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Newton-Krylov-Schwarz Techniques Applied to the Two-Dimensional Incompressible Navier-Stokes and Energy Equations

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ABSTRACT. We present research on Newton's method for the solution of the two-dimensional finite volume discretization of the incompressible Navier-Stokes and energy equations in primitive variables. Our previous research has employed a direct banded solver [6] and ILU preconditioned conjugate gradient-like algorithms (GMRES, CGS, QMRCGS, Bi-CGSTAB) [7, 8] to solve the linear systems arising on each Newton step. In this paper we show results from a preliminary investigation that uses domain decomposition to precondition the TFQMR conjugate gradient-like algorithm, showing the dependence of convergence rate on overlap, blocking strategy, and the additive/multiplicative trade-off.

1. Introduction

New numerical techniques are often tested on model problems that are over-simplified with respect to geometry, boundary conditions, and neglect of multiple scales. However, a better understanding of the processes at work, both physical and numerical, motivates solving more complicated model problems. Simulation codes for such models typically require solving very large systems of equations. Fully implicit discretization techniques that employ iterative solvers with a high degree of parallelism provide a viable mechanism for solving these more difficult problems.

In this work we present results obtained by applying a combination of numerical techniques of growing popularity for parallel computing, to model fluid flow and heat transfer problems involving both natural (free) and forced convection.

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The model problems addressed here include natural convection in an enclosed square cavity and internal flow past a backward facing step. The latter problem is defined on a physical domain with a large aspect ratio. Section 2 presents a description of these model problems. Section 3 presents a brief description of the numerical techniques, while results of our investigation are given in Section 4. Section 5 contains some conclusions and prospects.

2. Model Problems

The following are the governing equations for the model problems we solve:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{\partial uT}{\partial x}+\frac{\partial vT}{\partial y}=\frac{1}{Pe}\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right]$$

$$\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]$$

$$\frac{\partial uv}{\partial x}+\frac{\partial v^2}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{Re}\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]+Gr\ T$$

The backward facing step problem [1] is defined in the region $\Omega=[-.5,.5]\times[0,30]$ with $Re=100$, $Pe=Re\cdot Pr=70$, and $Gr=0$, and satisfies the following boundary conditions:

$$\text{For } x\in[0,.5],\quad \begin{cases} v(x,0)=\frac{3}{2}(4x)(2-4x)\\ T(x,0)=\left[1-(1-4x)^2\right]\left[1-\frac{1}{5}(1-4x)^2\right]\\ u(x,0)=0\end{cases}$$

$$\text{For } x\in[-.5,0],\quad u(x,0)=v(x,0)=\frac{\partial T}{\partial y}(x,0)=0$$

$$\text{For } y\in[0,30],\quad \begin{cases} u(\pm.5,y)=v(\pm.5,y)=0\\ \frac{\partial T}{\partial x}(\pm.5,y)=\mp\frac{32}{5}\end{cases}$$

$$\text{For } x\in[-.5,.5],\quad \frac{\partial u}{\partial y}(x,30)=\frac{\partial v}{\partial y}(x,30)=\frac{\partial T}{\partial y}(x,30)=0$$

The natural convection problem is defined in the region $\Omega=[0,1]\times[0,1]$ with $Re=1$, $Pe=Re\cdot Pr=0.71$, and $Ra=Gr\cdot Pr=10000$, and satisfies the following boundary conditions:

$$\text{For } (x,y)\in\partial\Omega,\quad u(x,y)=v(x,y)=0$$

$$\text{For } x\in[0,1],\quad \frac{\partial T}{\partial y}(x,0)=\frac{\partial T}{\partial y}(x,1)=0$$

$$\text{For } y\in[0,1],\quad \begin{cases} T(0,y)=0\\ T(1,y)=1\end{cases}$$

3. Numerical Techniques

The nonlinear governing equations are linearized utilizing an inexact Newton's Method [4]. The finite volume method on a staggered grid is used to discretize the model PDEs in primitive variables. Next, the Jacobian for each Newton iteration is formed by numerically evaluating the required derivatives [**6**]. New Newton iterates are computed until successive vector iterates change by less than 10^{-6} in Euclidean norm. This linear system is solved by the preconditioned iterative Krylov method, transpose-free quasi-minimal residual, (TFQMR) [**5**]. In an efficient inexact Newton approach, the Krylov solver is iterated until the scaled residual is less than 10^{-2}. For this work we use overlapping additive and multiplicative Schwarz block preconditioners [**2, 3**].

