Contemporary Mathematics
Volume 180, 1994

AN EFFICIENT COMPUTATIONAL METHOD FOR THE FLOW PAST AN AIRFOIL

GEORGE C. HSIAO, MICHAEL D. MARCOZZI and SHANGYOU ZHANG

ABSTRACT. A finite element–boundary element coupling procedure is applied to the computation of an incompressible flow past an airfoil. By utilizing a representation of the potential flow exterior to a circular auxiliary boundary, the reduced variational problem on an annular region is solved by the finite elment method alone. Two multigrid algorithms are introduced for the finite element equations. Both methods are optimal in the order of computation. The singularity at the trailing edge of the numerical solutions is corrected by the Kutta–Joukowski condition. Detailed numerical implementation is presented.

We consider a steady uniform two–dimensional fluid flow past a thin airfoil. As is customary, to arrive at a potential flow analysis, the compressibility and viscosity of the fluid are neglected, and the flow is assumed to be irrotational. The problem can then be formulated as an exterior boundary value problem for the velocity field $\mathbf{q} = (q_1, q_2)$:

$$
(1)\qquad
\begin{aligned}
\nabla \cdot \mathbf{q} := \frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} &= 0 && \text{in } \Omega^c := \mathbf{R}^2 \setminus \Omega \cup \Gamma,\\
\nabla \times \mathbf{q} := \frac{\partial q_2}{\partial x_1} - \frac{\partial q_1}{\partial x_2} &= 0 && \text{in } \Omega^c,\\
\mathbf{q} \cdot \mathbf{n} &= 0 && \text{on } \Gamma,\\
\mathbf{q} - \mathbf{q}_\infty &= o(1) && \text{as } |x| \to \infty,\\
\lim_{\mathbf{x} \to \mathbf{T}} |\mathbf{q}(\mathbf{x})| = |\mathbf{q}(\mathbf{T})| &< \infty && \text{(Kutta–Joukowski condition).}
\end{aligned}
$$

Here Γ is the profile of the thin airfoil Ω with one corner point at the trailing edge $\mathbf{T}$ (see Figure 1), and $\mathbf{q}_\infty$ is the given free stream velocity.

Alternatively, we can introduce a stream function ψ such that $\mathbf{q} = (\nabla\psi)^\perp := \left(\frac{\partial \psi}{\partial x_2}, -\frac{\partial \psi}{\partial x_1}\right)$. Denoting by $u(\mathbf{x})$ the disturbance stream function due to the airfoil, and setting $\psi(\mathbf{x}) = \psi_\infty(\mathbf{x}) + u(\mathbf{x})$, where $\psi_\infty(x) = -\mathbf{q}_\infty^\perp(\mathbf{x}) \cdot \mathbf{x}$, we may reformulate the problem (1) as an exterior boundary value problem for $u(\mathbf{x})$:

$$
(2)\qquad
\begin{aligned}
-\Delta u &= 0 && \text{in } \Omega^c,\\
u &= -\psi_\infty && \text{on } \Gamma,\\
u + \frac{\kappa}{2\pi} \log|\mathbf{x}| &= C_0 + o(1) && \text{as } |x| \to \infty.
\end{aligned}
$$

In this formulation, however, both constants C_0 and κ are unknown. Physically, κ is the circulation around the airfoil: $\kappa = \int_\Gamma \mathbf{q} \cdot d\mathbf{x}$, which will be determined

1991 *Mathematics Subject Classification.* 65N55, 65N30. 65F10.
Key words and phrases. Finite element, boundary element, Multigrid method.
This paper is in final form and no version of it will be submitted for publication elsewhere.

