigrid method (i.e. a local (0,1)-V-cycle). Altogether, this results in a Gauss-Seidel iteration for the overall system (2) with just a domain-wise traversal ordering. Since we have seen in the last section that the upper bound $c_1 < \infty$ of K_1 is independent of the traversal ordering, we directly obtain that the convergence rate of our domain-oriented Gauss-Seidel method is independent of k as well.

For the simple 1D case, Figure 3 shows the methodological difference between the level- and domain-oriented methods.

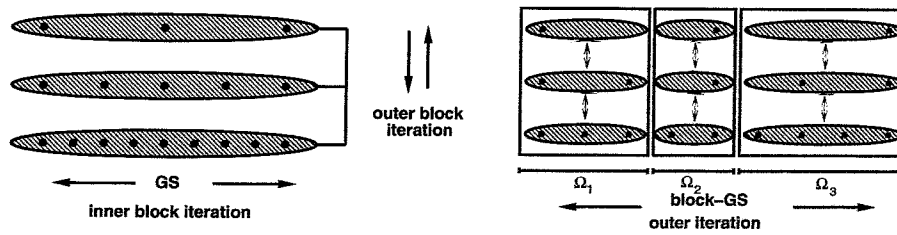
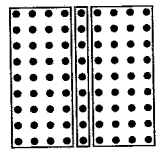


FIGURE 3. The level-oriented (left) and domain-oriented (right) GS-methods for (2) in 1D, $k = 3$.

Note that for the inner block iteration some alternatives to the level-wise traversal ordering exist. If we apply the domain decomposition principle *recursively* until in every domain only one grid point is contained, we obtain the so called point-block method as described in [**3, 4**]. Furthermore, if we restrict ourselves in each subdomain to the subsystem belonging to B_k (and keep the unknowns that belong to $B_l, l < k$ fixed) we can apply exact solvers as well. Note however, that for the outer block iteration all degrees of freedom of E_k take part in the residual computations. In contrast to the conventional domain decomposition method this allows information to travel over long distances as well and maintains fast multigrid-like convergence rates.

5. Schur Complement Preconditioning

Now, we use the generating system approach to derive a simple preconditioner for the Schur complement problem arising in conventional domain decomposition methods. This preconditioner results in a condition number that is independent of k.



For the ease of explanation only, we restrict ourselves to the simple situation depicted left. There, the grid points are split by the the middle line separator into the set of points Ω_k^2 situated on the separator and the set of remaining grid points $\Omega_k^1 = \Omega_k \backslash \Omega_k^2$ that belong to the interior to the two resulting subdomains.

The nodal basis system (1) is partitioned correspondingly, i.e.

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

where u_2 belongs to the separator and u_1 to the interior of the left and right subdomain. Then, the Schur complement reads $K_{22} = L_{22} - L_{21} L_{11}^{-1} L_{12}$.

Now, we will use the corresponding part B_k^1 of the nodal basis B_k for the grid points Ω_k^1 but the corresponding part E_k^2 of the generating system E_k for the separator. This results in a smaller generating system $\hat{E}_k = B_k^1 \cup E_k^2$. Figure 4 shows the center points and the supports of the contained functions.

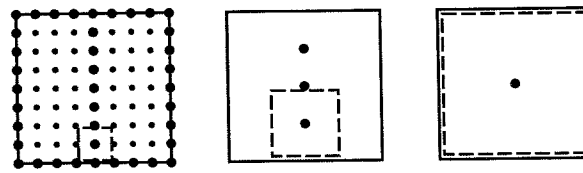


FIGURE 4. Support of the functions of $\hat{E}_k$.

Now, using $\hat{E}_k$, the Galerkin approach results in the semidefinite system

$$\begin{pmatrix} L_{11} & L_{12}^E \\ L_{21}^E & L_{22}^E \end{pmatrix} \begin{pmatrix} u_1 \\ u_2^E \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2^E \end{pmatrix}$$

and, since L_{11} involves only B_k^1 and is therefore invertible, we obtain the enlarged semidefinite Schur complement

$$K_{22}^E = L_{22}^E - L_{21}^E L_{11}^{-1} L_{12}^E.$$

Now, we could work directly with this semidefinite enlarged Schur complement K_{22}^E and the associated linear system $K_{22}^E u_2^E = f_2^E - L_{21}^E L_{11}^{-1} f_1$ like previously with L_k^E and (2). Moreover, we could apply level- or domain-oriented Gauss-Seidel methods that would give us a non-unique solution of the Schur complement system, etc.