4. Results

The finite volume discretization of the backward facing step problem uses a uniform grid with 24 cells along the x-axis and 96 cells along the y-axis. The natural convection problem uses a 48 by 48 uniform cell structure. Thus each implicit nonlinear system has 9216 degrees of freedom. The blocking for the Schwarz preconditioners are chosen to be a uniform checkerboard pattern. In the tables, the notation $(bx \times by)$ is used to indicate the blocking strategy which has bx blocks along the x-axis and by blocks along the y-axis. The blocking strategy is given in the first column of the tables. When overlapping is used each block is "grown" uniformly to give either a three or four cell overlap. The size of the overlap region is given in the first row of the tables. The values reported in the table are the average number of TFQMR iterations per Newton step required to meet the inexact Newton convergence criteria. For comparison, a global ILU(0) preconditioner requires an average of 178 TFQMR iterations per Newton step for the backward facing step problem, and 114 for the natural convection problem.

Our initial studies are designed to obtain a better understanding of the performance of these types of iterative methods applied to models with complicated physics. We identify three major issues that should be considered when using these methods.

The first issue concerns the partitioning into subdomains. With computationally complex problems, memory requirements will often dictate the minimum number of subblocks. The choice of blocking strategy can be very important in the convergence of the iterative algorithms. For example, Table 1 compares the results for the case of six blocks (rows 4-7) for the backward step problem. Using the additive Schwarz preconditioner with no overlap, the average TFQMR iterations per Newton step is 183 for 6×1 blocking, whereas 1×6 blocking only requires 14. Similar results hold for the multiplicative Schwarz preconditioner. The natural convection problem shows a similar but less pronounced dependence on the blocking strategy, as evidenced in Table 2. These results indicate that the

Table 1: Backward facing step problem results (ILU(0) requires 178 iterations).

	Additive		Multiplicative	
Blocking	No Overlap	3 Cell Overlap	No Overlap	3 Cell Overlap
1x4	7	8	4	3
2x2	29	21	14	7
4x1	91	35	48	17
1x6	14	13	4	4
2x3	31	22	15	8
3x2	63	30	30	12
6x1	183	70	82	30
3x3	64	40	33	13
4x4	100	68	47	20
6x6	187	118	85	34
Average TFQMR iterations per Newton step				

block aspect ratio significantly influences the performance of the preconditioner.

Another issue is the use of overlap. If a poor choice in blocking strategy cannot be avoided, the use of overlap can significantly reduce the number of required TFQMR iterations. Again, consider the 6×1 blocking case; the use of a three cell overlap reduces the average TFQMR iterations from 183 to 70 for the additive Schwarz preconditioner.

The use of the multiplicative Schwarz preconditioner instead of the additive Schwarz preconditioner can also provide performance benefits. The use of the multiplicative Schwarz preconditioner generally requires less than half the iterations needed by the additive Schwarz preconditioner. For example consider the backward facing step problem with 6×6 blocking and three cell overlap; the additive Schwarz preconditioner requires 118 average TFQMR iterations per Newton step while the multiplicative Schwarz preconditioner requires only 34. We note that the serial nature of the multiplicative Schwarz preconditioner is not a large deterrent to its use in a parallel computing environment since for the checkerboard blocking the preconditioner may be realized with only four serial steps through multicoloring.

5. Summary And Future Work

Our initial work indicates the blocking strategies play an important role in the performance of the algorithms. Both cell and block aspect ratios should be considered when selecting what blocking strategy to use. Several trials may be necessary to "tune" the preconditioner. Improvements may be obtained by the use of an overlap region and the use of the multiplicative Schwarz preconditioner.

Some future work will involve: extending these algorithms to systems of convection-reaction-diffusion equations, distributing the preconditioner over a heterogeneous network using PVM, adding a coarse grid solve, and performing

Table 2: Natural convection problem results (ILU(0) requires 114 iterations).

Blocking	No Overlap	4 Cell Overlap
1x4	10	5
2x2	9	5
4x1	9	5
1x6	17	7
2x3	12	6
3x2	12	6
6x1	16	8
1x8	23	9
2x4	13	8
4x2	15	8
8x1	21	9
2x8	24	13
4x4	22	12
8x2	26	14
6x6	36	15
8x8	51	19
Average TFQMR iterations per Newton step		

numerical eigenvalue analysis to study preconditioner effectiveness.

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