by the Kutta–Joukowski condition in (1). Following the principle of the finite element – boundary element coupling procedure in [6] and [8], we introduce a circular boundary Γ_0 of radius a as shown in Figure 1. Let $u = u_F$ in the annular (finite element) region Ω_F, and $u = u_B$ in the exterior (boundary element) region Ω_B. The coupling of the two parts of u is by the following transmission conditions:

$$(3) \qquad \frac{\partial u_F^-}{\partial n} = \frac{\partial u_B^+}{\partial n} =: \sigma \qquad \text{and} \qquad u_F^- = u_B^+$$

on Γ_0. By representing $u_B(\mathbf{x}) = \int_{\Gamma_0} \frac{\partial \gamma}{\partial \mathbf{n}_y}(\mathbf{x},\mathbf{y}) u_F^-(\mathbf{y})\, ds_{\mathbf{y}} - \int_{\Gamma_0} \gamma(\mathbf{x},\mathbf{y})\sigma(\mathbf{y})\, ds_{\mathbf{y}} + C_0$ in Ω_B, we obtain the following coupled differential – boundary integral system:

$$(4) \qquad \begin{cases} -\Delta u = 0 \quad \text{in } \Omega_F, \quad u_F|_\Gamma = -\psi_\infty, \quad \left.\frac{\partial u_F^-}{\partial n}\right|_{\Gamma_0} = \sigma, \\ \frac{1}{2}u_F^- - Ku_F^- + V\sigma - C_0 = 0 \quad \text{on } \Gamma_0 \\ \int_{\Gamma_0} \sigma = -\kappa, \qquad + \text{Kutta–Joukowski condition.} \end{cases}$$

Here $\gamma(\mathbf{x},\mathbf{y}) = (-1/2\pi)\log|\mathbf{x}-\mathbf{y}|$ is the fundermental solution, V and K are the simple–layer and double–layer potential operators on Γ_0 respectively (see, e.g., [6]).

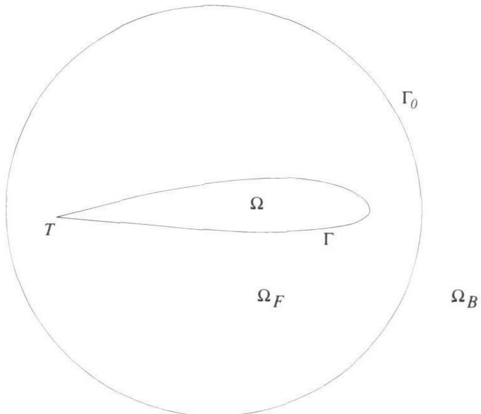

Figure 1. An airfoil and the auxiliary domains.

Because of the specially chosen auxiliary boundary Γ_0, the integral equation in (4) can be inverted exactly at the continuous level. Therefore the problem (4) is reduced to the following variational problem (cf. [7] and [8]): Find $u_F \in H^1_{-\psi_\infty}(\Omega_F) \bigcap H^2(\Omega_F)$ and $\kappa \in \mathbf{R}^1$ such that

$$(5) \qquad a(u_F, v) + 2\langle V\dot{u}_F, \dot{v}\rangle = f(v) \qquad \forall v \in H_0^1(\Omega_F),$$

where $f(v) = \langle -\frac{\kappa}{2\pi a}, v\rangle$. Here, $\dot{v}$ denotes the tangential derivative (on Γ_0), $a(u,v) = \int_{\Omega_F} \nabla u \cdot \nabla v\, dx$, $\langle v, \chi\rangle = \int_{\Gamma_0} v\chi\, ds$, $H_g^1(\Omega_F) = \{v \in H^1(\Omega_F) \mid v_{|\Gamma} = g(\mathbf{x})\}$, and $H^m = W^{m,2}$ are standard Sobolev spaces (cf. [1]). The Kutta–Joukowski condition ensures that the correct solution to (5) has to be a regular function even near the trailing edge $\mathbf{T}$. Therefore by assuming $u_F \in H^2(\Omega_F)$, the problems (4) and (5) are equivalent. For complete analysis of (4–5), we refer readers to [8].

To find the correct constant κ such that $u_F \in H^2(\Omega_F)$, we replace (5) by the following two basic problems: Find $u_i \in H_0^1(\Omega_F)$ for $i = 0, 1$, such that

$$(6) \quad a(u_i, v) + 2\langle V\dot{u}_i, \dot{v}\rangle = \begin{cases} a(\psi_\infty, v) + 2\langle V\dot{\psi}_\infty, \dot{v}\rangle & \text{if } i = 0, \\ \langle -1/2\pi a, v\rangle & \text{if } i = 1, \end{cases} \quad \forall v \in H_0^1(\Omega_F).$$