Here, however, we want to derive a preconditioner for K_{22}. A short calculation gives

$$K_{22}^E = \hat{S}_k^T \cdot (L_{22} - L_{21} L_{11}^{-1} L_{12}) \cdot \hat{S}_k^E = \hat{S}_k^T \cdot K_{22} \cdot \hat{S}_k^E$$

with $\hat{S}_k = S_k|_{E_k^2} : E_k^2 \to B_k^2$, i.e. the application of S_k to the separator only. Analogously to the BPX-preconditioner we can derive a preconditioner for K_{22} from K_{22}^E. We obtain

$$C_{22} = \hat{S}_k \hat{D}_k^{-1} \hat{S}_k^T \quad \text{with} \quad \hat{D}_k = \text{diag}(L_{22}^E).$$

Note that this construction principle (use the nodal basis in the interior of the subdomains but the subpart of the generating system E_k on the separator) works in the higher dimensional case as well. An analysis in [**5, 8**] shows $\kappa(C_{22}K_{22}) = O(1)$. A similar construction is given in [**9**]. See also [**11**], remark 10.3.

For our simple 2D example above and $L = \Delta$, Table 1 gives the condition numbers of K_{22} and $C_{22}K_{22}$ for different values of k. We clearly see that $\kappa(K_{22})$ behaves like $O(h_k^{-1})$ whereas $\kappa(C_{22}K_{22})$ behaves for sufficiently large k practically like $O(1)$. (The slight increase of the condition number is similar to that observed for the BPX-preconditioner in 2D.)

TABLE 1. Condition numbers of K_{22} and $C_{22}K_{22}$ for different k.

k	2	3	4	5	6	7	8
$\kappa(K_{22})$	1.95	3.84	7.64	15.26	30.51	61.02	122.04
$\kappa(C_{22}K_{22})$	1.65	2.08	2.42	2.67	2.85	2.99	3.09

REFERENCES

1. J. Bramble, and J. Pasciak and J. Xu, *Parallel multilevel preconditioners*, Math. Comp., 31, (1990), 333-390
2. M. Griebel, *Multilevel algorithms considered as iterative methods on indefinite systems*, SFB-Report 342/29/91 A, TUM-I9143, Institut für Informatik, TU München, 1991, also SIAM J. Sci. Comput., 15(3), (1994)
3. M. Griebel, *Grid- and point-oriented multilevel algorithms*, in Notes on Numerical Fluid Mechanics, 41, (1993), 32-46, Vieweg Verlag, Braunschweig
4. M. Griebel, *Multilevelmethoden als Iterationsverfahren über Erzeugendensystemen*, Teubner Skripten zur Numerik, (1994), Teubner Verlag, Braunschweig
5. P. Oswald, *Two remarks on multilevel preconditioners*, report Nr. Math/91/1, (1991), Mathematische Fakultät, FSU Jena
6. P. Oswald, *Norm equivalencies and multilevel Schwarz preconditioning for variational problems*, report Math/92/1, (1992), Mathematische Fakultät, FSU Jena
7. P. Oswald, *On discrete norm estimates related to multilevel preconditioners in the finite element method*, Proc. Int. Conf. Constr. Theory of Functions, (1991), Varna
8. P. Oswald, *Stable splittings of Sobolev spaces and fast solution of variational problems*, report Math/92/5, (1992), Mathematische Fakultät, FSU Jena
9. C. Tong, and T. Chan and C. Kuo, *A domain decomposition preconditioner based on a change to the multilevel nodal basis*, SIAM J. Sci. Stat. Comput., 12, (1991), 1486-1495
10. J. Xu, *Iterative methods by space decomposition and subspace correction: a unifying approach*, SIAM Review, 34(4), (1992), 581-613
11. J. Xu, *Theory of multilevel methods*, Report AM48, (1989), Department of Mathematics, Penn State Univ.
12. H. Yserentant, *Old and new convergence proofs for multigrid methods*, Acta Numerica, (1993)
13. X. Zhang, *Multilevel Schwarz methods*, Numer. Math., 63, (1992), 521-539

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