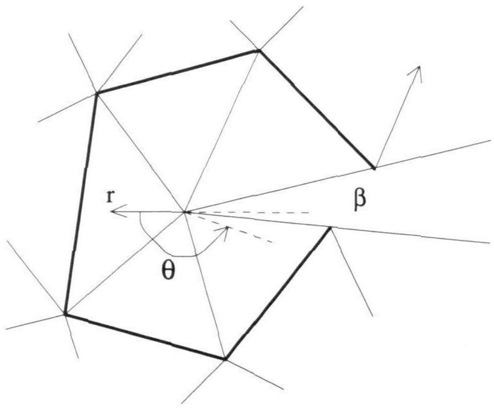

Figure 2. The polar coordinates near the trailing edge **T**.

We separate the singularity terms (at the trailing edge) from the solutions by $u_i = c_i u_s +$ (regular terms) (see [5] and [8]), where

$$u_s = r^{\frac{\pi}{2\pi-\beta}} \cos \frac{\pi\theta}{2\pi-\beta} \tag{7}$$

where polar coordinates are used at **T** (see Figure 2). Then $u_F = u_0 + \kappa u_1$ has no singular term u_s if $\kappa = -c_1/c_0$ can be found. The stress intensity factors c_0 and c_1 are computed according to a conventional method (see, e.g., [9]) in our computation, which will be presented below.

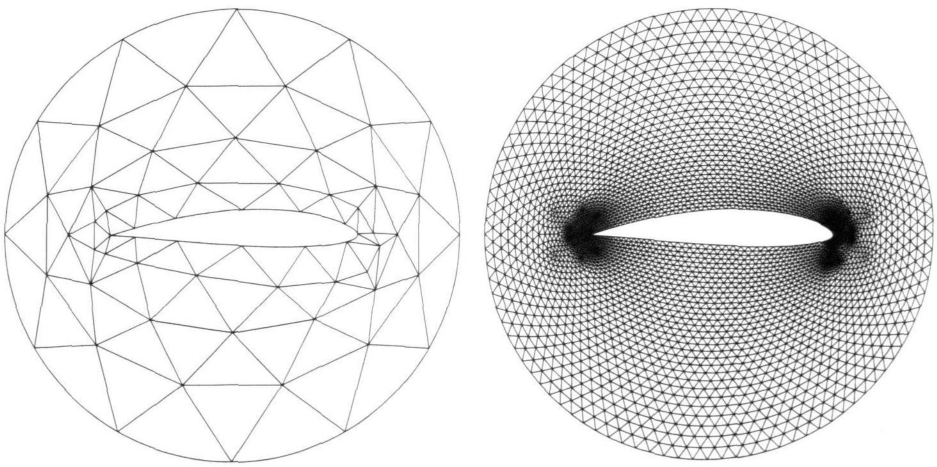

Figure 3. Nonnested triangulations (level 1 and 4) on Ω_F.

To discretize (6), we use piecewise linear finite elments (cf. [4]) on a family of nonnested triangulations (see Figure 3) $\{\mathcal{T}_k\}$, $\mathcal{T}_{k-1} \not\subset \mathcal{T}_k$ which are graded toward the trailing edge **T**. We refer readers to [3] and [10] for references on the multigrid method on nonnested grids. Defining V_k to be the intersection of the k-th level finite element space and $H_0^1(\Omega_F)$, (6) reads: Find $u_{i,k} \in V_k$, such that

$$A_k u_{i,k} + B_k V \dot{u}_{i,k} = f_i, \tag{8}$$

where $(A_k u, v) = a(u, v)$, $(B_k u, v) = 2\langle V\dot{u}_i, \dot{v}\rangle$, $f_0 = a(\psi_\infty, v) + 2\langle V\dot{\psi}_\infty, \dot{v}\rangle$ and $f_1 = \langle -1/2\pi a, v\rangle$ for all $u, v \in V_k$. We have two multigrid algorithms (cf. [7] for details) for solving the linear system (8). In one algorithm, we apply the multigrid method to the operator $A_k + B_k$. The operator $A_k + B_k$ is shown to be symmetric and positive definite in [7].

Definition 1 (A direct multigrid method). Given w_0 approximating the solution u_i in (8), one k–th level multigrid iteration produces a new approximation w_{m+1} as follows. First, m smoothings are performed:

$$w_l = w_{l-1} + \rho(C_k)^{-1}\left(f_i - C_k w_{l-1}\right), \qquad l = 1, \cdots, m, \tag{9}$$

where $C_k = A_k + B_k$, and $\rho(C_k)$ stands for the spectral radius of C_k or an upper bound of it. Then we solve the residual problem on the coarse level

$$C_{k-1} q = I_k^\top \left(f_i - C_k w_m\right) \tag{10}$$

by doing $p(> 1)$ $(k-1)$–st level multigrid iterations to get $\tilde{q}$. Finally

$$w_{m+1} = \begin{cases} w_m + I_k \tilde{q} & \text{if } k > 1, \\ w_m & \text{if } k = 1. \end{cases} \tag{11}$$

Here the I_k in (11) is the Lagrange nodal interpolation operator, and the $I_k^\top$ in (10) is the adjoint operator of I_k under the L^2 inner–product. This operator I_k is needed since the multilevel finite element spaces are not nested: $V_{k-1} \not\subset V_k$, caused by the nonnested grids and the curved boundaries.

In the second algorithm, we apply the multigrid method only to A_k, which is a discrete Laplacian. That is, from an iterative solution $u_{i,k}^{(j)} \in V_k$, the new solution is

$$u_{i,k}^{(j+1)} = u_{i,k}^{(j)} + \omega\epsilon, \tag{12}$$

where $\omega \le 1$ is a relaxation parameter and $\epsilon \in V_k$ is a solution for the following residual problem:

$$A_k \epsilon = f_i - A_k u_{i,k}^{(j)} - B_k u_{i,k}^{(j)}. \tag{13}$$

Definition 2 (A double iterative multigrid method). Given $u_{i,k}^{(j)}$ approximating the solution u_i in (8), one outer iteration produces a new approximation $u_{i,k}^{(j+1)}$ as defined in (12), where ϵ is obtained by doing $n(\ge 1)$ (inner) multigrid iteration(s) with initial guess 0. Here the (inner) multigrid iteration is defined in Definition 1 where f_i is replaced by the right hand side function in (13) and C_k replaced by A_k.

In [7], we proved that the speed of convergence for the multigrid method defined by Definition 1 is constant independent of the number of unknows in the linear system. Also in [7], we proved that the operator $A_k^{-1}(A_k + B_k)$ is well–conditioned. By the standard technique of [2], we have the following theorem of the optimal order of computation for the two multigrid methods (cf. [7]).

Theorem 1. *For both multigrid algorithms defined in Definitions 1 and 2, the number of arithmetic operations for solving the linear system (8), up to the order of truncation error, is proportional to the number of unknowns in the system.* □

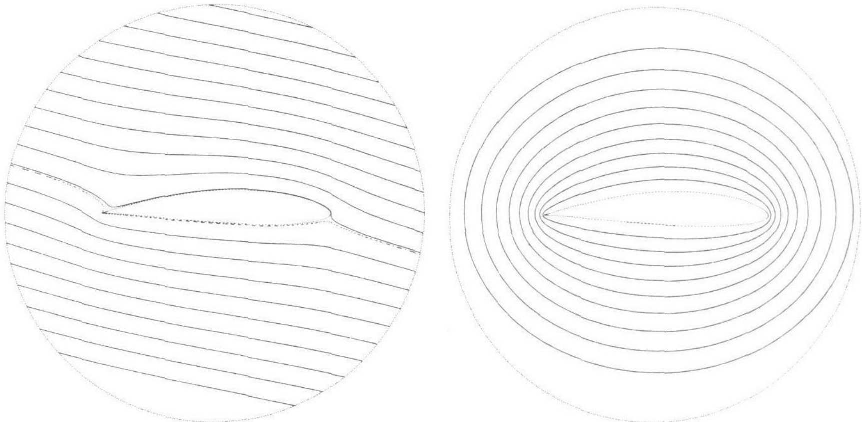

Figure 4. Contours of computed u_0 and u_1.

In our numerical test, we let $\mathbf{q}_\infty = (-1, 1/4)$ in (1) where the domain is described by a Kármán–Trefftz airfoil profile defined by the transformation

$$z = -nb\frac{(c-b)^n + (c+b)^n}{(c-b)^n - (c+b)^n}. \tag{14}$$

This maps the outside of the unit circle to the exterior of a thin airfoil on the complex plane. For our test, in (14), $b = 1$, $n = 1.9$, $d = 0.0707$, $i = \sqrt{-1}$, and $c = bd(1+i) + b\sqrt{(b+d)^2 + d^2}e^{i\theta}$ for $0 \le \theta < 2\pi$. The auxiliary (outer) boundary is the circle of radius $a = 3.5$ centered at the origin. We emphasize again that by the specially chosen outer boundary, the boundary element discretization does not appear in the computation. We remark that the relaxation parameter ω in (12) is necessary to ensure the convergence (cf. [7]). It appears the best ω is around 0.7 which is used in this numerical test (cf. [7] for the computational results on ω).

On the 4th level we have about 5000 unknowns in the linear system (8). By the first algorithm (Definition 1), we need about 10 multigrid V–cycle ($p = 1$ in Definition 1) iterations with 8 smoothings ($m = 8$ in Definition 1). The computed stream functions (on level 4) are depicted in Figure 4. We also tested the second algorithm (Definition 2), where we apply the V–cycle iteration with 8 smoothings in the inner iteration (stops after 4 or 5 cycles, rate is about 0.1 for the V–cycle iteration). Then 5 outer iterations reduce the error (to the smooth solution) to less than 1 percent. We comment that the work for both methods is about the same because one evaluation of $B_k v_k$ ($v_k \in V_k$) is more expensive than that of $A_k v_k$ (this depends on the implementation, and can be avoided). However, both methods are very efficient due to their optimal order of computation. In our test computation on a SPARC station IPX, it takes a few seconds to solve the linear system on level 4.

From the solution u_0 in Figure 4, we can see that due to the singularity at the trailing edge of the thin airfoil, the solution u_0 needs to be corrected by u_1 to obtain a physical solution that satisfies the Kutta–Joukowski condition. Since the shape of the thin airfoil is known, we know the exact singular term at the trailing edge as specified in (7), where $\beta \approx 3.2 \approx 18^o$ is the angle for our airfoil. After we computed

u_0 and u_1, we compute (cf. [9]) the coefficient c_i of the leading singular term in $u_i = c_i w +$ (smooth terms) by the formula:

$$c_i = \int_{\Gamma^*} (\nabla u_i \cdot \mathbf{n}) w^* \, ds - \int_{\Gamma^*} (\nabla w^* \cdot \mathbf{n}) u_i \, ds$$

for $i = 0$ and 1, where Γ^* is an arc surrounding the non–convex corner and $w^* := (-1/\pi) r^{-\pi/(2\pi-\beta)} \cos(\pi\theta)/(2\pi - \beta)$ is the dual function satisfying $\Delta w^* = 0$ in the region bounded by Γ^* and Γ. In our computation, Γ^* consists of the edges (depicted by bold lines in Figure 2) around the trailing edge T. The computed $c_0 \approx 0.928348$ and $c_1 \approx 0.220257$. After we found the singular terms in u_i, we can get the correct solution $u_F = u_1 + \kappa u_0 = u_1 - (c_1/c_0) u_0$, which is plotted in Figure 5.

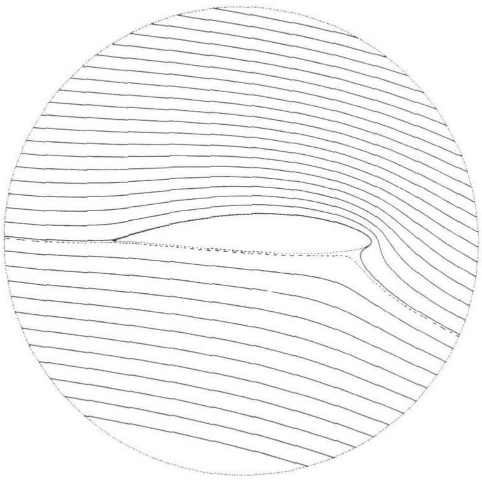

Figure 5. The computed solution u_F on the level 4 grid.

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Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA